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Twin buildings : characterization and classification results

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Mirror worlds

This thesis is a thesis on the world of twin buildings, an extension of the former building theory.
Building theory was orginally developed by J. Tits to give a geometrical interpretation of the theory of semi-simple linear algebraic groups. Implicitely buildings were born in 1965, in the famous paper [28]. Although at that time the geometries which J. Tits constructed were not called buildings yet. Buildings where officially established in the mathematical landscape in '74, with the appearence of the standard reference [29]. In this work J. Tits gives a complete classification of spherical buildings (i.e. buildings of finite diameter) of rank bigger than 3. Important in this classification was the fact that every spherical building of rank bigger than 3 is a Moufang building. (Moufang buildings can best be seen as buildings with a high degree of symmetry.)

A lot of techniques used in this work found their inspiration in algebraic group theory. Especially J. Tits succeeded in generalizing the concept of a root datum, the Galois cohomology of algebraic groups and relative algebraic group theory.

During the 90 new developments in physics (i.e. quantum gravity, super string theory) led to an interest of mathematicians into a new algebraic structure, namely Kac-Moody algebras. These structures arose as generalization of the former will known Lie algebras, symmetry groups of certain physical systems. Lie algebras are always defined in finite dimensional vector spaces. The difference in Kac-Moody theory is that Kac-Moody algebras can be defined in infinite dimensional vector spaces, making the theory in a lot aspects quite different from Lie algebra theory.

Every Lie algebra corresponds to an algebraic group and conversily every algebraic group induces a Lie algebra. As J. Tits already succeeded in giving a geometrical interpretation to linear algebraic groups he and M. Ronan started to think how a geometry associated to a Kac-Moody algebra would look like. As a result they introduced the concept of a twin building. One way to see twin buildings is as a couple of buildings, endowed with an opposition relation between them. This opposition relation is in fact a generalization from the fact that the diameter of a spherical building is finite. Intuitively this opposition relation is as if one would consider a building, put it before a mirror and consider the reflected building as its twin.

In the standard reference [32] J. Tits describes a possible classification program for twin buildings. The techniques he proposes hereby are quit related to the ones used in [29]. Having made a phd on a part of building theory that has strong connections with twin building theory, Bernhard Mühlherr got interested in this classification. In particular he wanted to get a classification of 2-spherical twin buildings, i.e. twin buildings where the diameter is locally finite.
A first major result needed to start the classification was proved in '92 by Bernhard Mühlherr and M. Ronan in [18]. Using this result they could prove that the two parts of a twin building are in the 2-spherical case Moufang buildings and that a twin building is completely determined by its local datum, namely a Moufang foundation.
The next important steps towards a classification were taken in [21] and [20]. In [21] Bernhard Mühlherr succeeded in extending the relative theory of algebraic groups and the Galois cohomology to the field of twin buildings. This led to [20] where the classification of 2-spherical twin buildings is reduced to a classification of three types of geometries. Namely twin buildings of type $\tilde{A}_2$, $\tilde{B}_2$ and 443.

As buildings of $\tilde{A}_2$ where studied before thoroughly (cfr. [35, 36]) this left to problem of classifying twin buildings of type $\tilde{B}_2$ and 443. This was the starting point of writing this thesis. In the end I got a classification of $\tilde{B}_2$ twin buildings where the geometries are locally classical or indifferent and a integrability criterion for twin buildings of type 443. During the process of finding the right tecniques the theory developed could in some cases also be used to solve other non related problems.

The thesis is organised in four Chapters.

Chapter 1 gives an overview of the definitions and notations used in the thesis.

Chapter 2 consists of the proofs of two theorems. The first theorem was a problem known to be true but no real formal proof was written down. It concerns the fact that every Moufoufang building can be seen as half of a twin building. This problem was stated in [29] with a strategy of a proof. But J. Tits mentions that the proof is not straightforward and some new concepts should be introduced. In order to get familiar with twin building theory, I had to solve this problem. At this point I have to mention this result was found independantly by P. Abramenko and I should thank him for the mathematical discussions and suggestions conderning this problem. The second theorem deals with a local characterization criterion for twin buildings which was also found independantly by P. Abramenko and H. Van Maldeghem.

In the setup of [29] and [20] a classification of 2-spherical twin buildings relies heavily on a carefull study of Moufang sets. Moufang sets are in fact the smallest twin buildings and form the building blocks of every twin buildings. Therefore we give in this chapter a classification of the Moufang sets needed to classify $\tilde{B}_2$ twin buildings which are locally classical of indifferent. The problems which arose here are quit related to Borel Tits theory (cfr. [2]), classical theory on orthogonal, hermitian and unitary groups of Witt index 1 (cfr. [6, 7]) and algebraic group theory (cfr. [27]). But as all these theories only work under restrictions, which had to be avoided for the classification, a completely new setup was developed and some new results on classical groups came out. (cfr. Theorem 127) As a byproduct of the theory developed in Chapter3, a local characterization of classical Moufang sets could be proved (cfr. Theorem 132).

The final chapter deals with classification and integrability conditions.
As already mentioned by the results in [18], twin buildings are completely determined by their local data which are called Moufang foundations. Hence

to classifiy twin buildings one has to give a list of existing Moufang foundations, and then try to see which Moufang foundations are integrable (i.e. isomorphic to the local data of a Moufang building).
As in almost all cases (by the results of Chapter 2 and [18]) twin buildings and Moufang buildings of type $\tilde{B}_2$ are the same objects we preferred working in this chapter with Moufang buildings instead of twin buildings. In particular we give a classification of $\tilde{B}_2$ Moufang buildings which are locally classical or indifferent and prove integrability conditions for Moufang foundations in Moufang buildings of type 443.
To give a list of existing foundations of type $\tilde{B}_2$ we rely heavily on the results of Chapter 3. To prove integrability we again rely on Chapter 3 and the results proved in [23]. Moreover to complete the classication we prove a theorem (cfr. Theorem 151) which uses representations of Moufang sets and properties of the geometry. Similar techniques could be used to prove a integrability criterion for Moufang foundations of type 443 (cfr. Theorem 158).

Contents

Chapter 1

Preliminaries

1.1 Conventions and notations

$\|X\|$	cardinality of a set X
$co_X(Y)$	complement of a set Y in X
$\mathbb{N}$	natural numbers
$\mathbb{Z}$	integers
$\mathbb{R}$	real numbers
$\mathbb{H}$	generalized quaternion algebra
$-$	standard involution of a certain generalized quaternion algebra $\mathbb{H}$
k	division ring
$char(k)$	characteristic of a division ring k
V, E	vector spaces
$dim(V)\|_k$	dimension of a k-vector space if k is clear from the context this is also denoted by $dim(V)$
$PG(E)$	projective space associated to a vector space E.
$GL_n(k)$	general linear group acting on a n-dimensional right k-vector space
G	group
$ord(g)$	order of an element g of a group
$Z(G)$	center of a group G
$[g_1, g_2]$	commutator of two elements g_1 and g_2 of a group G
$Fix_G(Y)$	$\{g \in G\|g(y) = y, \forall y \in Y\}$, where G acts on a set Y
$Stab_G(Y)$	$\{g \in G\|g(Y) = Y\}$, where G acts on a set Y
Id	identity map
$\Pi_{j=1}^m \theta_j$	$\theta_1\theta_2 \ldots \theta_m$
$\sum_{j=1}^m \theta_{1,j} \ldots \theta_{i(j),j}$	$\theta_{1,1}\theta_{2,1} \ldots \theta_{1,i(1)} + \ldots + \theta_{1,m}\theta_{2,m} \ldots \theta_{i(m),m}$

1.2 Definitions

In this section we list the most frequently used definitions in the sequel. Motivation of concepts will be indicated where it is possible.

1.2.1 Coxeter systems

Definition 1 Let I be a set. A *Coxeter matrix M over I* is a symmetric matrix $M = (m_{ij})_{i,j \in I}$ with $m_{ij} \in \mathbb{N} \cup \infty$ such that $m_{ij} \geq 2$, $i \neq j$ and $m_{ii} = 2$.
Given a Coxeter matrix M over I we denote by $E(M)$ the set $\{\{i,j\} \mid m_{ij} \geq 3\}$.

Definition 2 Let M be a Coxeter matrix over a set I. Then we denote by $G(M)$ the graph whose vertices are the elements of i and where $i,\ j \in I$ are adjacent whenever $\{i,j\} \in E(M)$.

For a $J \subset I$ we set $M_J = (m_{ij})_{i,j \in J}$.
Notice that by the above definition M_J is a Coxeter matrix over J.

Definition 3 Let $M = (m_{ij})_{i,j \in I}$ be a Coxeter over I and $\bar{M} = (\bar{m}_{k,l})_{k,l \in K}$ be a Coxeter matrix over K. An *isomorphism between M and $\bar{M}$* is defined as a bijection φ from I to K such that $m_{ij} = \bar{m}_{\varphi(i)\varphi(j)}$, $\forall\ i,j \in I$. A *embedding from M to $\bar{M}$* is an isomorphism from M to a Coxeter matrix of the form $\bar{M}_{K'}$ where $K' \subset K$

Definition 4 A Coxeter matrix $M = (m_{ij})_{i,j \in I}$ will be called *2-spherical* whenever m_{ij} is finite $\forall\ i,j \in I$.

Definition 5 Given a Coxeter matrix M, a *Coxeter system of type M* is a couple $(W, (s_i)_{i \in I})$ where W is a group with presentation $W = \langle s_i | (s_i s_j)^{m_{ij}} \rangle$. The *rank* of a Coxeter system is the cardinality of I. A Coxeter system $(W, (s_i)_{i \in I})$ is called *spherical* if the group W is finite.

The set $(s_i)_{i \in I}$ will also be denoted as S. The group W is called a Coxeter group of type M. Using standard theory as exposed in [14] and [3] one shows that if $(W, (s_i)_{i \in I})$ is a Coxeter system of type M and $J \subset I$, $(W, (s_i)_{i \in J})$ is a Coxeter system of type M_J.

Definition 6 Given a Coxeter matrix M and a Coxeter system (W, S) of type M we define for $w \in W$:

$$l(w) = min\{n \mid w = s_1 s_2 \dots s_n,\ s_j \in S, 1 \leq j \leq n\}.$$

An expression of the form $s_1 s_2 \dots s_m$, $s_j \in S$, $1 \leq j \leq m$, will be called *reduced* whenever $l(s_1 s_2 \dots s_m) = m$.
Given $w \in W$, $i, j \in I$ such that $m_{ij} < \infty$, we call w *right* $\{i, j\}$*-anti-reduced* if $l(ws_i) < l(w)$ and $l(ws_j) < l(w)$.

A standard example of a Coxeter group is provided by Euclidean reflection groups. Namely, consider a finite subgroup G of $GL_n(\mathbb{R})$ generated by m reflections $(r_i)_{1 \leq i \leq m}$. Then one can show that the group G has a presentation of the form $G = \langle s_i | (s_i s_j)^{m_{ij}} \rangle$. For more information about reflection groups we refer to [14].

1.2.2 Root systems

Let $M = (m_{ij})_{i,j \in I}$ be a Coxeter matrix over I and $(W, (s_i)_{i \in I})$ a Coxeter system of type $M = (m_{ij})_{i,j \in I}$. Remark that every element $z \in W$ defines an permutation (also denoted by z) W if we set :

$$z(x) = zx.$$

In this action we call elements of the form $ws_i w^{-1}$, $i \in I$, *reflections.*

Definition 7 Let $M = (m_{ij})_{i,j \in I}$ be a Coxeter matrix over I. Consider a Coxeter system (W, S) of type M with $S = (s_i)_{i \in I}$. Let $s_i \in S$. *The root defined by* s_i *(in* W*)* is defined as the set $\alpha_i = \{w \in W | l(s_i w) > l(w)\}$. All other roots in W are subsets of the form $w(\alpha_s) = \{wv \ | v \in \alpha_s\}$ for some $w \in W$. The *opposite root of a root* α is defined as $co_W(\alpha)$. Given a root α of the form $w(\alpha_i)$ we denote the reflection $ws_i w^{-1}$ by s_α. For every root α the *boundary of* α denoted by $\partial\alpha$, is the set of pairs $\{x, y\}$ such that $s_\alpha(x) = y$. Moreover given a root α, the *interior of* α is defined as the set $Int(\alpha) = \{\alpha \setminus \partial\alpha\}$. Roots are called positive or negative according to whether they contain 1 or not. If a root α is positive, this is denoted by $\alpha > 0$. Similarly $\alpha < 0$ means that the root α is a negative root. Remark that if α is a root in W then $s_\alpha(\alpha) = co_W(\alpha)$.

Definition 8 Let $M = (m_{ij})_{i,j \in I}$ be a Coxeter matrix over I and $(W, (s_i)_{i \in I})$ be a Coxeter system of type $M = (m_{ij})_{i,j \in I}$. Then we call the set of roots Φ a *the root system.*

Definition 9 Let Φ be a root system such that Φ consists of the roots in W where (W, S) is a Coxeter system of type M. Then we say that Φ is *of type* M' if and only if $M \cong M'$.

Definition 10 Let $M = (m_{ij})_{i,j \in I}$ be a Coxeter matrix over I. Given a root system Φ of type M such that $(W, (s_i)_{i \in})$ is the Coxeter system with Φ the set of roots in W. A *root base* or Φ is then defines as a set $\{w\alpha_i \mid i \in I\}$, where $w \in W$ and α_i is the root belonging to the reflection s_i.

Definition 11 Given a root system Φ, such that Φ consists of the roots in W, where (W, S) is a Coxeter system. Let Λ be a root base for Φ. Denote then the Coxeter matrix $\bar{M}_\Lambda = (\bar{m}_{\alpha\beta})_{\alpha\beta \in \Lambda}$ where $\bar{m}_{\alpha\beta} = ord(s_\alpha s_\beta)$.

The definition of root base implies that Φ is of type $\bar{M}_\Lambda$ for every root base $\Lambda \in \Phi$.

1.2.3 Buildings

Definition 12 Let $M = (m_{ij})_{i,j \in I}$ be a Coxeter matrix, (W, S) a Coxeter system of type M. A *building of type* M is a quadruple (Δ, W, S, d) where Δ is a set whose elements are called *chambers* and d is a function, called distance function, going from $\Delta \times \Delta$ to W satisfying :

Bu1 $d(x, y) = 1$ if and only if $x = y$

Bu2 Let $x, y \in \Delta$ with $d(x, y) = w$. If z is a chamber such that $d(y, z) = s$ with $s \in S$ then $d(x, z) \in \{w, ws\}$. Moreover if $l(ws) > l(w)$ then $d(x, z) = ws$.

Bu3 Let as above $x, y \in \Delta$ with $d(x, y) = w$. If $s \in S$ then there exists a chamber z of Δ such that $d(x, z) = ws$.

The *rank* of the building (Δ, W, S, d) is defined as the rank of (W, S).

Given a Coxeter system (W, S) of type $M = (m_{ij})_{i,j \in I}$, we can view it as a building in the following way. The chambers are the elements of W. Define the distance d_W on W by $d_W(x, y) = x^{-1}y$ for $x, y \in W$. Straightforward calculations show that (W, W, S, d_W) is a building of type $M = (m_{ij})_{i,j \in I}$.
To simplify notation a building (Δ, W, S, d) will sometimes be denoted as (Δ, d) or even as Δ when the rest of the data is clear.

The concept of a building is due to J.Tits as a result of his research to give a geometrical interpretation of the theory of semi-simple algebraic groups. A special class is provided by the buildings of spherical type i.e. buildings of the form (Δ, W, S, d) where (W, S) is a spherical Coxeter system. They present in a natural way the geometry associated to a simple algebraic group. More information about this subject can be found in the standard reference [29] where of all spherical buildings of rank bigger that 3 are classified.
An alternative way to look at buildings is described in [25]. Buildings are defined here as chamber systems with certain properties.

1.2.4 Chamber systems and buildings

Definition 13 Given a set I, a *chamber system over I* is a set C such that each element $i \in I$ determines a partition of C. The elements of C are also called chambers. Two chambers belonging to the same class of the partition defined by i, are called i-adjacent.

It is rather natural to consider galleries in chamber systems. Their structure expresses in some cases important topological invariants and certain properties.

Definition 14 Let C be a chamber system defined over a set I. A *gallery in C* is a sequence of chambers $\Gamma = c_1c_2 \ldots c_m$ such that each pair (c_i, c_{i+1}) is l_i-adjacent for some $l_i \in I$. The gallery Γ is said to be *non stammering* if $l_i \neq l_{i+1}$ for $1 \leq i \leq m$. The *type* of the gallery Γ is defined as the string $(l_1l_2 \ldots l_m)$.

Definition 15 Let C be a chamber system defined over a set I and $J \subset I$. A *J-gallery in C* is then defined as a gallery $\Gamma = c_1c_2 \, \ldots \, c_m$ with c_i l_i-adjacent to c_{i+1} such that $l_i \in J$, $1 \leq i \leq m$.

Definition 16 Consider a chamber system C over a set I. Let $J \subset I$. A *J-residue* is a set of chambers in C such that every two chambers of the set can be joined via a J-gallery. If a J-residue contains a chamber c we willdenote it by $R_J(c)$. A $\{i\}$-residue with is also called *i-panel* or sometimes a *panel* when i is clear from the context.

Given a set I a sequence of the form $(f_1, f_2, \ldots, f_m)$ with $f_i \in I$, $1 \leq i \leq m$, will be called *word over* I or simply *word* if I is clear from the context. Let $(W, (s_i)_{i\in I})$ be a Coxeter system of type $M = (m_{ij})_{i,j\in I}$. If $g = (j_1 j_2 \ldots j_t)$ is a sequence of elements of I then we define

$$r_g = s_{j_1} s_{j_2} \ldots s_{j_t}.$$

A word $f = (f_1, f_2, \ldots, f_m)$ will be called *reduced* if $l(r_f) = m$.
Let $(\Delta, W, (s_i)_{i\in I}, d)$ be a building of type $M = (m_{ij})_{i,j\in I}$. Then we call two chamber x and y *i-adjacent* whenever $d(x, y) \in \{1, s_i\}$. Set $C_\Delta = \Delta$. Then one easily checks that in this way we get a chamber system C_Δ over I. Thus every building gives rise to a chamber system. As to the connection between buildings and chamber systems we have the following theorem.

Theorem 17 *Let $(W, (s_i)_{i\in I})$ be a Coxeter system of type $M = (m_{ij})_{i,j\in I}$, C a chamber system over I. If every panel contains at least two chambers and the function d defined by :*

$$d_C(x, y) = r_f$$

where f is a reduced word if and only if there exists a gallery of type f from x to y is well defined then (C, W, S, d_C) is a building of type M.
Conversely let $(\Delta, W, (s_i)_{i\in I})$ be a building of type $M = (m_{ij})_{i,j\in I}$ and consider the chamber system C_Δ over I where $C_\Delta = \Delta$ and where chambers x and y are called i-adjacent if and only if $d(x, y) \in \{1, s_i\}$ then the following condition holds :

$$d(x, y) = r_f$$

where f is a reduced word if an only if there exists a gallery of type f in C from x to y.

proof :

Can be derive from section 1 of Chapter 3 in [25]. □

In the sequel we will not always explicitly mention whether we view a building as chamber system or not if this is clear from the context.
Using the chamber system approach of buildings we define the notion of morphisms between buildings.

Definition 18 Given two buildings (Δ, W, S, d) and (Δ', W, S, d') of the same type with $S = (s_i)_{i\in I}$, a *morphism* from (Δ, W, S, d) to (Δ', W, S, d') is a mapping φ going from Δ to Δ' such that x and y are i-adjacent if and only if

$\varphi(x)$ and $\varphi(y)$ are i-adjacent for $x, y \in \Delta$. An isomorphism is also called an *isometry*.

If (W, S) is a Coxeter system of certain type M, an isometry of W on itself, where we consider W as a building, is given by left multiplication with a fixed element of W.

1.2.5 Generalized n-gons

Let $\bar{M} = (\bar{m}_{ij})_{i,j \in I}$ be a spherical Coxeter system of rank 2. Suppose (Δ, W, S, d) is a spherical building of type $\bar{M}$. Then there is another way of defining the geometry of (Δ, W, S, d) using points and lines (cfr [37]). Firstly we define what is meant by geometry.

Definition 19 A *rank* 2 *geometry* Γ is a triple $(\mathcal{P}, \mathcal{L}, I)$, where $\mathcal{P}$, $\mathcal{L}$ are two sets, called the point set resp. line set, and $I \subset (\mathcal{P} \times \mathcal{L}) \cup (\mathcal{L} \times \mathcal{P}\ )$ a symmetric relation between $\mathcal{P}$ and $\mathcal{L}$.

If $(\mathcal{P}, \mathcal{L}, I)$ is a rank 2 geometry I is called the *incindence relation* of Γ. A point p and a line l are called *incident* whenever $(p, l) \in I$. The point p is said to lie on l and the line is said to pass through p. Two points lying on a line are called *collinear* and two lines intersecting in a point are called *concurrent*. A *flag* is a pair $(p, l) \in \mathcal{P} \times \mathcal{L}$ such that $(p, l) \in I$. The set of all flags in Γ is denoted by $\mathcal{F}$. If p is a point of a rank 2 geometry $\Gamma = (\mathcal{P}, \mathcal{L}, I)$ we denote $\Gamma(p) = \{h \in \mathcal{L} \mid (p, h) \in I\}$. Similarly $\Gamma(l) = \{\ q \in \mathcal{P} \mid (q, l) \in I\}$.

Definition 20 Given a rank 2 geometry $\Gamma = (\mathcal{P}, \mathcal{L}, I)$, a *subgeometry* Γ' is a rank 2 geometry $(\mathcal{P}', \mathcal{L}', I')$ where $\mathcal{P}' \subset \mathcal{P}$, $\mathcal{L}' \subset \mathcal{L}$ and $I' \subset I$.

Definition 21 Let $n \geq 2$ and $n \in \mathbb{N}$. A *generalized n-gon* is a rank 2 geometry $\Gamma = (\mathcal{P}, \mathcal{L}, I)$ such that the following axioms are satisfied :

(i) Γ contains no ordinary k-gon as subgeometry for $2 \leq k < n$.

(ii) Any two elements $v, u \in \mathcal{P} \cup \mathcal{L}$ are contained in some ordinary n-gon (viewed as subgeometry of Γ), a so called *apartment*.

(iii) For any element $u \in \mathcal{P} \cup \mathcal{L}$, $\Gamma(u)$ contains at least 3 elements.

Given a generalized n-gon $(\mathcal{P}, \mathcal{L}, I)$ for some $n \geq 2$ we can construct a spherical rank 2 building in the following way. Consider the Coxeter matrix $\bar{M}$

$$\begin{pmatrix} 1 & n \\ n & 1 \end{pmatrix}.$$

Let $(\bar{W}, (\bar{s}_i)_{i \in I})$ be the Coxeter system of type $\bar{M}$. Set $\bar{\Delta} = \mathcal{F}$. Define a distance function $\bar{d}$ on $\bar{\Delta} \times \bar{\Delta}$ in the following way. For two flags $F_1 = \{p, l_1\}$ and $F_2 = \{p, l_2\}$ define $\bar{d}(F_1, F_2) = \bar{s}_1$. Similarly if $F_1' = \{p_1, l\}$ and $F_2' = \{p_2, l\}$ we define $\bar{d}(F_1', F_2') = \bar{s}_2$. Let $F = \{p, l\}$ and $G = \{q, h\}$ be two flags. Consider a minimal sequence $x_1 x_2 \ldots x_m$ such that $x_i \in \mathcal{P} \cup \mathcal{L}$, $F_i = \{x_i, x_{i+1}\} \in \mathcal{F}$, $1 \leq i \leq m$. Define $\bar{d}(F, G) = \bar{d}(F_1, F_2).\bar{d}(F_2, F_3) \ldots \bar{d}(F_{m-1}, F_m)$. The following proposition holds.

Proposition 22 *With the notation from above the system* $(\mathcal{F},\bar{W},(\bar{s}_i)_{1 \leq i \leq 2})$, $\bar{d})$ *is a thick spherical building of rank* 2. *Conversely every thick spherical building* $(\bar{\Delta},\bar{W},\bar{S},\bar{d})$ *of rank* 2 *can be obtained in this way i.e. there exists a rank two geometry* Γ *such that* $\bar{\Delta}$ *is the set of all flags of* Γ *and* $\bar{d}$ *is as defined above.*

proof :

We refer to [29] and Theorem 1.3.8 in [37]. □

Definition 23 Let $\Gamma = (\mathcal{P}, \mathcal{L}, I)$ and $\Gamma' = (\mathcal{P}', \mathcal{L}', I')$ be generalized n-gons. An *isomorphism from* Γ *to* Γ' is a bijection β from $\mathcal{P}$ to $\mathcal{P}'$ and from $\mathcal{L}$ to $\mathcal{L}'$ preserving incidence i.e.

$$(x, y) \in I \Leftrightarrow (\beta(x), \beta(y)) \in I'.$$

A *duality from* Γ *to* Γ' is a bijection from $\mathcal{P}$ to $\mathcal{L}'$ and from $\mathcal{L}$ to $\mathcal{P}'$ preserving incidence. If there exists a duality from Γ to Γ' we say that Γ and Γ' are *dually isomorphic.*

1.2.6 BN-pairs

As already mentioned buildings arose from the geometrical structure of algebraic groups. It is therefore not surprising that they have a group theoretical counterpart.

Definition 24 Let G be a group with two subgroups B and N. Then (G, B, N, S) is *a Tits system or BN-pair* if the following axioms are satisfied :

BN0 $\langle B, N\rangle = G$.

BN1 $H = B \cap N \trianglelefteq N$ and N/H is a Coxeter group with generating set $S = \{(s_i)_{i\in I}\}$.

BN2 $Bs_iBwB \subset Bs_iwB \cup BwB$ whenever $w \in W$ and $s_i \in S$.

BN3 $s_iBs_i \neq B$ for $s_i \in S$

If G is a group with a BN-pair (B, N) one can show that $G = \bigsqcup_{w\in W} BwB$ (for a proof we refer to Lemma 5.1. in [25]). This is the so called *Bruhat decomposition* of G. Moreover we have the following theorem.

Theorem 25 *Every BN-pair (B, N) in a group G defines building, where chambers are left cosets of B and distance is given by :*

$$d(gB, hB) = w$$

where w is the unique element of W such that $g^{-1}h \in BwB$.

<u>proof</u> :

Follows Theorem 5.1 in [25] and Theorem 17 □

But not every building can be constructed in such a way. There is however a special condition that ensures a building (Δ, W, S, d) to come from a BN-pair. This is the condition of a group G acting strongly transitive on Δ (for more information we refer to p57 in [25]). A special condition which ensures that such a group exists is the Moufang condition. In order to give a proper definition of the Moufang condition we need some more terminology.

1.2.7 Moufang buildings

Let $M = (m_{ij})_{i,j\in I}$ be a Coxeter matrix over I and $(W, (s_i)_{i\in I})$ a Coxeter system of type M. Let Φ be a root system of type $M = (m_{ij})_{i,j\in I}$.

Definition 26 Two roots α and β in W are called *prenilpotent* if and only if $\alpha \cap \beta \neq \varnothing$ and $(-\alpha) \cap (-\beta) \neq \varnothing$. If two roots α and β in W are prenilpotent then *the interval* $[\alpha, \beta]$ is defined as the set

$$\{\gamma \in \Psi \mid \alpha \cap \beta \subset \gamma \text{ and } (-\alpha) \cap (-\beta) \subset (-\gamma)\}$$

If $\{\alpha, \beta\}$ is a prenilpotent pair of roots, the set $[\alpha, \beta] \setminus \{\alpha, \beta\}$ will be denoted by (α, β).

Definition 27 An *apartment* Σ in a building (Δ, W, S, d) of type $M = (m_{ij})_{i,j \in I}$ is an isometric copy of the Coxeter system (W, S), viewed as the building (W, W, S, d_W), in Δ. A *root in* Δ is defined as an isometric copy of a root α in Δ. The boundary of a root in Δ is defined in a similar way. Given an apartment Σ in Δ, and $c \in \Sigma$, we can define *positive* and *negative* roots with respect to this chamber as follows. *Positive roots* with respect to c are those containing c, while *negative roots* are those not containing c. When the chamber c is clear from the context we will also simply speak about positive and negative roots in Σ.

One can prove that apartments always exist and that they characterize the geometry of the building (cfr. Theorem 3.11. in [25]).

Definition 28 Start with a building (Δ, W, S, d) of a certain type $M = (m_{ij})_{i,j \in I}$. Fix an apartment Σ_0 and denote the set of all roots in Σ_0 by Φ. Then we call the building (Δ, W, S, d) a *Moufang building* if there exists a family of automorphism groups $(U_\alpha)_{\alpha \in \Phi}$ (called *root groups*) such that :

Mo1 Every element $u \in U_\alpha$ fixes all chambers of α. If π is a panel on $\partial\alpha$ and c is the chamber of π lying in α then U_α fixes c and acts regularly on all the chambers of $\pi \setminus \{c\}$.

Mo2 If $\{\alpha, \beta\}$ is a pair of prenilpotent distinct roots then :

$$[U_\alpha, U_\beta] \subset U_{(\alpha,\beta)}.$$

Mo3 For each $u_\alpha \in U_\alpha \setminus \{1\}$ there exists an element $m(u_\alpha) \in U_{-\alpha} u_\alpha U_{-\alpha}$ stabilizing Σ.

Mo4 If $n = m(u_\alpha)$ then for every root $\beta \in \Phi$ we have $nU_\beta n^{-1} = U_{s_\alpha(\beta)}$.

The apartment Σ_0 will also be called the *standard apartment* of (Δ,W,S,d). Given a Moufang building Δ with root groups $(U_\alpha)_{\alpha\in\Phi}$, we define the group $G = \langle U_\alpha\rangle_{\alpha\in\Phi}$, N the group generated by all $m(u_\gamma)$ with $u_\gamma \in U_\gamma$ for a root γ in Φ. If Δ is a generalized polygon, the group G is also called the *little projective group* and the root group elements are called *root elations*.

Definition 29 Let (Δ, W, S, d) be a Moufang building with root groups $(U_\alpha)_{\alpha\in\Phi}$ and standard apartment Σ_0 and (Δ', W', S', d') a Moufang building with root groups $(U_{\alpha'})_{\alpha'\in\Phi'}$ and standard apartment Σ'_0. An *isomorphism from Δ to Δ' seen as Moufang buildings* is an isomorphism φ from Δ to Δ' such that $\varphi(\Sigma_0) = \Sigma'_0$ and for every $\alpha \in \Phi$

$$\{\varphi u_\alpha^{-1}\varphi^{-1} | u_\alpha \in U_\alpha\} = U_{\varphi(\alpha)}.$$

Remark that general theory as exposed in sections 1-4 in Chapter 6 of [25] show that if (Δ, W, S, d) is a spherical Moufang building the root group systen $(U_\alpha)_\alpha$ is uniquely determined by Δ. It follows therefore that every isomorphism between two spherical Moufang buildings will define automatically an isomorphism between those buildings seen as Moufang buildings.
Let (Δ, W, S, d) be a Moufang building with root groups system $(U_\alpha)_{\alpha\in\Phi}$. Fix a root base in Φ, and call it Λ. Choose for every root $\beta \in \Lambda$ a fixed element $u_\beta \neq 1 \in U_\beta$. Then we define S as the set $\{m(u_\beta)|\beta \in \Lambda\}$. Fix a chamber $c_+ \in \Sigma_0$, and use this chamber to call roots positive or negative. Denote the subgroup of elements of N that fix Σ_0 by H, the torus in the classical sense. It is easy to check that $H \subset N_G(U_\alpha)$ for all root groups U_α. We denote the group $\langle H, U_\alpha\rangle_{\alpha>0}$ by B_+, $\langle H, U_\alpha\rangle_{\alpha<0}$ by B_- and for $\alpha \in \Phi$ $\langle H, U_\alpha\rangle$ as B_α.
The group B_+ also has a geometrical meaning : it is the full stabilizer in G of the standard chamber c_+ in Δ and N is the stabilizer of the apartment Σ_0 in G. The first fact is not obvious to show. It follows mainly from Lemma 4 in section 5 in [31]. (In fact this lemma yields that $G = \cup(B_+wB_+)_{w\in W}$). We have the following theorem.

Theorem 30 *Let (Δ, W, S, d) be a Moufang building with root groups $(U_\alpha)_{\alpha\in\Phi}$. Then the quadruple (G, B_+, N, S) with notations as above is a BN-pair.*

proof :

See Proposition 6.16 of [25]. □

1.2.8 Twin buildings

After the book [29] appeared in 1974 the classification of spherical buildings was a fact. A natural question that arose was whether a generalization of the concept of a spherical building could be found. The answer was given in the late 80's. At that time J. Tits and M. Ronan introduced the concept of a twin building. This definition was motivated by the theory of Kac-Moody groups. The twin buildings appeared in this theory in a group theoretical context namely as twin BN-pairs.
We give the formal definitions.

Definition 31 Let (W, S) be a Coxeter system of type $M = (m_{ij})_{i,j\in I}$ with $S = (s_i)_{i\in I}$. *A twinned pair of buildings or a twin building of type* M is a pair of buildings (Δ_+, W, S, d_+) and (Δ_-, W, S, d_-) endowed with a codistance funcion d^* going from $\Delta_+ \times \Delta_- \sqcup \Delta_- \times \Delta_+$ to W satisfying ($\epsilon \in \{-1, 1\}$, $x \in \Delta_\epsilon, y \in \Delta_{-\epsilon}$ and $d^*(x, y) = w$)

Tw1 $d^*(y, x) = w^{-1}$.

Tw2 If $z \in \Delta_{-\epsilon}$ is such that $d_{-\epsilon}(y, z) = s_i \in S$ and $l(ws_i) < l(w)$ then $d^*(x, z) = ws_i$.

Tw3 For every $s_i \in S$ there exists at least one chamber $z \in \Delta_{-\epsilon}$ with $d^*(x, z) = ws_i$.

The *rank* of a twin building is defined as the rank of the associated Coxeter system (W, S).

Given a twin building $((\Delta_+, W, S, d_+), (\Delta_-, W, S, d_-), d^*)$, two chambers $x \in \Delta_\epsilon$ and $y \in \Delta_{-\epsilon}$ are called *opposite* whenever $d^*(x, y) = 1$. Opposition defines a symmetric relation on $\Delta_+ \times \Delta_-$, sometimes denoted by $\mathcal{O}$.

Definition 32 Let (Δ_+, W, S, d_+) and (Δ_-, W, S, d_-) be two buildings of type M and $\mathcal{O}$ a symmetric binary relation on $\Delta_+ \times \Delta_-$. Then $\mathcal{O}$ is called a *twinning* between Δ_+ and Δ_- if there exists a codistance function d^* from $(\Delta_+ \times \Delta_-) \sqcup (\Delta_- \times \Delta_+)$ to W producing a twin building $((\Delta_+, W, S, d_+)$, $(\Delta_-, W, S, d_-), d^*)$ such that :

$$\mathcal{O} = \{(x, y) \in (\Delta_+ \times \Delta_-) \sqcup (\Delta_- \times \Delta_+) \mid d^*(x, y) = 1\}.$$

1.2.9 Twin BN-pairs

As in the case of ordinary buildings, certain systems of groups will also yield twin buildings. As already mentioned these are the twin BN-pairs.

Definition 33 Let (W, S) be a Coxeter system of type $M = (m_{ij})_{i,j \in I}$, G a group with subgroups B_+, B_-, N and S a subset of G/N. Then we call the tuple (G, B_+, B_-, N, S) a *twin BN-pair* (of type M) if the following axioms are satisfied ($\epsilon \in \{-1, 1\}$):

TBN1 (G, B_+, N, S) and (G, B_-, N, S) are BN-pairs of type M with $W \cong N/(B_+ \cap N) \cong N/(B_- \cap N)$.

TBN2 $B_\epsilon w B_{-\epsilon} s_i B_{-\epsilon} = B_\epsilon w s_i B_{-\epsilon}$ for all $w \in W$ and $s_i \in S$ such that $l(ws) < l(w)$.

TBN3 $B_+ s_i \cap B_- = \emptyset$ for all $s_i \in S$.

In a similar way as for BN-pairs one can prove that every twin BN-pair has an associated twin building $((\Delta_+, W, S, d_+), (\Delta_-, W, S, d_-), d^*)$. More precisely firstly one proves that $G = \bigsqcup_{w \in W} B_+ w B_- = B_- \bigsqcup_{w \in W} w\ B_+$. This is the so called *Birkhoff decomposition* of G. Using this decomposition one proves the following theorem.

Theorem 34 *Every twin BN-pair (G, B_+, B_-, W, N, S) of type M in a group G defines a twin building $((\Delta_+, W, S, d_+), (\Delta_-, W, S, d_-), d^*))$ where (Δ_+, W, S, d_+) is the building associated to the BN-pair (G, B_+, N, S), (Δ_-, W, S, d_-) is the buildings associated to the BN-pair (G, B_-, N, S) and d^* is given by :*

$$d^*(gB_+, hB_-) = w$$

where w is the unique element of W such that $g^{-1}h \in B_+ w B_-$.

proof :

We refer to example 6 on p 23 in [1]. □

1.2.10 Moufang sets

The following concept which will appear frequently in this thesis is the notion of Moufang set. These objects where formally introduced by J.Tits in the standard reference [31], though a lot of important examples already existed in other terminologies. Moufang sets turned out to be of great importance in the study of twin buildings. In Chapter 2 we show that under some restrictions twin buildings and Moufang buildings are the same objects. Given such a Moufang building the root group structure induces on every panel a permutation group which turns this panel into a Moufang set. In order to classify twin buildings it is thus necessary to carefully study the Moufang sets which arise.

Definition 35 A *Moufang set* is a set X of points such that $|X| > 2$ together with a family of groups U_x called *root groups* satisfying :

Mos1 For every $x \in X$ the group U_x acts regularly on $X \setminus \{x\}$.

Mos2 Every group U_x stabilizes the set of groups $\{U_y | y \in X\}$ via conjugation.

Definition 36 Let $(X, (U_x)_{x\in X})$ be a Moufang set. Then we denote for $x, y, z \in X$, $u(x; y, z)$ as the unique element of U_x sending y to z. Elements of root groups are also called *root elations* and the group $\langle U_x \mid x \in X\rangle$ is called *transvection group* and is denoted by TX.

Definition 37 Given a Moufang set $(X, (U_x)_{x\in X})$ a *Moufang subset* is a sub set $Y \subset X$ such that the system $(Y, (Stab_{U_y}(Y))_{y\in Y})$ forms a Moufang set.

Proposition 38 *Assume that Y is a Moufang subset of the set $(X, (U_x)_{x\in X})$. Then $Z \subset Y$ is a Moufang subset of X if and only if it is a Moufang subset of Y.*

proof :

The proposition follows from the equality

$$Stab_{Stab_{U_y}(Y)}(Z) = Stab_{U_y}(Z),\ \forall y \in Z.$$

□

Another property of Moufang subsets is the following.

Proposition 39 *Let $(X, (U_x)_{x \in X})$ be a Moufang set and $(Y_i)_{i \in I}$ a family of Moufang subsets indexed over the set I. If $\bigcap_{i \in I} Y_i \neq \emptyset$, and $|\bigcap_{i \in I} Y_i| > 3$, it is a Moufang subset of $(X, (U_x)_{x \in X})$.*

proof :

Follows from similar arguments as above. □

Morphisms are defined in the following way.

Definition 40 Let $(X, (U_x)_{x \in X})$ and $(Y, (U_y)_{y \in Y})$ be two Moufang sets. An *isomorphism* between $(X, (U_x)_{x \in X})$ and $(Y, (U_y)_{y \in Y})$ is defined as a bijection β from X to Y such that for every $x \in X$ the map

$$u_x \mapsto \beta u_x \beta^{-1}$$

defines a group isomorphism of U_x onto $U_{\beta(x)}$.
A *morphism* between $(X, (U_x)_{x \in X})$ and $(Y, (U_y)_{y \in Y})$ is defined as an isomorphism of $(X, (U_x)_{x \in X})$ onto a Moufang subset of $(Y, (U_y)_{y \in y})$.

Given two Moufang sets $(X, (U_x)_{x \in X})$ and $(Y, (U_y)_{y \in Y})$ and a morphism β from X to Y, then β induces an *injection of TX into TY*, which we will denote in the sequel by superscript β and which is defined as :

$$g^\beta = \beta \circ g \circ \beta^{-1}, \forall g \in TX.$$

The following condition will in a lot of cases simplify the calculations to prove that a bijection between point sets defines a isomorphism between Moufang sets.

Lemma 41 *Let $(X, (U_x)_{x \in X})$ and $(X', (U_{x'})_{x' \in X'})$ be two Moufang sets. Then a bijection β from X to X' defines a Moufang set isomorphism if and only if there exist two points x and y in X such that the mappings β_x and β_y with :*

$$\begin{aligned} \beta_x(u_x) &= \beta \circ u_x \circ \beta^{-1}, \forall u_x \in U_x \\ \beta_y(u_y) &= \beta \circ u_y \circ \beta^{-1}, \forall u_y \in U_y \end{aligned}$$

define bijections between U_x and $U'_{\beta(x)}$ and between U_y and U'_{β_y}.

proof :

If β is a Moufang set isomorphism we have by definition that β_x and β_y define groups isomorphisms.
Conversely suppose β is a bijection such that β_x and β_y are bijections between the groups. Remark that β_x and β_y define by construction group morphisms. In order to show that β is a Moufang set isomorphism we have to prove that for any $z \in X$ the map β_z with $\beta_z(u_z) = \beta \circ u_z \circ \beta^{-1}$ defines a group isomorphism from U_z to $U_{\beta(z)}$.
Let $z \in X$, then we choose the unique $u_y \in U_y$ with $u_y(x) = z$ and $u_y U_x u_y^{-1} = U_z$.
If $\bar{u}_z \in U_z$ there thus exists $\bar{u}_x \in U_x$ such that $\bar{u}_z = u_y \bar{u}_x u_y^{-1}$ and we find :

$$\begin{aligned} \beta_z(\bar{u}_z) &= \beta \circ \bar{u}_z \circ \beta^{-1} \\ &= (\beta \circ u_y \circ \beta^{-1})(\beta \circ \bar{u}_x \circ \beta^{-1})(\beta \circ u_y^{-1} \circ \beta^{-1}) \\ &= \beta_y(u_y) \circ \beta_x(\bar{u}_x) \circ \beta_y(u_y^{-1}). \end{aligned}$$

As β_x and β_y are group isomorphisms this shows β_z defines a group isomorphism form U_z to $U_{\beta(z)}$. □

Definition 42 Let $(X, (U_x)_{x \in X})$ be a Moufang set. Then it is called *abelian or commutative* whenever $Z(Fix_{TX}\{x, y\}) = Fix_{TX}\{x, y\}$ for any two points x, $y \in X$.

In the following chapter we will investigate the connection between twin BN-pairs and twin buildings. It turns out that both objects are equivalent if the residue's of the buildings involved are big enough.

1.2.11 Moufang foundations

Motivated by the outline of the classification of twin buildings as described in [32] and [20] we give the definition of a Moufang foundation in the sense in [20]. It turns out that a great deal of the classification of twin buildings depends on a classification of Moufang foundations. Moufang foudations can best be seen as representations of the local data one can extract given a Moufang building.

Definition 43 Let $M = (m_{ij})_{i,j\in I}$ be Coxeter matrix over I. A *Moufang foundation of type* M is a triple $((\Delta_{ij})_{\{i,j\}\in E(M)}, (c_{ij})_{\{i,j\}\in E(M)}, (\beta_{ijk})_{\{i,j\},\{j,k\}\in E(M)})$ such that :

MoFo1 For every $\{i,j\} \in E(M)$, Δ_{ij} is a Moufang building of type $M_{\{i,j\}}$ with root groups $(U_{\alpha_k^{ij}})_{\alpha_k^{ij}\in\Phi_{ij}}$ where $U_{\alpha_k^{ij}}$ is the root group acting on the k-panel in Δ_{ij} containing c_{ij} and Φ_{ij} is a root system of type $M_{\{i,j\}}$.

MoFo2 For every $\{i,j\} \in E(M)$, c_{ij} is a chamber of Δ_{ij} and $c_{ij} = c_{ji}$, $\forall$ $\{i,j\} \in E(M)$.

MoFo3 For $\{i,j\}$, $\{j,k\} \in E(M)$, β_{ijk} defines a Moufang set isomorphism from the induced Moufang set $\mathcal{M}_{R_j(c_{ij})}(\Delta_{ij})$ to the induced Moufang set $\mathcal{M}_{R_j(c_{jk})}(\Delta_{jk})$.

Definition 44 Let $((\Delta_{ij})_{\{i,j\}\in E(M)}, (c_{ij})_{\{i,j\}\in E(M)}, (\beta_{ijk})_{\{i,j\},\{j,k\}\in E(M)})$ be a Moufang foundation of type M and $((\Delta'_{ij})_{\{i,j\}\in E(M')}, (c'_{ij})_{\{i,j\}\in E(M')}, (\beta'_{ijk})_{\{i,j\},\{j,k\}\in E(M')})$ be a Moufang foundation of type M'. An *isomorphism from* $((\Delta_{ij})_{\{i,j\}\in E(M)}, (c_{ij})_{\{i,j\}\in E(M)}, (\beta_{ijk})_{\{i,j\},\{j,k\}\in E(M)})$ *to* $((\Delta'_{ij})_{\{i,j\}\in E(M')}, (c'_{ij})_{\{i,j\}\in E(M')}, (\beta'_{ijk})_{\{i,j\},\{j,k\}\in E(M')})$ is defined as a tuple $((\gamma_{ij})_{\{i,j\}\in E(M)}, \gamma)$ with γ an isomorphism from M to M' such that γ_{ij} defines for every $\{i,j\} \in E(M)$ a isomorphism from Δ_{ij} to $\Delta_{\gamma(i),\gamma(j)}$ seen as Moufang buildings such that $\gamma_{ij}(c_{ij}) = c_{\gamma(i)\gamma(j)}$ and $\gamma_{jk}^{-1}\beta'_{\gamma(i)\gamma(j)\gamma(k)}\gamma_{ij} = \beta_{ijk}$, $\forall\{i,j\}$, $\{j,k\} \in E(M)$.

1.3 Algebraic prerequisites

We recall and prove some lemma's that will be used in the the sequel. Throughout this section k denotes a division ring endowed with an involution σ i.e. σ is a permutation of k satisfying :

$$\begin{aligned}(x+y)^\sigma &= x^\sigma + y^\sigma,\ \forall x,y \in k\\ (xy)^\sigma &= y^\sigma x^\sigma,\ \forall x,y \in k\\ 1^\sigma &= 1^\sigma.\end{aligned}$$

We set $Tr(\sigma) = \{t + t^\sigma \mid t \in k\}$, $Fix(\sigma) = \{t \in k \mid t^\sigma = t\}$, $k_{(\sigma,\epsilon)} = \{t - t^\sigma\epsilon \mid t \in k\}$ and $k^{(\sigma,\epsilon)} = k/k_{(\sigma,\epsilon)}$ where $\epsilon \in k$.
Remark that the following holds :

$$Tr(\sigma) \subseteq Fix(\sigma).$$

The following result concerning equality in this equation can be found in Chapter 8 in [29].

Lemma 45 *Let k be a division ring with involution σ. If $char(k) \neq 2$ or $\sigma|_{Z(k)} \neq 1$ then $Fix(\sigma) = Tr(\sigma)$.*

proof :

We refer to section 8.1.5. on p120 in [29]. □

Let $c \in k$. Then we will denote by σ^c the involution of k determined by :

$$\lambda^{\sigma^c} = c\lambda^\sigma c^{-1}, \forall \lambda \in k.$$

We have the following Lemma which can be derived from section 8.2.1 on p 122 of [29].

Lemma 46 *For c, $\epsilon \in k$ we have :*

$$ck_{(\sigma,\epsilon)} = k_{(\sigma^c,\epsilon')},$$

where $\epsilon' = c(c^{-1})^\sigma \epsilon$. In particular there always exists $c \in k$ such that $1 \in Tr(\sigma^c)$.

proof :

The first claim follows from the equality :

$$ct - ct^\sigma\epsilon = (ct) - (ct)^{\sigma^c}(cc^{-1})^\sigma\epsilon, \forall t \in k.$$

Suppose σ is an involution such that $1 \notin Tr(\sigma)$. Let $\theta \in Tr(\sigma)$. Consider $\sigma^{\theta^{-1}}$. We have setting $\theta = c$:

$$1 \in Tr(\sigma^{\theta^{-1}}) = \theta^{-1}k_{\sigma,-1} = \theta^{-1}Tr(\sigma).$$

□

Lemma 47 *Let k be a division ring with $Z(k) \neq k$, σ an involution of k such that $1 \in Tr(\sigma)$. If k is not generated as a ring by $Tr(\sigma)$ it is a generalized quaternion algebra with σ its standard involution.*

proof :

See 8.14 on p 150 in [29]. □

As a corollary one deduces the following result concerning commutativity.

Corollary 48 *Let k be a division ring not equal to a generalized quaternion algebra, σ an involution of k with $1 \in Tr(\sigma)$. Then*

$$Z(k) = k \text{ if and only if } \theta_1\theta_2 = \theta_2\theta_1, \forall\theta_1, \theta_2 \in Tr(\sigma).$$

proof :

Follows from Lemma 47 as $Tr(\sigma)$ generates k unless its a generalized quaternion algebra. □

Lemma 49 *Let k be a division ring with involution σ such that $Z(k) \neq k$. Then k is a generalized quaternion algebra with standard involution σ if and only if :*

$$[\theta_1, \theta_2] \in Z(k), \, \forall\theta_1, \theta_2 \in Tr(\sigma).$$

proof :

If k is a generalized quaternion algebra with standard involution σ, the condition on the traces is clearly satisfied as in this case $Tr(\sigma) = Z(k)$.
Conversily suppose that the condition of the Lemma holds.
Choose $\theta_0 \in Tr(\sigma)$.
We find :

$$\theta\theta_0 = \theta_0\theta z_\theta, \, \forall\theta \in Tr(\sigma)$$

where z_θ is an element of $Z(k)$ possibly depending on θ. If $\theta \in Z(k)$, we have $z_\theta = 1$.
So suppose $\theta \notin Z(k)$.
Then :

$$\begin{aligned}(1+\theta)\theta_0 &= \theta_0(1+\theta)z_{1+\theta} \\ &= \theta_0 z_{1+\theta} + \theta_0\theta z_{1+\theta} \\ &= \theta_0 + \theta\theta_0 \\ &= \theta_0 + \theta_0\theta z_\theta\end{aligned}$$

shows :

$$z_{1+\theta} + \theta z_{1+\theta} = 1 + \theta z_\theta.$$

As $\theta \notin Z(k)$ this is only possible if $z_{1+\theta} = z_\theta = 1$.
As θ_0 was chosen arbitrarily this implies :

$$[\theta, \theta'] = 1, \, \forall \theta, \theta' \in Tr(\sigma).$$

Without loss of generality we can assume $1 \in Tr(\sigma)$. (cfr. see Lemma 46).
The Lemma follows from Lemma 47. □

Lemma 50 *If k is a division ring then*

$$[x, y] \in Z(k), \, \forall x, y \in k$$

if and only if $Z(k) = k$.

proof :

Complelety analogous as the proof of Lemma 49 □

Lemma 51 *Let k be a division ring such that $Z(k) \neq k$. If every element of k satisfies a quadratic equation over $Z(k)$, then $k \cong \mathbb{H}$.*

proof :

If $[k : Z(k)] < \infty$ the proof can be found on p103 in [10]. Choose $\theta_1 \notin Z(k)$. Then there exists a $\theta_2 \in k$ such that $[\theta_1, \theta_2] \neq 1$. Let $Z(C_k\{\theta_1, \theta_2\})$ be the center of the centralizer of θ_1 and θ_2 in k. Consider the $Z(C_k\{\theta_1, \theta_2\})$-algebra generated by θ_1 and θ_2. Call it L. The condition on k implies that L is a division ring. We prove that L is generated over $Z(k)$ by $\{1, \theta_1, \theta_2, \theta_1\theta_2\}$. Denote the $Z(k)$-algebra generated by $\{1, \theta_1, \, \theta_2, \, \theta_1 \, \theta_2\}$ as S. To this end we show by induction on m that every product $\Pi_{i(j)=1}^m \, \theta_{i(j)}$ with $i(j) \in \{1, 2\}$, $\forall$ j is inside the S.
Let $m = 2$.

By assumption we have that θ_1^2, θ_2^2 and $\theta_1\theta_2 \in S$. Consider $\theta_1 + \theta_2$. The conditions on k implies that there exist z_1, $z_2 \in Z(k)$ such that :

$$\begin{aligned}(\theta_1+\theta_2)^2 &= \theta_1^2+\theta_1\theta_2+\theta_2\theta_1+\theta_2^2 \\ &= (\theta_1+\theta_2)z_1+z_2. \end{aligned} \tag{1.1}$$

This equation clearly implies that $\theta_2\ \theta_1 \in S$.
Suppose that the induction hypthesis is thrue for m. Consider a product $\Pi_{i(j)=1}^m\ \theta_{i(j)}$. Then we have to show that $\theta_{i(m+1)}\ \Pi_{i(j)=1}^m\theta_{i(j)} \in S$. Without loss of generality we can assume that $\theta_{i(m+1)} = \theta_1$. Two cases occurs :

First case : $\theta_{i(1)} = \theta_1$.
Then we find $\theta_1\ \Pi_{i(j)=1}^m\ \theta_{i(j)} = \theta_1^2\ \Pi_{i(j)=2}^m\theta_{i(j)}$. As $\theta_1^2 = \theta_1 u_1 + u_2$ for some $u_i \in Z(k)$ we find, the induction hypothesis implies that $\theta_1\ \Pi_{i(j)=1}^m\theta_{i(j)} \in S$.

Second case $\theta_{i(1)} = \theta_2$.
If in this case $\theta_{i(2)} = \theta_2$ the induction hypothesis implies as in the foregoing case that $\theta_1\ \Pi_{i(j)=1}^m\ \theta_{i(j)} \in S$.
Hence we are left with the case where $\theta_{i(2)} = \theta_1$.
We find using equation (1.1) :

$$\begin{aligned}&\theta_1\Pi_{i(j)=1}^m\theta_{i(j)} \\ &= \theta_1\theta_2\Pi_{i(j)=2}^m\theta_{i(j)} \\ &= (-\theta_2\theta_1 - \theta_1^2 - \theta_2^2 + (\theta_1+\theta_2)z_1 + z_2)\Pi_{i(j)=2}^m\theta_{i(j)}\end{aligned}$$

By what we already proved and the induction hypothesis we find that $(-\theta_2\theta_1 - \theta_1^2 - \theta_2^2 + (\theta_1+\theta_2)z_1 + z_2)\Pi_{i(j)=2}^m\ \theta_{i(j)} \in S$.
Hence we find that also in this case $\theta_1\ \Pi_{i(j)=1}^m\ \theta_{i(j)} \in S$.

By this we proved that every product $\Pi_{i(j)=1}^m\ \theta_{i(j)}$, $i(j) \in \{1,2\}$ is contained in S, and hence $S = L$. As $Z(k) \subseteq Z(C_k\{\theta_1,\theta_2\})$, L is a finite dimensional division ring of dimension n^2 over $Z(L)$ for a natural number n. We find $Z(L) = Z(C_k\{\theta_1,\theta_2\}) = Z(k)$ and L is a generalized quaternion algebra. This implies in particular that $\theta_2\theta_1 = z_1 + \theta_1 z_2 + \theta_2 z_3 + \theta_1\theta_2 z_4, z_i \in Z(k), 1 \leq i \leq 4$.
Let z be arbitrary in k. If $z \notin L$ we consider the $Z(k)$-algebra generated by z, θ_1, θ_2. Denote this algebra by R. Due to the properties of L and k it follows that R is generated over $Z(k)$ by $\{1, \theta_1, \theta_2, z, \theta_1\theta_2, \theta_1 z, \theta_2 z, \theta_1\theta_2 z\}$,

yielding$[R : Z(R)] < \infty$. As R is generated over $Z(k) \subset Z(R)$ by at most 8 elements, $[R : Z(R)]$ is necessarily 4. As the finite dimensional case of the lemma holds R is a quaternion algebra with standard involution which we denote by a bar sign.
If $x \in R$ it is therefore solution of the quadratic equation $x^2-(x+\bar{x})x-\bar{x}x = 0$ with coëfficïents in $Z(R)$. But by assumption x is also solution of a quadratic equation $x_2+xc_1+c_2 = 0$ with coëficcïents in $Z(k)$. If $x \notin Z(R)$ this implies that $x+\bar{x} = c_1 \in Z(k)$ and $\bar{x}x = c_2 \in Z(k)$. For any quaternion algebra the map $R \mapsto Z(R)$ that sends t to $t+\bar{t}$ is surjective. Hence we find $Z(R) = Z(k)$. This means that the set $\{1, \theta_1, \theta_2, \theta_1\theta_2, z\}$ would be linearly dependant over $Z(k)$ contradicting the choice of z. This shows $L = k$, and k is a generalized quaternion algebra.
That the converse holds follows from standard theory of quaternion algebra's.
□

The strategy of proof above can be used to show the following.

Corollary 52 *Let k be a division ring, σ an involution of k. Then every element of $Tr(\sigma)$ is solution of a quadratic equation over $Z(k)$ if and only if k is a generalized quaternion algebra.*

proof :

If k is not a generalized quaternion algebra Lemma 47 shows it is generated as a ring by $Tr(\sigma)$. Following an analogous strategy as the proof of Lemma 51 where $\theta_i, z \in Tr(\sigma)$ leads to a contradiction. □

Lemma 53 *Let k be a generalized quaternion algebra with standard involution σ and $a \in k$. Suppose a satisfies :*

$$\lambda^\sigma a\lambda = \in (Z(k)(a))$$

where $(Z(k)(a))$ is the $Z(k)$-subalgebra of k generated by a. Then a is an element of $Z(k)$.

proof :

Let a be as in the Lemma. Then we find for every $\lambda \in k$, z_1 and $z_2 \in Z(k)$ such that :

$$\lambda^\sigma a\lambda = az_1 + z_2.$$

Equivalently :

$$\lambda^{-1}a\lambda = az_1(\lambda^\sigma\lambda)^{-1} + z_2(\lambda^\sigma\lambda)^{-1}.$$

Set $z_1' = z_1(\lambda^\sigma\ \lambda)^{-1}$ and $z_2' = z_2(\lambda^\sigma\ \lambda)^{-1}$. As $\lambda^\sigma\ \lambda \in Z(k)$ we can write for $\lambda \in k$:

$$\lambda^{-1}a\lambda = az_1' + z_2',\ \text{with}\ z_1', z_2' \in Z(k).$$

Which is equivalent to :

$$a\lambda + \lambda a z_1' + \lambda z_2' = 0 \in Z(k).$$

Adding this equation with $(\lambda^\sigma + \lambda)a \in Z(k)(a)$ implies :

$$\lambda(a(1 + z_1^\lambda) + z_2^\lambda) \in Z(k)(a)\ \ (2).$$

Suppose $z_1' \neq 1$. As $a \notin Z(k)$ equation (2) is only possible if $\lambda \in Z(k)(a)$. But as $Z(k)(a)$ is a field this implies $\lambda a = a\ \lambda$ and $z_1^\lambda = 1$, a contradiction. This means that :

$$z_1^\lambda = 1,\ \forall\lambda \in k \text{ and } z_2' = 0$$

Thus for every $\lambda \in k$ we have :

$$a\lambda + \lambda a = z_2^\lambda \in Z(k).$$

As $a\lambda +\ \lambda^\sigma a \in Z(k)$ this yields :

$$(\lambda + \lambda^\sigma)a \in Z(k),\ \forall\lambda \in k.$$

Hence $a \in Z(k)$.

□

To end this section we give a usefull Lemma on semi-linear transformations.

Lemma 54 *Let k, k' be division rings V a right k-vector space and V' a right k'-vector space. Suppse β is a bijection from V to V' such that :*

$$\begin{aligned} \beta(v + w) &= \beta(v) + \beta(w),\ \forall v, w \in V \\ \beta(v\lambda) &= \beta(v)\lambda_v',\ \forall v \in V, \forall\lambda \in k \end{aligned}$$

with $\lambda_v' \in k'$ might depend on v. If $dim(V') \geq 2$, β defines a semi-linear transformation from V to V'.

proof :

We first show that for $\lambda \in k$, $v \in V$ the element λ'_v does not depend on v.
Let $w \in V$.
Suppose firstly that $\beta(w)$ is linearly independant from $\beta(v)$.
Then the equation :

$$\begin{aligned} \beta(v)\lambda'_v + \beta(w)\lambda'_w &= \beta((v+w)\lambda) \\ &= \beta(v+w)\lambda_{v+w} \\ &= \beta(v)\lambda_{v+w} + \beta(w)\lambda_{v+w} \end{aligned}$$

shows that $\lambda_v = \lambda_w = \lambda_{v+w}$.
If $\beta(w)$ is linearly dependant on $\beta(v)$ we choose a $u \in V$ such that $\beta(u)$ is linearly independant from $\beta(v)$. By what we already proved we then have $\lambda_u = \lambda_v = \lambda_w$.
By this we can thus write for $\lambda \in k$:

$$\beta(v\lambda) = \beta(v)\lambda', \forall v \in V,$$

where λ' does not depend on v. Define the bijcetion α from k to k' by :

$$\beta(v\lambda) = \beta(v)\lambda^\alpha, \forall v \in V.$$

We check that α determines a field isomorphism from k to k'.
By definition of α we have $\alpha(1) = 1$ and $\alpha(0) = 0$. Moreover the equations :

$$\begin{aligned} \beta(v)(\lambda^\alpha + \mu^\alpha) &= \beta(v(\lambda + \mu)) \\ &= \beta(\lambda+\mu)^\alpha \\ \beta(v)\lambda^\alpha\mu^\alpha &= \beta(v(\lambda\mu)) \\ &= \beta(v)(\lambda\mu)^\alpha \end{aligned}$$

show that α defines an isomorphism from k to k' and it follows that β determines a semi-linear transformation from V to V' with associated isomorphism α from k to k'.

□

Chapter 2

General theory

2.1 Twin buildings and Moufang buildings

2.1.1 Introduction

Motivation of this section is a remark made by J.Tits in the standard reference on twin buildings [31]. Paragraph 3 of loc.cit. deals mainly with a group theoretical approach of twin buildings. In the theory of buildings it is also well known that every Moufang building Δ has a natural BN-pair. In his paper J. Tits gives the description of certain group systems called RD-systems. When one verifies the axioms of an RD-system it is not hard to see that from every Moufang building such a system arises. Proposition 4 of loc. cit. says then that whenever an RD-system is given one can construct a twin BN-pair. The proof is left over to the reader but the author mentions this is not easy. This proposition implies in particular that every Moufang building Δ has an associated twin BN-pair. Hence in view of the above remarks we can state that from every Moufang building one can construct a twin building. Moreover from the calculations it turns out that Δ is isomorphic to one half of the twin building. As there is no real proof written down yet of this well known fact it will be exposed in this section. We will use a slightly different approach than the one used in [31]. In this way it is hoped

that the geometry of the twin buildings will become more clear. During the exposition of the proof it will also be indicated how to extract a concrete proof of Proposition 4 in section 3.3. in [31].
The converse of the problem namely "Is every half of a twin building a Moufang building?" also holds under some mild conditions. This follows from [18]. Hence it follows that twin buildings and Moufang buildings are in a lot of situations the same objects. The Moufang condition characterizes buildings as automorphism groups of certain geometries. The above result implies that in a lot of cases twin buildings can be seen as couples of buildings with a well defined opposition relation. Or they are represented as group geometries defined in the automorphism group of a buildings. The main theorem of this section will be :

Theorem 55 *Every Moufang building (Δ, W, S, d) is half of a twin building i.e. there exists a building (Δ_-, W, S, d_-) and a codistance function d_* from $\Delta \times \Delta_- \sqcup \Delta_- \times \Delta$ to W such that $((\Delta, W, S, d), (\Delta_-, W, S, d_-), d_*)$ is a twin building.*

Something about the proof. As already mentioned a proof can be extracted from [29]. It essentially boils down to checking that given a Moufang building (Δ, W, S, d) with root groups $(U_\alpha)_{\alpha\in\Phi}$ the system $(G, (U_\alpha)_{\alpha\in\Phi}$ with G $\langle U_\alpha$ $|\alpha \in \Phi\rangle$ forms an RD-system. One has to check the five axioms $RD1$ up to $RD5$. That $RD1$ till $RD4$ are satisfied is rather easy. The problem is $RD5$. This axiom requires that for every $\alpha > 0$, with α a fundamental root of a root system, $B_\alpha \not\subset B_-$ and $B_{-\alpha} \not\subset B_+$. That $B_{-\alpha} \not\subset B_+$ there holds is rather easy to check. It follows essentially from the equality $B_+ = Stab_B(c_+)$. If $B_{-\alpha}$ would be fully contained in B_+ then every element of $B_{-\alpha}$ has to fix c_+. This contradicts the regular action of $B_{-\alpha}$ on a the s_α-panel through α. To exclude the other inclusion one cannot use the same argument. The difference here is that B_- has no interpretation in terms of the buildings geometry. For this we will have to look deeper into the structure of Δ.

2.1.2 Properties of Moufang buildings

Most of the facts given here can be found in [29]. We list some known properties and theorems of Moufang buildings and give proofs where necessary. The notations used here are the ones introduced in Chapter 1. Consider a Moufang building (Δ, W, S, d) with a system of root groups U_α and B, N and

S, c_+ as before. We list the axioms for an RD-system as described in section 5.2. in [29].

Definition 56 Let G be a group, $M = (m_{i,j})_{i,j\in I}$ a Coxeter matrix and Φ a root system of type M, $(B_\alpha)_{\alpha\in\Phi}$ a generating system of subgroups of G. Then $(G, (B_\alpha)_{\alpha\in\Phi})$ forms an *RD-system* if the following axioms are satisfied (where B_+ stands for $\langle B_\alpha|\alpha > 0\rangle$ and B_- stands for $\langle B_\beta|\beta < 0\rangle$, H is the intersection of all B_α, and $\Lambda = \{\alpha_i|i \in I\}$ is a fundamental root system for Φ and $s_i = s_{\alpha_i}$ for all $i \in I$)

RD1 If $\{\alpha, \beta\}$ is a prenilpotent pair of roots, there is an order $(\alpha = \beta_1, \beta_2, \ldots \beta_m)$ on $[\alpha, \beta]$ starting at α such that $B_{\beta_1}B_{\beta_2}\ldots B_{\beta_m}$ is a group.

RD2 For $i \in I$, $B_{\alpha_i} \cap B_{-\alpha_i} = H$.

RD3 For $i \in I$, the group B_{α_i} has two double cosets in the group it generates with $B_{-\alpha_i}$.

RD4 For $i \in I$, there exists an element in $\langle B_{\alpha_i}, B_{-\alpha_i}\rangle$ which maps B_β onto $B_{s_i(\beta)}$ for all $\beta \in \Phi$.

RD5 For all $i \in I$, $B_{\alpha_i} \not\subset B_-$ and $B_{-\alpha_i} \not\subset B_+$.

As already mentioned above a candidate of an RD-system is provided by every Moufang building.

Lemma 57 *Let $M = (m_{ij})_{i,j\in I}$ be a Coxeter matrix, Φ a root system of type M and (Δ, W, S, d) a Moufang building of type M with associated root groups $(U_\alpha)_{\alpha\in\Phi}$. Consider the group system $(G, (B_\alpha)_{\alpha\in\Phi})$. Then $(G, (B_\alpha)_{\alpha\in\Phi})$ satisfies axioms $RD1$, $RD2$, $RD3$ and $RD4$.*

proof :

1. Axiom **RD1**.
Consider the set $B_\alpha B_{\beta_2}\ldots B_{\beta_m}$ where $\{\alpha, \beta\}$ is a prenilpotent pair of roots. Using exercise 15 on p82 in [25] there exists an ordering on $[\alpha, \beta]$ such that $[\alpha, \beta] = \{\beta_1, \beta_2, \ldots, \beta_m\}$ and $[\beta_i, \beta_j] \subset \{\beta_i, \beta_{i+1}, \ldots, \beta_j\}$ for $i \leq j$. By the definition of H it follows that for every $h \in H$ and every $\alpha \in \Phi$

$$hU_\alpha h^{-1} = U_\alpha.$$

We prove by induction on m, the number of roots in the interval $[\alpha, \beta]$ that :

$$B_\alpha B_{\beta_2} \dots B_{\beta_m} = HU_\alpha U_{\beta_2} \dots U_{\beta_m}.$$

Pick an arbitrary element u_{β_j} of U_{β_j} for some j. Then

$$u_{\beta_j} B_\alpha B_{\beta_2} \dots B_{\beta_m} \subset B_\alpha u_{\beta_j} HU_\gamma B_{\beta_2} \dots B_{\beta_m}$$

where γ is one of the roots in $\{\beta_2, \dots \beta_m\}$. The induction hypothesis yields :

$$B_\alpha u_{\beta_j} HU_\gamma B_{\beta_2} \dots B_{\beta_m} \subset B_\alpha u_{\beta_j} B_{\beta_2} \dots B_{\beta_m}.$$

In a similar way we can switch u_{β_j} with B_{β_i} in the product if $j \leq i$. From this it follows that

$$u_{\beta_j} B_\alpha B_{\beta_2} \dots B_{\beta_m} = B_\alpha B_{\beta_2} \dots B_{\beta_m}$$

hence $B_\alpha B_{\beta_2} \dots B_{\beta_m}$ is a group. This proves $RD1$.

2. Axiom **RD2**.

Consider $x \in B_{\alpha_i} \cap B_{-\alpha_i}$ for some $i \in I$. Then we can write $x = hu_{-\alpha_i}$. As $x \in B_{\alpha_i} \subset Stab_G(c_+)$ it follows that :

$$u_{-\alpha_i}(c_+) = c_+.$$

But then the regular action of $U_{-\alpha_i}$ on $R_i(c_+) \backslash \{s_i(c_+)\}$ implies that $u_{-\alpha_i} = 1$. Hence $x \in H$, showing that $B_{\alpha_i} \cap B_{-\alpha_i} \subseteq H$. That the converse inclusion $H \subset B_{\alpha_i} \cap B_{-\alpha_i}$ holds is clear.

3. Axiom **RD3**.

Choose $\bar{u}_{\alpha_i} \in U_{\alpha_i}$ such that $s_i = m(\bar{u}_{\alpha_i}) \in S$. Then we show that $B_{-\alpha_i} \subset H \cup B_{\alpha_i} s_i B_{-\alpha_i}$. From this inclusion one can deduce easily that

$$\langle B_{\alpha_i} B_{-\alpha_i} \rangle = B_{\alpha_i} \sqcup B_{\alpha_i} s_i B_{\alpha_i}.$$

Let $x \in B_{-\alpha_i}$. If $x \in B_{-\alpha_i}$ then $x \in H$. If $x \notin B_{\alpha_i}$ then $x = hu_{-\alpha_i}$ for $u_{-\alpha_i} \in U_{-\alpha_i}$ and $u_{-\alpha_i} \neq 1$. But then $m(u_{-\alpha_i}) s_i \in H$ and $x = u_{\alpha_i} m_{-\alpha_i} u'_{\alpha_i}$ for certain $u_{\alpha_i}, u'_{\alpha_i} \in U_{\alpha_i}$ is contained in $B_{\alpha_i} s_i B_{\alpha_i}$.

Similar arguments show that

$$\langle B_{\alpha_i} B_{-\alpha_i} \rangle = B_{-\alpha_i} \sqcup B_{-\alpha_i} s_i B_{-\alpha_i}.$$

4. Axiom **MR4**.

This follows from **Mo4**. □

Theorem 58 *Given a Moufang building (Δ, W, S, d) of type M (with notations as above) then there is a unique homomorphism $\nu : N \mapsto W$ such that for $n \in N$ and $\alpha \in \Phi$*

$$nB_\alpha n^{-1} = B_{\nu(n)(\alpha)}.$$

The kernel of ν is H. This implies that $N/H \cong W$ and N/H is generated by a set $\tilde{s}_i H$ where $\{\tilde{s}_i\}$ is a set of $m(u_{\alpha_i})$ with $u_{\alpha_i} \neq 1$ and $\Lambda = \{\alpha_i \,|\, i \in I\}$ a fundamental root system in Φ.

proof :

This is a restatement of Lemma $3(i), (iii)$ in paragraph 5 in [29]. As the proof given in loc. cit. follows from axioms $(RD2)$, $(RD3)$ and $(RD4)$ the proof is still valid. □

Let (Δ, W, S, d) be a Moufang building with root groups $(U_\alpha)_{\alpha\in\Phi}$ such that $S = (s_i)_{i\in I}$ and $\Lambda = \{\alpha_i|\alpha_i \in I\}$ a fundamental root system for Φ. Choose for every $i \in I$ a fixed $u_{\alpha_i} \neq 1$. For the sequel we will identify s_i with $\{m(u_{\alpha_i})$ using the isomorphism ν as in Theorem 58. $|i \in I\}$. Granted this identification the notation $wB_\alpha w^{-1}$ for $w \in W$ makes sense.

Theorem 59 *Given a Moufang building (Δ, W, S, d) of type M then G acts transitively on Δ and the system (G, B_+, N, S), defined as above, forms a BN-pair.*

proof :

Suppose that for some $B_{-\alpha_i} \subset B_+$. Then in particular $U_{-\alpha_i} \subset B_+$. This contradicts the regular action of $U_{-\alpha_i}$ on $R_i(c_+) \setminus \{s_i(c_+)\}$. Hence for all $i \in I$ we find that $B_{-\alpha_i} \not\subset B_+$. As the axiom $(RD1)$ holds for $(G, (B_\alpha)_{\alpha\in\Phi})$ Lemma 4 in paragraph 5 in [30] is still valid. Following the strategy of Proposition $4(i)$ one deduces that (G, B_+, N, S) is a BN-pair . □

The strategy we follow from now on will differ from the one suggested in [31]. We start by constructing a chamber system $\mathcal{C}^-$ as in the sense in [25]. It turns out that $\mathcal{C}^-$ is a building on which the group G acts. Then we define an opposition relation between (Δ, W, S, d) and $\mathcal{C}^-$. Using a result of B. Mühlherr [19] on twin buildings we deduce that the opposition relation defines a twinning. This implies that (Δ, W, S, d) is half of a twin building.

2.1.3 The chamber system $\mathcal{C}^-$

In this paragraph we construct a chamber system $\mathcal{C}^-$ using the groups. First we need some lemma's.

Lemma 60 *Given a negative root α_i with $i \in I$ then*

$$B_\alpha B_{\alpha_i} \tilde{s}_i B_- \subset B_{\alpha_i} \tilde{s}_i B_-$$

for every negative root $\alpha \in \Phi$.

proof :

The proof is completely analogous to Lemma 4 in section 5 in [31]. One replaces all positive roots by negative roots. □

Lemma 61 *Let $w \in W$ (with (W, S) a Coxeter group) , and $s_{i_1} \dots s_{i_m}$ a reduced expression of w. Set for $j \in \{1, \dots, m\}$ $w_j = s_{i_1} \dots s_{i_j}$, $w_0 = 1$ and $\beta_j = w_{j-1}(\alpha_j)$ then $\{\beta_1, \dots, \beta_m\}$ is the set of all positive roots sent by w^{-1} to a negative root.*

proof :

This lemma is a restatement of Proposition $3(i)$ in [31] section 5. The proof can be found there. □

Lemma 62 *Given any $w \in W$ and a reduced expression $s_{i_1} \dots s_{i_m}$ of w then the set*

$$U_{-\beta_1} \dots U_{-\beta_m}$$

is a group U_{-w} only depending on w. The group B_{-w} satisfies $B_{-w} w B_- = B_- w B_-$. The same statements hold for U_w and $B_w w B_+$.

proof :

The statement of this lemma is analogous to the statement of Proposition $3(ii), (iv)$ in section 5 in [31]. The only difference is that the groups here are parametrized by negative roots. One can easily check that the proof given in loc.cit. remains valid if positive roots are replaced by negative roots. That the same statements hold if all roots are positive follows from Proposition 3 of in section 5 of [31]. □

Using the groups U_{-w} we construct the following chamber system $\mathcal{C}^-$. Let $U_- = \langle U_{-\alpha}\rangle_{\alpha>0}$. For a given $w \in W$ the group U_{-w} defines a coset structure on U_-. We define $\mathcal{C}^-_w$ as the set of all right cosets of U_{-w} in U_-. The set of chambers of $\mathcal{C}^-$ is the disjoint union $\sqcup\mathcal{C}^-_w$. As we want the chamber system $\mathcal{C}^-$ to be defined over the set I we have to define an i-adjacency relation for every $i \in I$. To do this we first fix some terminology which is used in [31] in section 5.11.
Given $J \subseteq I$ such that $W_J = \langle s_i | i \in J\rangle$ is finite and an element $w \in W$, then w is called *right J-anti-reduced* if $l(w) = max\{l(u)|u \in wW_J\}$. For $w \in W$ and $i \in I$, w^i stands for the unique right $\{i\}$-anti-reduced element in the i-panel in W containing w. For adjacency we state the following rule :
two chambers xU_{-w} and yU_{-v} are i-adjacent if and only if

(1) $w^i = v^i$,

(2) $xU_{-w^i} = yU_{-w^i}$.

It is easily checked that $\mathcal{C}^-$ equipped with this adjacency relation is indeed a chamber system over I in the sense in [25] chapter 1.
We also remark that the group U_- acts on the chamber system $\mathcal{C}^-$ by left multiplication. It is easily checked that under this action i-adjacent chambers are send to i-adjacent chambers. This means that the group U_- acts as a group of type preserving automorphisms of the chamber system $\mathcal{C}^-$.
The next step is to construct a chamber systems morphism between $\mathcal{C}^-$ and (Δ, W, S, d).

Lemma 63 *The map κ between $\mathcal{C}^-$ and (Δ, W, S, d) that sends xU_w to $xw(c_+)$ is a type preserving morphism between the chamber systems $\mathcal{C}^-$ and (Δ, W, S, d) (i.e. it sends i-adjacent chambers to i-adjacent chambers).*

proof :

We have to check that κ is well defined and that, if xU_{-w} and yU_{-v} are i-adjacent, then also $\kappa(xU_{-w})$ and $\kappa(yU_{-v})$ are i-adjacent. To see this we rely on the following property :

$$U_{-w} \subset Stab_G(w(c_+)). \tag{2.1}$$

Let's first check this property. By Theorem 58 and Lemma 60 the group $w^{-1}U_{-w}w$ is contained in B_+. As $Stab_G(c_+) = B_+$ formula (3.15.1) is clear. Because of property (3.15.1) the map κ is well defined, i.e. if $xU_{-w} = x'U_{-w}$ then $x(w(c_+)) = x'(w(c_+))$.
Suppose that xU_{-w} and yU_{-v} are i-adjacent, i.e. $w^i = v^i$ and $xU_{-w^i} = yU_{-w^i}$. From $w^i = v^i$ it follows that $w(c_+)$ and $v(c_+)$ are i-adjacent and belong to the i-panel containing w^i. From $y^{-1}x \in U_{-w^i}$ we deduce that $y^{-1}x$ stabilizes $w^i(c_+)$, hence also stabilizes the i-panel through $w^i(c_+)$. This means that $y^{-1}x(w(c_+))$ and $v(c_+)$ are i-adjacent, hence also $x(w(c_+))$ and $y(v(c_+))$ are i-adjacent. This completes the proof of the lemma. □

2.1.4 Properties of κ

In this paragraph we show that κ is a 2-covering from $\mathcal{C}^-$ onto (Δ, W, S, d) i.e. κ sends every spherical rank 2 residue in $\mathcal{C}^-$ isomorphically onto a rank 2 residue in (Δ, W, S, d). We start by showing that κ is surjective. For this we need an additional property of Moufang buildings.

Proposition 64 *Given a Moufang building (Δ, W, S, d) with standard apartment Σ_0, then the orbit of Σ_0 (as a set of chambers) under B_- is the full building Δ, i.e. $B_-(\Sigma_0) = \Delta$.*

<u>proof</u> :

The proposition follows from the decomposition $G = B_-WB_+$ regarded the fact that $\{w(c_+)|w \in W\} = \Sigma_0$. First we show that $G = \cup_{w\in W}(B_-wB_-)$. Using Lemma 62 we write :

$$B_-s_iB_-wB_- = B_-s_iB_{-w}wB_-.$$

Two cases occur :
(1) $l(s_iw) > l(w)$.
Then $s_iB_{-w}s_i \subset B_-$ and

$$B_-s_iB_-wB_- = B_-(s_iB_{-w}s_i)s_iwB_- = B_-s_iwB_-.$$

(2) $l(s_iw) < l(w)$.
Hence

$$\begin{aligned} B_-s_iB_-wB_- &= B_-s_iB_-s_is_iwB_- \\ &\subset \{B_-s_iB_-, B_-\}s_iwB_- \\ &\subset B_-wB_- \cup B_-s_iwB_-. \end{aligned}$$

As for every $j \in I$ $U_{\alpha_j} = s_{\alpha_j}\ U_{-\alpha_j}\ s_{\alpha_j}$ and $G = \langle U_{\alpha_i}\ | i \in I\rangle$ one deduces that $\cup_{w\in W}(B_-wB_-) = G$.
Subsequently we show that for every $w \in W$ and $s_i \in S$

$$B_-s_iB_-wB_+ \subseteq B_-swB_+ \cup B_-wB_+.$$

As above again two cases can occur :
(1) $l(s_iw) < l(w)$.
This means that the root $w^{-1}(\alpha_i)$ is negative, hence

$$\begin{aligned} B_-s_iB_-wB_+ &= B_-s_iB_{-\alpha_i}wB_+ \\ &= B_-s_iw(w^{-1}B_{-\alpha_i}w)B_+ \\ &= B_-s_iwB_+. \end{aligned}$$

(2) $l(s_iw) > l(w)$.
The we use the above equation and calculate :

$$\begin{aligned} B_-s_iB_-wB_+ &= B_-s_iB_-s_is_iwB_+ \\ &\subset B_-\{1, s_i\}B_-s_iwB_+ \\ &= B_-s_iwB_+ \cup B_-wB_+. \end{aligned}$$

By similar arguments as for $\cup_{w\in W}\ B_-wB_-$ it follows that $B_-WB_+ = G$. □

Corollary 65 *The morphism κ is surjective.*

proof :

Consider an arbitrary chamber a in Δ. Then by Proposition 1 we have $a = b_-v(c_+)$ for some $b_- \in B_-$ and $v \in W$. As for every root α, $H \subset Stab_G(U_\alpha)$ we can write b_- as u_-h for $u_- \in U_-$ and $h \in H$. Because H fixes every chamber of Σ_0 we can write $a = u_- v(c_+)$. If we consider the element $u_- U_{-v}$ of $\mathcal{C}^-$ then clearly $\kappa(u_- U_{-v}) = a$. □

The only problem that remains to prove is that κ maps rank 2 residues isomorphically onto rank 2 residues.

Theorem 66 *The map κ is 2-covering from $\mathcal{C}^-$ to Δ i.e. it sends spherical rank 2 residues isomorphically onto spherical rank 2 residues.*

proof :

To prove this we remark that the action of U_- on $\mathcal{C}_-$ and Δ is compatible with κ, i.e. for all $xU_{-w} \in \mathcal{C}_-$ and $u_- \in U_-$ we have $\kappa(u_- xU_{-w}) = u_- \kappa(xU_{-w})$. In order to prove that κ is a 2-covering, it will then be enough to show that κ induces an isomorphism between every $\{i,j\}$ residue containing a chamber U_{-w}, with w an $\{i,j\}$-anti-reduced element in W, and its image in Δ. To see this we remark that every rank 2 residue in $\mathcal{C}_-$ always contains a chamber xU_{-w} where w is $\{i,j\}$-anti-reduced and $x \in U_-$. The morphism determined by x^{-1} will then send the given rank 2 residue to another rank 2 residue that contains U_{-w}.
Fix a certain rank 2 residue in $\mathcal{C}_-$ of spherical type $\{i,j\}$ (hence $m_{ij} < \infty$). Call this residue R_-^{ij}. Suppose that R_-^{ij} contains a chamber U_{-w} with w $\{i,j\}$-anti-reduced. As $U_{-w} \in R_-^{ij}$, we see that $w(c_+) \in \kappa(R_-^{ij})$. If we denote by R^{ij} the $\{i,j\}$-residue in Δ which contains $w(c_+)$ then we have to show that κ induces an isomorphism between R_-^{ij} and R^{ij}.
1. The map κ induces a surjection between R_-^{ij} and R^{ij}.
This will follow from the fact that κ induces a surjection between rank 1 residues. Consider a fixed $i \in I$ and a chamber a in Δ. Using Proposition 1 and the action of U_- on Δ we can assume that $a = v(c_+)$, $v \in W$. Then every chamber of the i-residue containing a can be written under the form $v(u_{\alpha_i} s_i(c_+))$ with $u_{\alpha_i} \in U_{\alpha_i}$ and $\alpha_i > 0$.
Two cases occur :
(i) $l(vs_i) < l(v)$.
Then $vu_{\alpha_i} = vu_{\alpha_i}v^{-1}v$ with $vu_{\alpha_i}v^{-1} \in U_{v(\alpha_i)}$. Granted the condition on v, one has $U_{v(\alpha_i)} \subset U_{-v^i}$. If we consider in $\mathcal{C}^-$ the chamber $vu_{\alpha_i}v^{-1}U_{-vs_i}$, then this chamber is i-adjacent to U_{-v} and $\kappa(vu_{\alpha_i}v^{-1}U_{-v}) = a$.
(ii) $l(vs_i) > l(v)$.
Using Lemma 1, one starts by rewriting u_{α_i} as $u_{-\alpha_i}s_i b_{-\alpha_i}$ with $u_{-\alpha_i} \in U_{-\alpha_i}$ and $b_{-\alpha_i} \in B_{-\alpha_i}$. As we also know that $s_i b_{-\alpha_i} s_i \subset B_{\alpha_i}$ the chamber

a coincides with $vu_{-\alpha_i}(c_+)$. Because of the condition on v we have that $vu_{-\alpha_i}v^{-1} \in U_{-v(\alpha_i)} \subset U_{-v^i}$. Hence the chamber $vu_{-\alpha_i}v^{-1}U_{-vs_i}$ is i-adjacent to U_{-v} and $\kappa(vu_{-\alpha_i}v^{-1}U_{-vs_i}) = a$.
This completes the proof that κ induces a surjection between rank 1 residues in $\mathcal{C}^-$ and Δ. Because rank 2 residues are connected it is clear that κ induces a surjection of R_-^{ij} onto R^{ij}.
2. The morphism κ induces an injection of $R_-^{i,j}$ into R^{ij}.
Suppose that we have two chambers $u'_-U_{-w'}$ and $u''_-U_{-w''}$ in R_-^{ij} such that $\kappa(u'_-U_{-w'}) = \kappa(u''_-U_{-w''})$. This means that $u'_-w'(c_+) = u''_-w''(c_+)$ and both w' and w'' belong to the $\{i,j\}$-residue in W determined by w, where w is the unique right $\{i,j\}$-anti-reduced element of this residue. Because of the conditions on w it is easy to check that both u'_- and u''_- belong to U_{-w}. We rewrite the above equality as :

$$(w^{-1}u'_-w)w^{-1}w'(c_+) \;=\; (w^{-1}u''_-w)w^{-1}w''(c_+).$$

As both u'_- and u''_- belong to U_{-w} the elements $w^{-1}u'_-w$ and $w^{-1}u''_-w$ belong to B_+. Call the first one b'_+ and the second one b''_+, then we find :

$$b'_+w^{-1}w'(c_+) \;=\; b''_+w^{-1}w''(c_+).$$

But this implies by the Bruhat decomposition of the group G (as we have a BN-pair in G) that $w^{-1}w' = w^{-1}w''$, yielding $w' = w''$.
There remains to show that $u'_-U_{-w'} = u''_-U_{-w''}$.
From the equality $u'_-w'(c_+) = u''_-w'(c_+)$ one deduces that $w'^{-1}u_-''^{-1}u'_-w' \in B_+$. The element $u_-''^{-1}u'_-$ is contained in U_{-w} and we call it u_{-w}. Then u_{-w} satisfies $w'^{-1}u_{-w}w' \in B_+$. Consider the set of positive roots sent by w^{-1} into negative roots, namely $\{\gamma_1,\dots,\gamma_n\}$. Because of the properties of w we can divide this set into two subsets (after possibly reordering) $\{\gamma_1,\dots\gamma_{l-1}\}\sqcup\{\gamma_l,\dots,\gamma_n\}$. Here $\{\gamma_1,\dots\gamma_{l-1}\}$ is the set of positive roots sent by w' to a negative root and $\{\gamma_l,\dots,\gamma_n\}$ is the set of remaining roots. With this notation in mind we write u_{-w} as $u_{-w'}u_{-r}$ with $u_{-w'} \in U_{-w'}$ and $u_{-r} = u_{-\gamma_l}\dots u_{-\gamma_n}$. We rewrite the formula $w'^{-1}u_{-w}w' \in B_+$ as :

$$w'^{-1}u_{-r}w' \;=\; w'^{-1}u_{-w'}^{-1}w'\tilde{b}_+$$

for a $\tilde{b}_+ \in B_+$. The right hand side of this equation shows that the element $w'^{-1}u_{-r}w'$ belongs to B_+. Suppose that $w = w'\underbrace{s_js_i\dots s_j}_{m \text{ terms}}$ with $l(w) = l(w') +$

m.
Then :

$$w'^{-1}\{\gamma_l, \ldots, \gamma_n\} = \{\alpha_j, s_j(\alpha_i), \ldots, s_j s_i \ldots s_i(\alpha_j)\}.$$

Hence we can write $w'^{-1}u_{-r}w'$ as
$u_{-\alpha_j}u_{-s_j(\alpha_i)} \ldots u_{-s_j s_i \ldots s_i(\alpha_j)}$ yielding that $w^{-1}u_{-r}w' \in U_- \cap B_+$. Now we look at the rank 2 building Γ_{ij} determined by $B_{\alpha_i}, B_{-\alpha_i}, B_{\alpha_j}$ and $B_{-\alpha_j}$ (i.e. the rank 2 building we get by considering the group $\langle B_{\alpha_i}, B_{\alpha_j}, B_{-\alpha_i}, B_{-\alpha_j}\rangle$ and the induced BN-pair in it.). It follows that $w'^{-1}u_{-r}w'$ is inside the group generated by these four groups. But $w'^{-1}u_{-r}w'$ fixes the fundamental chamber c_+^{ij} in this polygon. Hence this element is inside $U_-^{ij} \cap B_+^{ij}$ where the groups B_+^{ij} and B_-^{ij} are similarly as above. The proof that κ is a 2-covering will be done if we show the following lemma.

Lemma 67 *If we are given a spherical rank* 2 *building with Coxeter group* $\langle\ s_1, s_2\ |(s_1 s_2)^{m_{12}}\ \rangle$ *then*

$$B_+ \cap B_- = H$$

proof :

If we consider a spherical rank 2 Moufang building, the groups B_+ and B_- both have a geometric meaning. Indeed, in the standard apartment Σ there will be two chambers c_+ and c_- such that the $l(d(c_+, c_-))$ is maximal in the Coxeter group. The group B_+ will then be the stabilizer of c_+ in G, B_- will be the stabilizer of c_- and H will be the stabilizer of the standard apartment in G. As (G, B_+, N, S) and (G, B_-, N, S) are both BN-pairs in this case the Bruhat decompositions $G = \cup_w B_+ w B_+ = \cup_{w \in W} B_- w B_-$ implies that $B_- \cap B_+ \subseteq H$. As $H \subseteq B_+ \cap B_-$ we have :

$$H = B_+ \cap B_-.$$

□

This lemma implies that $w' u_{-r} w'^{-1}$ lies in H. Moreover by properties of spherical Moufang buildings explained in [25] on pages 75 and 76 it follows that $u_{-r} = 1$. This yields $u_- \in U_{-w'}$ or $u_-''^{-1} u_-' \in U_{-w'}$, hence $u_-'' U_{-w'} = u_-' U_{-w'}$ what we wanted to show. This completes the proof of Theorem 3. □

As already mentioned the group U_- acts on both $\mathcal{C}^-$ and Δ in a way compatible with κ. This implies that $Stab_{U_-}(a) = Stab_{U_-}(\kappa(a))$ with $a \in \mathcal{C}^-$. If we do this for c_+ then $\kappa(1) = c_+$ and $Stab_{U_-}(1) = 1$ and $Stab_{U_-}(c_+) = U_- \cap B_+$. This gives us $U_- \cap B_+ = \{1\}$, which is a very strong condition. Consider $B_- \cap B_+$. Every element in this intersection can be written as hu_- for $h \in H$ and $u_- \in U_-$. But then $u_- = 1$ and the element is contained in H. As also $H \subseteq B_- \cap B_+$ we find :

$$B_- \cap B_+ = H.$$

Using the universal properties of buildings we get the following corollary.

Corollary 68 *The chamber system $\mathcal{C}^-$ is a building of type M isomorphic to (Δ, W, S, d) under κ.*

proof :

This follows from the results in [30]. It is shown in this paper that every building is a universal object with respect to 2-coverings. This means that if we have a chamber system X over I and a 2-covering φ from X to a building (Δ, W, S, d) of type $M = (m_{ij})_{i,j \in I}$ then φ is necessarily an isomorphism. □

Corollary 69 *The pair (G, B_-, N, S) is a BN-pair.*

proof :

The proof is similar to the proof of Theorem 2 as $B_\alpha \not\subset B_-$, $\forall \alpha > 0$. □

Another consequence of the identity 2.1.4 that gives a concrete proof of Proposition 4 in [32] is the following.

Corollary 70 *Given a Moufang building (Δ, W, S, d) of type $M = (m_{ij})_{i,j \in I}$ with root groups $(U_\alpha)_{\alpha \in \Phi}$ then with the notations as before (G, B_+, B_-, N, S) is a twin BN-pair.*

proof :

This is the same as the proof of Proposition 4 in section 5 in [31]. But we rephrase it for completeness.

It follows from the calculations already made that the systems (G, B_+, N, S) and (G, B_-, N, S) are BN-pairs of type M. Moreover $B_+ \cap N = B_- \cap N = H$. Hence axiom **TBN1** is satisfied.
We prove **TBN2**.
For given s_i with $i \in I$ and $w \in W$ one has

$$B_+ s_i B_+ w B_- = \begin{cases} B_+ s_i w B_- & \text{if } l(ws_i) < l(w) \\ B_+ w B_- \cup B_+ s_i w B_- & \text{if } l(ws_i) > l(w) \end{cases}$$

Two cases can occur :
$l(s_i w) < l(w)$.
The root $w^{-1}(\alpha_i)$ is a negative root. Using lemma 4 we see that

$$\begin{aligned} B_+ s_i B_+ w B_- &= B_+ s_i B_{\alpha_i} w B_- \\ &= B_+ s_i w B_{w^{-1}(\alpha_i)} B_- \\ &= B_+ s_i w B_-. \end{aligned}$$

If $l(s_i w) > l(w)$ then by what we just proved

$$\begin{aligned} B_+ s_i B_+ w B_- &= B_+ s_i B_+ s_i s_i w B_- \\ &\subset B_+ \{1, s_i\} B_+ w B_- \\ &= B_+ \{sw, w\} B_-. \end{aligned}$$

That the symmetric formula :

$$B_- s_i B_- w B_+ = \begin{cases} B_- s_i w B_+ & \text{if } l(ws_i) < l(w) \\ B_- w B_+ \cup B_- s_i w B_+ & \text{if } l(ws_i) > l(w) \end{cases}$$

holds follows by similar arguments.
Remains to show that axiom **TBN3** is satisfied.
If for $i \in I$, $B_+ s_i \cap B_- \neq \emptyset$ we find $b_+ \in B_+$, $b_- \in B_-$ such that $s_i = b_+ b_-$. But then $s_i U_{\alpha_i} s_i = U_{-\alpha_i}$ implies that $b_- U_{-\alpha_i} b_-^{-1} \subset B_+ \cap B_- = H$. Hence $U_{-\alpha_i} \subset b_- H b_-^{-1}$. This contradicts the regular action of $U_{-\alpha_i}$ on the i-panel containing B_- in the building provided by the BN-pair (G, B_-, N, S). □

2.1.5 The relation $\mathcal{O}$

We start from a Moufang building (Δ, W, S, d). Denote the set of all roots in W by Φ. is given by $\Phi = \{\alpha\}$. The root groups are denoted by U_α.

We use notation as before. Then we know that there are two BN-pairs involved, (G, B_+, N, S) and (G, B_-, N, S). The first BN-pair yields a building (Δ_+, W, S, d_+) isomorphic to (Δ, W, S, d). From the second one, the building (Δ_-, W, S, d_-) is constructed. As the chambers of Δ_+ and Δ_- correspond to cosets of B_+ respectively B_-, the group G acts in a natural way on both buildings. Let c_+ and c_- be the chambers of Δ_+ and Δ_- corresponding to B_+ and B_-. We define the relation $\mathcal{O} \subset \Delta_+ \times \Delta_- \cup \Delta_- \times \Delta_+$ by the following rules :
($(x_+, y_-) \in \Delta_+ \times \Delta_-$, $(y_-, x_+) \in \Delta_- \times \Delta_+$)

$$(x_+, y_-) \in \mathcal{O}$$
$$\Updownarrow$$
$$\exists g \in G \text{ such that } (g(x_+), g(y_-)) = (c_+, c_-)$$

$$(y_-, x_+) \in \mathcal{O}$$
$$\Updownarrow$$
$$(x_+, y_-) \in \mathcal{O}$$

We describe the relation $\mathcal{O}$ for rank 2 Moufang buildings.

Theorem 71 *Suppose that (Δ, W, S, d) is a rank 2 Moufang building of spherical type then the relation $\mathcal{O}$ defines a twinning between Δ_+ and Δ_-.*

proof :

As the building Δ is of spherical type there exists a unique element w_0 in W such that $l(w_0) > l(w)$ $\forall w \in W$. We make the following construction. Set $(\Delta_1, W, S, d_1) = (\Delta, W, S, d)$, $(\Delta_2, W, S, d_2) = (\Delta, W, S, w_0 d w_0)$. Define a codistance function d^* between Δ_1 and Δ_2 by :
($(x_1, x_2) \in \Delta_1 \times \Delta_2$, $(x_2, x_1) \in \Delta_2 \times \Delta_1$)

$$\begin{aligned} d^*(x_1, x_2) &= w_0 d(x_1, x_2) \\ d^*(x_2, x_1) &= d(x_1, x_2) w_0. \end{aligned}$$

It follows from Proposition 1 in [31] that the couple $((\Delta_1, W, S, d_1), (\Delta_2, W, S, d_2))$ with the codistance function d^* is a twin building. It can be shown that this is the only possible twinning on Δ.
We know that two BN-pairs (G, B_+, N, S) and (G, B_-, N, S) can be constructed. Each of these BN-pairs has an associated building. Denote them by (Δ_+, W, S, d_+) and (Δ_-, W, S, d_-). We give a short description of (Δ_+, W, S, d_+).

The set of chambers Δ_+ is given by the set $\{gB_+ \mid g \in G\}$. Let $s \in S$ then g_1B_+ is s-adjacent to g_2B_+ if and only if $B_+g_1^{-1}g_2B_+ = B_+sB_+$. To define the distance between two chambers one uses the Bruhat decomposition of the group G. This means that the group G has a decomposition :

$$G = \sqcup(B_+wB_+)_{w\in W}.$$

Moreover if $B_+w'B_+ = B_+w''B_+$ then it follows that $w' = w''$. For two chambers g_1B_+ and g_2B_+ of Δ_+ the distance $d(g_1B_+, g_2B_+)$ is defined as the unique element $v \in W$ such that :

$$B_+g_1^{-1}g_2B_+ = B_+vB_+.$$

Using standard arguments it follows that (Δ_+, W, S, d_+) is a building. The same can be done for (G, B_-, N, S). This gives the building $(\Delta_- W, S, d_-)$. From the construction of (Δ_+, W, S, d_+) it can be proved that it is isomorphic to (Δ, W, S, d). The isomorphism is given by :

$$\begin{array}{rcl}\varphi_1 : \Delta_+ & \to & \Delta \\ \varphi_1(gB_+) & \mapsto & g(c_+).\end{array}$$

Consider the map φ_2 from (Δ_-, W, S, d_-) to (Δ_2, W, S, d_2) determined by :

$$\varphi_2(gB_-) = gw_0(c_+).$$

One checks that φ_2 defines an isomorphism from (Δ_-, W, S, d_-) to (Δ_2, W, S, d_2). To finish the proof we show the following equivalence :
$((x_+, y_-) \in \Delta_+ \times \Delta_-)$

$$(x_+, y_-) \in \mathcal{O} \Leftrightarrow d^*(\varphi_1(x_+), \varphi_2(y_-)) = 1.$$

(1) If $(x_+, y_-) \in \mathcal{O}$ then $x_+ = gB_+$ and $y_- = gB_-$, with $g \in G$. Hence $\varphi_1(x_+) = g(c_+)$ and $\varphi_2(y_-) = gw_0(c_+)$. We calculate :

$$\begin{array}{rcl}d(g(c_+), gw_0(c_+)) & = & d(c_+, w_0(c_+)) \\ & = & d(c_+, w_0c_+) \\ & = & w_0.\end{array}$$

This implies that $d^*(\varphi_1(x_+), \varphi_2(y_-)) = 1$.
(2) Suppose gB_+ and hB_- are such that $d^*(\varphi_1(gB_+), \varphi_2(hB_-)) = 1$. This

means that $d(g(c_+), hw_0(c_+)) = w_0$. Using the isomorphism φ_1 and the Bruhat decomposition of G it follows that :

$$hb_- = gb_+$$

for appropriate $b_- \in B_-$ and $b_+ \in B_+$. This means that $(gB_+, hB_-) \in \mathcal{O}$.
□

Remains to prove the same result for non-spherical rank 2 Moufang buildings. Let (Γ, W, S, d) be such a building. We consider a graph whose vertex set V is the set of all residues in Γ. Two vertices are joined by an edge if and only if they lie in a chamber. In this way we get a bipartite graph (V, E), which turns out to be a tree. It can also be easily checked that every isomorphisms of Γ as building induces an isomorphism of the tree (V, E). For more information about non-spherical rank 2 Moufang building we refer to [26]. The result we will prove is :

Theorem 72 *Given a non-spherical rank 2 Moufang building (Γ, W, S, d) then the relation $\mathcal{O}$ defines the opposition relation of a twinning between Δ_+ and Δ_-.*

proof :

First we fix some notations and terminologies.
Denote $W = \{s, t\}$. The chambers of Γ will be considered as pairs $\{x, x'\}$, where x and x' stand for the simplices in the chamber $\{x, x'\}$. We assume that the standard chamber is given by $c_0 = \{x_0, x_1\}$ and the standard apartment Σ_0 is the sequence $\ldots c_{-2} \overset{t}{\sim} c_{-1} \overset{s}{\sim} c_0 \overset{t}{\sim} c_1 \overset{s}{\sim} c_2 \ldots$. Write $c_i = \{x_i, x_{i+1}\}$, $\forall i$. Then the standard apartment Σ_0 corresponds to a sequence $\ldots x_{-2} \sim x_{-1} \sim x_0 \sim x_1 \sim x_2 \ldots$ in the tree (V, E).
As to the Moufang structure on Γ we keep the notations from above.
Let α_i^+ be the positive root of Σ_0 such that x_i lies on it's boundary. Similarly α_i^- is the negative root of Σ_0 such that x_i lies on $\partial\alpha_i^-$. By calculations already made there are two BN-pairs involved ; (G, B_+, N, S) and (G, B_-, N, S). They give rise to two buildings (Δ_+, W, S, d_+) and (Δ_-, W, S, d_-). To prove that $\mathcal{O}$ is the opposition relation of a twinning between $\Delta+$ and Δ_- we refer to Proposition 5.4. in [18]. In order to use this proposition we show the following :
(i) The relation $\mathcal{O}$ defines a 1-twinning between Δ_+ and Δ_-.

(ii) For any four chambers y_-,c_-^1 and c_-^2 in Δ_- and $e_+ \in \Delta_+$ such that $(e_+, c_-^1) \in \mathcal{O}$, $(e_+, c_-^2) \in \mathcal{O}$ and :

$$\begin{aligned} l(d_-(c_-^1, y_-)) &= l(d_-(c_-^2, y_-)) \\ &= min\{l(d_-(a_-, y_-))|(e_+, a_-) \in \mathcal{O}\} \end{aligned}$$

we have $d_-(c_-^1, y_-) = d_-(c_-^2, y_-)$.
(iii) For any four chambers $y_- \in \Delta_-$, $y_+^1, y_+^2, e_+ \in \Delta_+$ such that $(y_+^1, y_-) \in \mathcal{O}$, $(y_+^2, y_-) \in \mathcal{O}$ and :

$$\begin{aligned} l(d_+(e_+, y_+^1)) &= l(d_+(e_+, y_+^2)) \\ &= min\{l(d_+(a_+, c_+))|(a_+, y_-) \in \mathcal{O}\} \end{aligned}$$

we have $d_+(y_+^1, e_+) = d_+(y_+^2, e_+)$.
(iv) Given chambers $y_-, a_- \in \Delta_-$, e_+ and $b_+ \in \Delta$ such that a_- is as in (ii), $l(d(a_-, y_-))$ is minimal, b_+ is as in (iii) and $l(d(e_+, b_+))$ is minimal then

$$d_+(e_+, b_+) = d_+(a_-, y_-).$$

If $(i), (ii), (iii)$ and (iv) are satisfied we define for every $x \in \Delta_\epsilon$ ($\epsilon \in \{1, -1\}$) a codistance function $d_x : \Delta_- \mapsto W$ in the following way. For every $z \in \Delta_{-\epsilon}$, $d_x(z)$ equals $d_{-\epsilon}(x_{-\epsilon}, z)$ with $(x, x_{-\epsilon}) \in \mathcal{O}$ such that $l(d(x_{-\epsilon}, z)$ is minimal as in (ii) or (iii). One easily checks this defines a codistance function for every x.
Remains to check these 4 properties :
(i) Because of the definition of $\mathcal{O}$ it suffices to check that $\mathcal{O}$ defines a 1 twinning between the s-residue R_+^s in Δ_+ containing c_+ and the s-residue R_-^s in Δ_- containing c_-. We check that for all the chambers x_- of R_-^s satisfy $(x_-, c_+) \in \mathcal{O}$ except $s(c_-)$.
Every element of R_-^s has the form $u_{-\alpha_s}s(c_-)$ for $u_{-\alpha_s} \in U_{-\alpha_s}$. Suppose that $u_{-\alpha_s} \neq 1$. Granted the properties of the BN-pair (G, B_-, N, S) we can write $u_{-\alpha_s}sc_- = u_{\alpha_s}su'_{\alpha_s}sc_-$ for appropriate u_{α_s} and $u'_{\alpha_s} \in U_{\alpha_s}$. But then $u_{-\alpha_s}s(c_-) = u_{\alpha_s}(c_-)$. And $(c_+, u_{-\alpha_s}(c_-)) = (u_{\alpha_s}(c_+), u_{\alpha_s}(c_-))$. Hence $(c_+, u_{-\alpha_s}(c_-)) \in \mathcal{O}$.
Consider the chamber $s(c_-)$. If $(c_+, s(c_-)) \in \mathcal{O}$ then there would exist a $g \in G$ such that $g(c_+) = c_+$ and $g(s(c_-)) = c_-$. But then $g \in B_+$ and $gs \in B_-$ or $s = b_+b_-$ for $b_+ \in B_+$ and $b_- \in B_-$. This contradicts the fact that s stabilizes the standard apartment Σ_0 as we are working in a tree. Hence $(s(c_-), c_+) \notin \mathcal{O}$.

Granted the action of G on Δ_+ and Δ_- we may assume that $d_+ = c_+$ in $(ii), (iii)$ and (iv).
(ii) Suppose that y_-, c_-^1 and c_-^2 are chambers as in (ii) with $(c_+, y_-) \notin \mathcal{O}$. Then $y_- = gB_-$, $c_-^1 = b_+^1 B_-$ and $c_-^2 = b_+^2 B_-$ for $g \in G, b_+^i \in B_+$. Let $d_-(c_-^1, y_-) = w_1$ and $d_-(c_-^2, y_-) = w_2$. It follows from the assumptions that $l(w_1) = l(w_2)$.
Assume $w_1 \neq w_2$.
Because we work in a non spherical Coxeter group two possibilities occur. Namely $w_1^2 = w_2^2 = 1$ or $w_1^2 \neq 1$ and $w_2^2 \neq 1$.
Expressing that the distances from c_-^1 and c_-^2 to y_- are w_1 and w_2 gives :

$$\begin{aligned} gB_- &= b_+^1 b_-^1 w_1 B_- \\ &= b_+^2 b_-^2 w_2 B_- \end{aligned}$$

for $b_-^i \in B_-$.
Hence

$$b_+^1 b_-^1 w_1 = b_+^2 b_-^2 w_2 b_-$$

for $b_- \in B_-$.
Using the properties of the BN-pair (G, B_-, N, S) we find :

$$(b_+^2)^{-1} b_+^1 = b_-^2 w_2 b_- w_1^{-1} (b_-^1)^{-1}.$$

If $w_1^2 = w_2^2 = 1$ then

$$b_-^2 w_2 b_- w_1^{-1} (b_-^1)^{-1} = b'_- w_2 w_1 b''_-$$

for $b'_-, b''_- \in B_-$.
If $w_1^2 \neq 1$ and $w_2^2 \neq 1$ then $w_1 w_2 = 1$ and the properties of the BN-pair (G, B_-, N, S) yield :

$$b_-^2 w_2 b_- w_1^{-1} (b_-^1)^{-1} = b'_- w_2^2 b''_-$$

for $b'_-, b''_- \in B_-$.
In all cases we find that if $w_1 \neq w_2$ then for a $v \neq 1$, b'_- and $b''_- \in B_-$

$$b'_- v b''_- \in B_+,$$

with $l(v) = 0 \, mod \, 2$. This means that $b'_- v b''_-$ has to fix the chamber c_+. Write $b'_- v b''_- = u'_- v u''_- h$ for $h \in H$. Then $u'_- v u''_-$ has to fix c_+.

Two cases occur :
(a) $u''_- = 1$.
Then we have $u'_-(v(x_0) = x_0$ and $u'_-(v(x_1)) = x_1$. This is only possible if $v = 1$ and $u'_- = 1$.
(b) The element $u''_- \neq 1$.
Suppose that $W = \{s, t\}$, $\partial\alpha_s = x_0$, $\partial\alpha_t = x_1$.
If $u''_- \in U_{-\alpha_s}$ we find that $u'_- v u''_-(x_0) = x_0$. Granted the condition on u''_- this implies that $u'_-(v(x_0)) = x_0$. Again a contradiction.
Hence there $u''_- \notin U_{-\alpha_2}$ and there exists an index j such that $x_j \sim y_1 \sim y_2 \sim \ldots \sim u''_-(x_0) \sim u''_-(x_1)$ is the gallery in Γ from Σ_0 to $u''_-(x_1)$.
Suppose that $j < 0$ (we already excluded the case where $j = 0$).
Because $l(v) = 0 \, mod \, 2$, v acts as a translation of Σ_0 i.e. :

$$v(x_l) \;=\; x_{l+k_0},\ \forall l$$

for a fixed $k_0 \in \mathbb{Z}$.
Let $v(x_j) = x_m$.
If $m \leq 0$ then $d_+(x_j, u''_-(x_0)) \neq d_+(x_m, x_0)$. One easily checks that there cannot exist a $u'_- \in U_-$ with $u'_-(vu''_-(x_0)) = x_0$.
If $m \geq 1$ then :

$$d_+(x_m, vu''_-(x_0)) < d_+(x_m, vu''_-(x_1))$$

One checks that for no $u'_- \in U_-$ we can have $u'_-(vu''_-(x_0)) = x_0$.
If $j > 0$ one uses similar arguments to deduce a contradiction.
(3) If (y_+^1, y_-) and $(y_+^2, y_-) \in \mathcal{O}$ then

$$\begin{aligned} y_- &= g(c_-) \\ y_+^1 &= g(c_+) \\ y_+^2 &= gb_-(c_+) \end{aligned}$$

for $g \in G$ and $b_- \in B_-$.
A symmetric proof completely analogous to (2) gives $d_+(y_+^1, c_+) = d_+(y_+^2, c_+)$.
(4) Let y_- and c_-^1 be chambers of Δ_- with $(c_+, c_-^1) \in \mathcal{O}$ and $d(c_-^1, y_-)$ is minimal as in (ii). Then we look for a chamber y_+ in Δ_+ such that $(y_+, y_-) \in \mathcal{O}$ and $d_+(c_+, y_+) = d_-(c_-^1, y_-)$. This will imply (iv). Without loss of generality we can assume that $c_-^1 = c_-$.
Let the minimal gallery in Δ between c_- and y_- be :

$$y_-^0 = c_- \overset{s}{\sim} y_-^1 \overset{t}{\sim} y_-^2 \overset{s}{\sim} \ldots \overset{t}{\sim} y_-^m = y_-$$

If $y_-^1 = u_{-\alpha_s}s(c_-)$ let y_+^1 be $u_{-\alpha_s}s(c_+)$. If $y_-^2 = u_{-\alpha_t}tu_{-\alpha_s}s(c_-)$ let y_+^2 be $u_{-\alpha_t}tu_{-\alpha_s}s(c_+)$.
If we do this for all y_-^i we get a gallery :

$$y_+^0 = c_+ \overset{s}{\sim} y_+^1 \overset{t}{\sim} y_+^2 \overset{s}{\sim} \ldots \overset{t}{\sim} y_+^m$$

from c_+ to y_+^m. One shows with a proof similar as in (2) that for no $v \in W$ and $b_-, b'_- \in B_-$ we can have that $b_-vb'_- \in B_+$. This ensures us that all the y_+^j are different. The gallery is therefore non-stammering and $d_+(c_+, y_+^m) = d_-(c_-^1, y_-)$. By construction we have $(y_+, y_-) \in \mathcal{O}$.
This completes the proof that (i),(ii),(iii) and (iv) are satisfied for $\mathcal{O}$. Hence $\mathcal{O}$ is the opposition relation of a twinning between Δ_+ and Δ_-. □

2.1.6 Constructing a 2-twinning

In this paragraph we will show that the building (Δ, W, S, d) is half of a twin building using a result of B. Mühlherr in [18]. We restate the main result of loc. cit.

Theorem 73 *Let M be a Coxeter matrix over I, let (Δ_+, W, S, d_+) and (Δ_-, W, S, d_-) be two thick buildings of type M and let $\mathcal{O} \subseteq (\Delta_+ \times \Delta_-) \cup (\Delta_- \times \Delta_+)$ be a non-empty symmetric relation. Then $\mathcal{O}$ is the opposition relation of a twinning between $(\Delta_+, W, S, \delta_+)$ and $(\Delta_-, W, S, \delta_-)$ if and only if the following condition is satisfied :*
If $J \subseteq I$ is of cardinality at most 2 and if $R_+ \subseteq \Delta_+$ and $R_- \subseteq \Delta_-$ are J-residues, then either $\mathcal{O} \cap ((R_+ \times R_-) \cup (R_- \times R_+)) = \varnothing$ or $\mathcal{O} \cap ((R_+ \times R_-) \cup (R_- \times R_+))$ is the opposition relation of a twinning between R_+ and R_-.

We now have :

Theorem 74 *Given a Moufang building (Δ, W, S, d) with root groups $(U_\alpha)_{\alpha \in \Phi}$ then Δ is half of a twin building i.e. there exists a building (Δ_-, W, S, d_-) and a codistance function d_* such that $((\Delta, W, S, d), (\Delta_-, W, S, d_-), d^*)$ is a twin building.*

proof :

By Theorem 2 and Corollary 3 we know that there are two BN-pairs involved. Namely (G, B_+, N, S) and (G, B_-, N, S). The building (Δ_+, W, S, d_+) associated to (G, B_+, N, S) is by construction isomorphic to Δ. We define the symmetric relation $\mathcal{O}$ between Δ_+ and Δ_- as before. Consider $s_i, s_j \in S$. Let $R^+_{s_i s_j}$ and $R^-_{s_i s_j}$ be the $\{s_i, s_j\}$-residues in Δ_+ and Δ_- containing c_+ and c_- respectively. Then it follows from Theorem 4 and Theorem 5 that $\mathcal{O}$ defines the opposition relation of a twinning between $R^+_{s_i s_j}$ and $R^-_{s_i s_j}$. By construction this implies that $\mathcal{O}$ satisfies the conditions of Theorem 6. Hence $\mathcal{O}$ defines a twinning between Δ_+ and Δ_-. This means that $\Delta \cong \Delta_+$ is half of a twin building. □

2.2 Characterization

2.2.1 Introduction

As usual $M = (m_{i,j})_{i,j \in I}$ stands for a certain Coxeter matrix and ϵ is an element of the set $\{1, -1\}$. When considering buildings we will not always explicitly mention the type if this is not relevant in the context. The following definition can be found in [19]

Definition 75 A (thick) 1-*twinning* between a pair of (thick) buildings $(\Delta+, W, S, d_+)$, $(\Delta-, W, S, d_-)$ of the same type is a symmetric binary relation $\mathcal{O} \subset \Delta_+ \times \Delta_- \sqcup \Delta_- \times \Delta_+$ satisfying :
if $(c_\epsilon, c_{-\epsilon}) \in \mathcal{O}$, every panel in Δ_ϵ through c_ϵ contains exactly one chamber z with $(z, c_{-\epsilon}) \notin \mathcal{O}$.

Given the notion of a 1-twinning, galleries between chambers of Δ_ϵ and $\Delta_{-\epsilon}$ can be introduced.

Definition 76 Let $\mathcal{O}$ be a (thick) 1-twinning between (Δ_+, W, S, d_+) and (Δ_-, W, S, d_-). A *gallery* between $c_0 \in \Delta_\epsilon$ and $c_n \in \Delta_{-\epsilon}$ is a sequence of chambers $(c_0, c_1, \ldots, c_n)$ such that :
(i) $(c_1, c_2, \ldots, c_n) \in \Delta_{-\epsilon}$ is a gallery in $\Delta_{-\epsilon}$ and $(c_0, c_1) \in \mathcal{O}$ or
(ii)$(c_0, c_1, \ldots c_{n-1}) \in \Delta_\epsilon$ is a gallery in Δ_ϵ and $(c_{n-1}, c_n) \in \mathcal{O}$. If $\Gamma = (c_0, c_1, \ldots, c_m)$ denotes a gallery, its *length* is defined as m.

As a consequence of the connectedness of Δ_+ and Δ_- every chamber of Δ_ϵ can be joined via a gallery to every chamber of $\Delta_{-\epsilon}$. It thus makes sense to consider *minimal galleries* between chambers of Δ_ϵ and $\Delta_{-\epsilon}$ (i.e. galleries of minimal length).
Let $x \in \Delta_\epsilon$ then we denote :

$$x^o = \{y \in \Delta_{-\epsilon} | (x, y) \in \mathcal{O}\}.$$

2.2.2 First result

Theorem 77 *A thick* 1*-twinning* $\mathcal{O}$ *between two buildings* (Δ_+, W, S, d_+) *and* (Δ_-, W, S, d_-) *of type* M *defines a twinning if and only if :*
(i) *Given* $x \in \Delta_\epsilon$ *and* $y \in \Delta_{-\epsilon}$ *then* $d_{-\epsilon}(x_y, y) = d_{-\epsilon}(\bar{x}_y, y)$ *whenever* x_y *and* $\bar{x}_y$ *are two chambers in* x^o *satisfying :*

$$l(d_{-\epsilon}(x_y, y)) = l(d_{-\epsilon}(\bar{x}_y, y)) = min\{l(d_{-\epsilon}(z, y) | z \in x^o\}.$$

(ii) *If* $y_x \in y^o$ *such that the distance* $d_\epsilon(y_x, x)$ *is minimal, and* $x_y \in x^o$ *such that the distance* $d_{-\epsilon}(x_y, y)$ *is minimal then :*

$$d(x_y, y) = (d(y_x, x))^{-1}.$$

Under these conditions we can define a function d^* *from* Δ_+ *and* Δ_- *to* W*. Namely for* $x \in \Delta_\epsilon$ *and* $y \in \Delta_\epsilon$*, we set* $d^*(x, y)$ *as* $d(x_y, y)$ *where* x_y *a chamber of* x^o *at minimal distance from* y*. Under these conditions* d^* *defines a codistance function of a twinning between* (Δ_+, W, S, d_+) *and* (Δ_-, W, S, d_-)*.*

proof :

Suppose that $\mathcal{O}$ is a twinning between (Δ_+, W, S, d_+) and (Δ_-, W, S, d_-). This means that there exists a codistance function d^* going from $\Delta_+ \sqcup \Delta_-$ such that $\mathcal{O} = \{(x, y) \in \Delta_+ \times \Delta_- \sqcup \Delta_- \times \Delta_+ \, | d^*(x, y) = 1\}$. Let $x, x_y, \bar{x}_y$ and y_x be as in the theorem. Set $w = d^*(x, y)$. It is a general observation that for $x_y, \bar{x}_y, y_x$, $w = d_{-\epsilon}(x_y, y) = d_{-\epsilon}(\bar{x}_y, y) = d_\epsilon(x, y_x)$. Hence conditions (i) and (ii) are satisfied form every $x \in \Delta$.
We check axioms $(\mathbf{Tw1}), (\mathbf{Tw2})$ and $(\mathbf{Tw3})$ for d^*.
Conversily let $\mathcal{O}$ be as in the thoerem. Then we check that it defines a twinning between Δ_+ and Δ_-. $(\mathbf{Tw1})$.

This follows from property (ii).
(**Tw2**).
Suppose $d^*(x,y) = w$, $x \in \Delta_\epsilon, y \in \Delta_{-\epsilon}$, $s_i \in S$ such that $l(ws_i) < l(w)$. Let z be any chamber of $\Delta_{-\epsilon}$ s_i-adjacent to y.
The definition of d^* implies that we can choose a chamber $x_y \in x^o$ such that $d_{-\epsilon}(x_y, y) = w$. Consider in $\Delta_{-\epsilon}$ a minimal gallery $\Gamma_w = y^0(= x_y) \sim y^1 \sim \ldots \sim y^{m-1} \sim y^m(= y)$ of type w from x_y to y. Without loss of generality we can assume that $d^*(x, y^{m-1}) = w'$, $w's_i = w$ and $l(w') < l(w)$. It is not hard to check that there exists a gallery $\Gamma_{w'}^{-1} = y^{m-1} \sim x^1 \sim \ldots \sim x$ of type h' with $r_{h'} = w'^{-1}$ in Δ_ϵ with $(x_1, y^{m-1}) \in \mathcal{O}$. But then $d^*(x_1, y) = s$ and granted the condition of a 1-twinning we find $d^*(x_1, z) = 1$. This yields $l(d^*(z,x)) \leq m-1$. As $l(d^*(x,y)) = m$ we find $l(d^*(z,x)) = m-1$. Hence $d^*(z,x) = w'^{-1}$ and $d^*(x,z) = w' = ws$.
(**Tw3**).
Consider two chambers $x \in \Delta_\epsilon$ and $y \in \Delta_{-\epsilon}$ with $d^*(x,y) = w$. Two cases occur.
First case : $l(ws_i) < l(w)$.
As we saw in the proof for (**Tw2**) every chamber z in $\Delta_{-\epsilon}$, s_i-adjacent to y satisfies $d^*(x,z) = ws_i$.
Second case : $l(ws_i) > l(w)$.
Choose a chamber $y_x \in y^o$ such that $d(x, y_x) = d^*(x,y)$. There exists in the s_i-residue containing y exactly one chamber, call it p with $d^*(y_x, p) = s_i$. We show that $d^*(x,p) = ws_i$.
Set $\tilde{w} = d^*(x,p)$. The hypothesis on w implies that $l(\tilde{w}) = l(w)$ or $l(\tilde{w}) = l(w)+1$.
Suppose that $l(\tilde{w}) = l(w)$.
Choose $p_x \in p^o$ such that $d(x, p_x) = d^*(x,p)$.
We have two possibilities.

(1) $d^*(p_x, y) = 1$. Granted the condition on the opposition relation we deduce $w = \tilde{w}$.

(2) $d^*(p_x, y) = s_i$.
Choose a third chamber q of the s_i-panel through y. (Such a chamber always exists as the buildings under consideration are thick.) As $d^*(p_x, y) = s_i$, q lies opposite p_x. Choose a minimal gallery $\Gamma_{\tilde{w}}$ in Δ_ϵ of type $\tilde{h}$, with $r_{\tilde{h}} = \tilde{w}$ from x to p_x.

Consider $l(d^*(x,q))$.
Then $l(\tilde{w}) - 1 \leq l(d^*(x,q)) \leq l(\tilde{w})$. If $l(d^*(x,q)) = l(\tilde{w}) - 1$, there exists a gallery from x to p of type $\tilde{h}'s$ such that $r_{\tilde{h}'}s = \tilde{w}$. But then $l(\tilde{w}s) = l(\tilde{w}) - 1$. This in turn implies that there exists a gallery $\Gamma_{\tilde{w}}$ of type g with $r_g = \tilde{w}$ from x_p to y.
Hence $\tilde{w} = w$.
If $l(d^*(x,q)) = l(\tilde{w})$ we have $d^*(x,q) = \tilde{w}$.
Completely similar arguments for x, y_x, y, q, p imply $d^*(x,q) = d^*(x,y) = d^*(x,p)$.
Putting these two equalities together gives $d^*(x,q) = d^*(x,p) = d^*(x,y) = w = \tilde{w}$.
In any case we find that if $l(\tilde{w}) = l(w)$, then $w = \tilde{w}$. Consider a minimal gallery $\Gamma_w = x_0(= x) \sim x_1 \sim \ldots \sim x_m(= y_x)$ in Δ_ϵ of from x to y_x. If $d(x_i, x_{i+1}) = s_{i+1}$ it follows $d^*(x_i, p) = ws_1s_2\ldots s_i$. In particular $d^*(x_m, p) = 1$ a contradiction.
We conclude $l(\tilde{w}) = l(w) + 1$ and $d^*(x,p) = ws = \tilde{w}$.

□

2.2.3 The local approach

In what follows we prove a local condition on a thick 1-twinning sufficient and necessary for a the 1-twinning to be a twinning.

Definition 78 *Given a thick* 1*-twinning* $\mathcal{O}$ *between* (Δ_+, W, S, d_+) *and* (Δ_-,W,S,d_-) *we say it satisfies condition Ptw for a chamber* $c \in \Delta_\epsilon$ *if :*
$\forall y \in \Delta_{-\epsilon}, \forall c_y, \bar{c}_y \in c^o$ *such that* $l(d_{-\epsilon}(c_y, y)) = l(d_{-\epsilon}(\bar{c}_y, y)) = min\{l(d_{-\epsilon}(z,y))|z \in c^o\}$,
$\forall y_c \in y^o$ *with* $l(d_\epsilon(c, y_c) = min\{l(d_\epsilon(v,c)|v \in c^o\}$

$$d_{-\epsilon}(c_y, y) = d_{-\epsilon}(\bar{c}_y, y) = d_\epsilon(c, y_c).$$

Under these conditions we can define a function f going from $\{c\} \times \Delta_{-\epsilon}$ to W. If $y \in \Delta_{-\epsilon}$ then $f(c,y) = d_{-\epsilon}(c_y, y)$ for a chamber $c_y \in c^o$ at minimal distance from y.
Let c be as in the definition. Then we denote in the sequel the induced function f also by d^c_*. A first observation is :

Lemma 79 *A thick* 1*-twinning* $\mathcal{O}$ *between* (Δ_+, W, S, d_+) *and* (Δ_-, W, S, d_-) *is a twinning if and only if condition Ptw is satisfied for every chamber* $c_\epsilon \in \Delta_\epsilon$.

As before $\mathcal{O}$ is a 1-twinning between (Δ_+, W, S, d_+) and (Δ_-, W, S, d_-). Next step is to impose condition Ptw on one chamber $c \in \Delta_\epsilon$. Then we want to prove that condition Ptw is valid for every chamber $z \in \Delta_\epsilon \sqcup \Delta_{-\epsilon}$.

Lemma 80 *Let* $\mathcal{O}$ *be a* 1*-twinning between* (Δ_+, W, S, d_+) *and* (Δ_-, W, S, d_-). *Suppose condition Ptw is satisfied for some chamber* $x \in \Delta_+$. *If* $d^*(x, y) = w$, z *is* s_i*-adjacent to* y *and* $l(ws_i) < l(w)$ *we have* $d^x_*(x, z) = ws_i$.

proof :

As $l(ws_i) < l(w)$ the w equals $w_1 s$ with $l(w_1 s) = l(w_1) + 1$. Hence every minimal gallery Γ_w of type h $(r_h = w)$ from x to y via x^o can be replaced by a gallery $\Gamma_{w_1 s} = x \sim y_1 \sim \ldots y_{m-1} \sim y$ of type $h'i$ with $r_{h'} = w_1$ and $(x, y_1) \in \mathcal{O}$.
Two cases occur.
First case : $z = y_{m-1}$.
Then we have $d^x_*(x, z) = d^x_*(x, y_{m-1}) = ws_i$.
Second case : $z \neq y_{m-1}$.
Consider a chamber $y_x \in y^o_{m-1}$ at minimal distance from x. There exists a minimal gallery $\Gamma_{w_1^{-1}}$ in δ_+ of type h_1 with $r_{h_1} = w_1$ from x to y_x. The chamber z should satisfy $(y_x, z) \in \mathcal{O}$. Otherwise $(y_x, y) \in \mathcal{O}$ and $d^x_*(x, y) = w_1 = w$ a contradiction. It follows that $d^x_*(x, z) = w_1$. □

Lemma 81 *Suppose we are given a* 1*-twinning between* (Δ_+, W, S, d_+) *and* (Δ_-, W, S, d_-) *such that condition Ptw is satisfied for* $x \in \Delta_\epsilon$. *Let* $d^x_*(x, y) = w$ *for* $y \in \Delta_-$. *If* z *is a chamber of* $\Delta_{-\epsilon}$, s_i*-adjacent to* y *then* $d^x_*(x, z) \in \{w, ws_i\}$.

proof :

Set $l(w) = m$. There are two possibilities.
First case $l(ws) = m - 1$.
Then the claim follows from *Lemma* 80

Second case $l(ws) = m+1$.
If $l(d_*^x(x,z)) = m+1$ then we know as there is a gallery of type hi with $r_h s_i = w$ from x to z that $d^*(x,z) = ws$.
Remains to prove the lemma if $l(d_*^x(x,z)) = m$.
(Remark that these are the only possible values for $l(d^*(x,z))$ granted the condition on w.)
Let $d_*^x(x,z)$ be $\tilde{w}$. Consider elements $z_x, y_x \in \Delta_\epsilon$, $x_z, x_y \in \Delta_{-\epsilon}$ such that $d_\epsilon(x, y_x) = d_{-\epsilon}(x_y, y) = w$ and $d_\epsilon(x, z_x) = d_{-\epsilon}(x_z, z) = \tilde{w}$. As we are working with a 1-twinning there are two possibilities :

1. First possibility : $(z_x, y) \in \mathcal{O}$ or $(y_x, z) \in \mathcal{O}$.
   Then it is clear that $d_*^x(x,z) = d_*^x(x,y)$.

2. Second possibility whenever (z_x, y) and $(y_x, z) \neq \mathcal{O}$.
   Consider a third chamber r which is s-adjacent to both z and y. It follows that $(z_x, r) \in \mathcal{O}$. Hence $l(d_*^x(x,r)) \in \{m, m-1\}$.
   Suppose that $l(d_*^x(z_x,r)) = m-1)$. Then $l(ws) = m-1$ as there would be a gallery $\Gamma_{w's} = x \sim x^0 \sim x^1 \sim \ldots \sim r \sim y$ of length m with $(x, x^0) \in \mathcal{O}$. The type of $\Gamma_{w's}$ is $h'i$. Thus $d_*^x(x,y) = r_{h'} s_i = w$ with $l(r_{h'}) < l(w)$.
   We conclude that $l(d_*^x(x,r)) = m$ and thus $d_*^x(x,r) = \tilde{w}$. An analogous reasoning gives $d_*^x(x,r) = w$. Hence $\tilde{w} = w$ and $d_*^x(x,z) \in \{w, ws\}$. □

Before proving the main theorem we give an important lemma.

Lemma 82 *Let $\mathcal{O}$ is a 1-twinning between (Δ_+, W, S, d_+) and (Δ_-, W, S, d_-). Suppose $x \in \Delta_\epsilon$, $y \in \Delta_{-\epsilon}$, and Γ_h a minimal gallery from x to y via x^o of type $h = (h_1 h_2 \ldots h_m)$. Then every chamber $z \in \Delta_{-\epsilon}$ joined by a gallery $\Gamma_h^{-\epsilon}$ of type h to y lies opposite x.*

proof :

Set $\Gamma_h = x y_0 y_1 \ldots y_m (= y)$. Construct a special gallery Γ_h^ϵ in Δ_ϵ. Denote x by x_0. Consider the s_{h_1} panel through x. Then there is a unique chamber in this panel not opposite y_0. This is the chamber x_1. Suppose we already constructed a gallery $x_0 x_1 \ldots x_i$ of type $(h_1 h_2 \ldots h_i)$ with $(x_j, y_j) \in \mathcal{O}$, $0 \leq j \leq i$ and $(x_j, y_{j+1}) \notin \mathcal{O}$ for $0 \leq j \leq i-1$. Then we choose as x_{i+1} the unique chamber of the $s_{h_{i+1}}$ panel through x_i not opposite y_i. Proceeding in this way we end up with a gallery $\Gamma_h^\epsilon = x_0 x_1 \ldots x_m$ in Δ_ϵ of type h such that $(x_j, y_j) \in \mathcal{O}$, $0 \leq j \leq m$, and $(x_j, y_{j+1}) \notin \mathcal{O}$ for $0 \leq j \leq m-1$. Let z_0 be another chamber of $\Delta_{-\epsilon}$ joined to y by a gallery $\Gamma_h^{-\epsilon} = z_0 z_1 \ldots z_m$ of type h. Then it one easily deduces using the properties of 1-twinnings and the fact that $x y_0 y_1 \ldots y_m$ is minimal that $(x_j, z_j) \in \mathcal{O}$ for $0 \leq j \leq m$. This implies in particular that $(x, z_0) \in \mathcal{O}$. □

Theorem 83 *Given a thick* 1*-twinning* $\mathcal{O}$ *between* (Δ_+, W, S, d_+) *and* (Δ_-, W, S, d_-)*. Then* $\mathcal{O}$ *defines a twinning if and only if condition Ptw is satisfied for some element* $x \in \Delta_\epsilon$*.*

proof :

That this condition is necessary is follows theorem 79.

To show the converse we use the following strategy.
Fix a chamber x' of Δ_ϵ, s_i-adjacent to x for $s_i \in S$. Then we prove the chamber x' obeys condition Ptw.
As buildings are connected chamber systems this implies that every chamber of Δ_ϵ satisfies condition Ptw. From this we deduce then that also all chambers of $\Delta_{-\epsilon}$ satsisfy condition Ptw. Theorem 79 implies then that $\mathcal{O}$ defines a twinning between Δ_+ and Δ_-.
Consider $y' \in \Delta_{-\epsilon}$. Let $\Gamma_{\tilde{h}} = x' \tilde{y}_0 \tilde{y}_1 \ldots \tilde{y}_{m-1}\, \tilde{y}_m (= y)$ and $\Gamma_{\bar{h}} = x'\, \bar{y}_0 \bar{y}_1 \ldots \bar{y}_{m-1}$ $\bar{y}_m$ $(= y)$ be two minimal galleries from x' to y via $(x')^o$. Suppose the type of $\Gamma_{\tilde{h}}$ is $\tilde{h} = (\tilde{h}_1 \tilde{h}_2 \ldots \tilde{h}_m)$ and the type of $\Gamma_{\bar{h}}$ is $\bar{h} = (\bar{h}_1 \bar{h}_2 \ldots \bar{h}_m)$. Let $r_{\tilde{h}} = \tilde{w}$ and $r_{\bar{h}} = \bar{w}$. Then $l(\tilde{w}) = l(\bar{w}) = m$. Remains to show that $\tilde{w} = \bar{w}$.
Two cases occur :

1. First case : there exists no minimal gallery from x' to y of type g, such that $l(s_i r_g) < l(r_g)$.
   According to the length of the distance between x and y tree sub cases occur.

   First subcase : $l(d_*^x(x, y)) = m$
   We assume that $d_*^x(x, y) = s_1 s_2 \ldots s_m = w$. Next thing to do is calculate the relation between x and $\tilde{y}_0$ and between x and $\bar{y}_0$. As a generic case we consider x and $\tilde{y}_0$.

Suppose $d^*(x,\tilde{y}_0) \notin \mathcal{O}$. On going down from y to $\tilde{y}_0$ via $\tilde{y}_{m-1} \sim \tilde{y}_{m-2} \sim \ldots \sim \tilde{y}_0$ we get (use Lemma 80) :

$$\begin{aligned} d^x_*(x,\tilde{y}_m) &= w \\ d^x_*(x,\tilde{y}_{m-1}) &= w \text{ or } w\tilde{s}_{i_m} \\ &\vdots \\ d^x_*(x,\tilde{y}_0) &= w\tilde{s}_{i_m}\ldots\hat{\tilde{s}}_{i_p}\ldots\tilde{s}_{i_1} \end{aligned}$$

where the hat stands for possible omitting a certain generator.
This means $w = s\tilde{s}_1\ldots\hat{\tilde{s}}_p\ldots\tilde{s}_m$. Thus there is a minimal gallery $xz_0z_1\ldots z_m$ of type $(ii_1\ldots i_{p-1}i_{p+1}\ldots i_m) = \tilde{g}$, $(r_{\tilde{g}} = \tilde{w})$, from x to y via x^o. But as this type starts with i and we are working with a 1-twinning, x' lies opposite z_1. Thus there is a gallery $x'z_1\ldots y$ from x' to y of length $m-1$. A contradiction to the minimality of m. This implies $(x,\tilde{y}_0) \in \mathcal{O}$. In a completely analogous way we find $(x,\bar{y}_0) \in \mathcal{O}$. As the chamber x satisfies condition Ptw we deduce the equality $\tilde{w} = \bar{w}$.

Second subcase : $l(d^*(x,y)) = m+1$
Suppose $d^x_*(x,\tilde{y}_0) = 1$ or $d^x_*(x,\bar{y}_0) = 1$ We get a contradiction with $l(d^x_*(x,y)) = m+1$.
Hence :

$$\begin{aligned} d^x_*(x,\tilde{y}_0) &= s_i \\ d^x_*(x,\bar{y}_0) &= s_i \end{aligned}$$

Consider arbitrary chambers lets call them $\tilde{y}_{-1}$ and $\bar{y}_{-1}$ in Δ_-, s_i-adjacent to $\tilde{y}_0$ and $\bar{y}_0$ respectively.
We have :

$$d^x_*(x,\tilde{y}_{-1}) = d^x_*(x,\bar{y}_{-1}) = 1$$

The hypothesis x yields $s\tilde{w} = s\bar{w}$ or equivalently $\tilde{w} = \bar{w}$.

Third subcase : $l(d^*(x,y)) = m-1$
Then there exists a minimal gallery Γ of length $m-1$ from x to y via x^o namely $xy_0y_1\ldots y$. Then $d^x_*(x',y_0)$ equals s_i. Let y_{-1} be an arbitrary chamber in Δ_-, s_i-adjacent to y_0. Then x' lies opposite y_{-1} and thus the sequence $x'y_{-1}y_0y_1\ldots y_{m-1}$ is a minimal gallery from x' to y

of a certain type g_0 such that $l(s_i r_{g_0}) < l(g_0)$, a contradiction with the hypothesis.

2. Second case : there exists a minimal gallery from x' to y of type u with $l(s_i r_u) < l(r_u)$.
Notations are as above. Without loss of generality we may assume that $\Gamma_{\tilde{h}}$ is a gallery of type $u = \tilde{h} = (i\tilde{h}_2 \dots \tilde{h}_m)$. Under these assumptions it follows that $d^x_*(x,y) = s_{\tilde{h}_2} \dots s_{\tilde{h}_m}$. As before we denote $d^*(x,y)$ by w. We claim all minimal galleries from x' to y via x^o have type v such that with $r_v = r_{\tilde{h}}$. Consider as above the other minimal gallery $\Gamma_{\bar{h}} = x'\bar{y}_0\bar{y}_1 \dots \bar{y}_{m-1}\ \bar{y}_m = y$. Assume that the type of $\Gamma_{\bar{h}} = (\bar{h}_1\bar{h}_2 \dots \bar{h}_m)$.
First possibility : $l(s_i r_{\bar{h}}) < l(r_{\bar{h}})$.
Without loss of generality we can assume in this case that $\bar{s}_{\bar{h}_1} = s_i$. We have that x lies opposite $\bar{y}_1$ and the hypothesis on x gives $s_{\tilde{h}_1} s_{\tilde{h}_2} \dots s_{\tilde{h}_m} = s_{\bar{h}_1} s_{\bar{h}_2} \dots s_{\bar{h}_m}$. Hence $r_{\tilde{h}} = r_{\bar{h}}$.

The second case occurs when $l(s_i r_{\bar{h}}) > l(r_{\bar{h}})$.
Consider $d^*(x, \bar{y}_0)$.
Suppose that $d^*(x, \bar{y}_0) = s_i$. On going down from y to $\bar{y}_0$ the distance d^x_* should stutter two times. (Remember that $d^x_*(x,y) = \tilde{s}_2\tilde{s}_3 \dots \tilde{s}_m$).
Two possbilities occur.
There exists p, $1 \le p \le m$ such that :

$$w = \hat{s}_i s_{\bar{h}_1} \dots s_{\hat{\bar{h}}_p} \dots s_{\bar{h}_m}$$

or there exist p_1, p_2 with $1 \le p_1$, p_2 such that :

$$w = s_i s_{\bar{h}_1} \dots s_{\hat{\bar{h}}_{p_1}} \dots s_{\hat{\bar{p}}_2} \dots s_{\bar{h}_m}$$

The second possibility cannot occur as we have that $l(s_i w) > l(w)$. If $p \neq m$ we can replace the gallery $\Gamma_{\tilde{h}}$ by a gallery $\Gamma_{\ddot{w}} = \ddot{y}_0\ddot{y}_1 \dots \ddot{y}_{p-1}\bar{y}_p \dots y$ of type $\tilde{h}$. We can restrict ourselves in particular to the galleries $\ddot{y}_0\ddot{y}_1 \dots \bar{y}_p$ and $\bar{y}_0\bar{y}_1 \dots \bar{y}_p$ and we have reduced the situation to the case where $p = m$.
So we can assume without loss of generality that $p = m$.
From the calculation from above gives :

$$w = s_{\bar{h}_1} s_{\bar{h}_2} \dots s_{\bar{h}_{m-1}}.$$

To proceed we construct a special gallery in Δ_ϵ starting in x.
Consider a chamber $x_2 \in \Delta_\epsilon$ with $d_\epsilon(x, x_2) = s_{\bar{h}_1}$. Then $(x_2, \tilde{y}_2) \in \mathcal{O}$. Choose a chamber $x_3 \in \Delta_\epsilon$, $s_{\bar{h}_3}$-adjacent to x_2. As before we find $d^*(x_3, \tilde{y}_3) \in \mathcal{O}$. Continuing in this way we build up a gallery $\Gamma = xx_2x_3 \ldots x_m$ in Δ_ϵ such that $d_\epsilon(x_i, x_{i+1}) = s_{\bar{h}_i}$ and $(x_i, \tilde{y}_i) \in \mathcal{O}$ for $1 \le i \le m$.
Choose a third chamber $z_0 \in \Delta_{-\epsilon}$ of the $s_{\bar{h}_m}$-panel through y, not opposite x_m. As we know $d^x_*(x, y) = w$ we have using Lemma 81 that $d^x_*(x, z_0) \in \{w, ws_{\bar{h}_m}\}$.
Suppose that $d^x_*(x, z_0) = w$.
This yields $(x_m, z_0) \in \mathcal{O}$ as $w = s_{\bar{h}_1} s_{\bar{h}_2} \ldots s_{\bar{h}_{m-1}}$, a contradiction.
It follows that $d^x_*(x, z_0) = ws_{\bar{h}_m}$.
Suppose $z_0 = \bar{y}_{m-1}$. On going down from $\bar{y}_{m-1}$ to $\bar{y}_0$ we get granted the assumption $d^x_*(x, \bar{y}_0) = s_i$ that :

$$ws_{\bar{h}_m} w^{-1} = s_i$$

or equivalently :

$$s_{\bar{h}_1} s_{\bar{h}_2} \ldots s_{\bar{h}_m} = s_i w.$$

This contradicts the assumption $l(s_i r_{\bar{h}}) > l(r_{\bar{h}})$. Therefore $z_0 \neq \bar{y}_{m-1}$. But then $d_{-\epsilon}(\bar{y}_0, z_0) = ws_{\bar{h}_m}$ and granted Lemma 80 we find $d^x_*(x, \bar{y}_0) = 1$, contradicting the hypothesis that $d^x_*(x, \bar{y}_0) = s_i$.
The initial assumption that $d^x_*(x, \bar{y}_0) = 1$ is false and the only possibility is that $d^x_*(x, \bar{y}_0) = s_i$. We saw $d^*_x(x, y) = s_{\tilde{h}_2} s_{\tilde{h}_3} \ldots s_{\tilde{h}_m}$. Hence $l(d^x_*(x, y))$ equals $m - 1$. As $d_{-\epsilon}(\bar{y}_0, y) = s_{\bar{h}_1}\ s_{\bar{h}_2}\ \ldots\ s_{\bar{h}_m}$ and two possibities occur on going down from y to $\bar{y}_0$.
First possbilitie :

There exists p_1, p_2, $1 \le p_i \le 2$ such that :

$$ws_{\bar{h}_m} \ldots \hat{s}_{\bar{h}_{p_1}} \ldots \hat{s}_{\bar{h}_{p_2}} \ldots s_{\bar{h}_1} = s_i$$

(Remark that in this case $p_1 \neq m$ as $d^x_*(x, \bar{y}_0) = s_i$.) or :

$$ws_{\bar{h}_m} s_{\bar{h}_{m-1}} \ldots s_{\bar{h}_1} = s_i.$$

The first possibility contradicts the fact that $l(ws_{\bar{h}_m}) = m$ and. If the second possbilitye would occur we have $l(s_i(ws_{\bar{h}_m})) = l(s_i r_{\bar{h}}) < l(r_{\bar{h}})$ contradicting the initial assumption on $r_{\bar{h}}$.
This means that the possbility that $l(s_i r_{\bar{h}}) > l(r_{\bar{h}})$ is thus excluded.

The above discussion shows the following property. If $x' \in \Delta_\epsilon$ is s_i-adjacent to x and $y \in \Delta_{-\epsilon}$. Consider two minimal galleries $\Gamma_{\tilde{h}}$ and $\Gamma_{\bar{h}}$ of types $(\tilde{h}_1\tilde{h}_2\dots\tilde{h}_m)$ and $(\bar{h}_1\bar{h}_2\dots\bar{h}_m)$ respectively. Then $r_{\tilde{h}} = r_{\bar{h}}$. This is one half of condition Ptw for x'.
We show that the other half of condition Ptw for x' also holds. This follows from the following observation. Let $y \in \Delta_{-\epsilon}$ and $\Gamma_f = yx_0x_1\dots x'$ be a minimal gallery from y to x' via y^o of type $(f_1f_2\dots f_n)$. Using Lemma 2.2.3 one can easily construct a gallery $\Gamma_{f^{-1}} = x'y_0y_1\dots y$ of type $(f_nf_{n-1}\dots f_1)$ from x' to y. As we know that all minimal galleries from x' to y have the same type up to homotopies we're done.
By the connectedness of the buildings Δ_+ and Δ_-, condition Ptw is valid for all chambers of Δ_ϵ.
Let $z \in \Delta_{-\epsilon}$ and $u \in \Delta_\epsilon$. As every minimal galery from z to u defines a minimal galery from u to z, one easily checks that z also satisfies condition Ptw. Hence $\mathcal{O}$ defines a twinning between Δ_+ and Δ_- by Theorem 79. □

Chapter 3

Moufang sets

3.1 Introduction

In the standard reference [32] J. Tits introduces the basic concepts to be used in a possible classification program for twin buildings. Of major importance here is the notion of Moufang set. Loc. cit. is the first place where a formal definition appears. In view of the work of B. Mühlherr [20] the class of the so called induced Moufang sets need special attention. Induced Moufang sets are obtained as local data derived from the global geometry of Moufang buildings. In order to carry out the classification program for the $\tilde{B}_2$ case in the spirit in [20] one needs to solve the following question. "Suppose we are given two Moufang quadrangles Γ_1 and Γ_2 having isomorphic induced Moufang sets. Does this yield any relation between Γ_1 and Γ_2 ?"
In this chapter we will develop an alternative setup concerning induced Moufang sets which emphasizes less the quadrangle. As a consequence the question above is translated to a more algebraic one. Namely in a lot of cases the Moufang sets we consider are closely related to the endomorphism groups of a certain vector space endowed with a quadratic form of Witt index 1. Using the geometry provided by the form one could also view these endomorphism groups as auto-morphism groups of this geometry.

Hence the question concerning isomorphic induced Moufang sets is translated to a question concerning isomorphic linear groups preserving a form of Witt index 1 or to a question of isomorphisms of the related geometries.
As the groups and geometries which arise here are in some cases classical, some work is already done concerning isomorphisms.
Nevertheless most of the theorems we find in the literature provide partial answers that are valid under restrictions which we had to overcome.

In a lot of cases one could use Borel-Tits theory [2]. One disadvantage is that this works only if the groups are algebraic ones. This yields in particular that the vector space in which the form is defined should be finite dimensional over a field of characteristic not 2. As the induced Moufang sets we consider do not always arise from algebraic groups (e.g. if characteristic of ground field is 2) this still leaves a gap.

Classical theory (cfr. e.g. [6], [7], [8], [12], [11], [17], [24]) also gives answers in some cases. As above these results mostly work only if the vector space is finite dimensional, the form is non-degenerate, and the groups are of the same type. Especially the question concerning isomorphic orthogonal Moufang sets in characteristic 2 was problematic. (cfr. Proposition 127) A partial answer was given in [5] but the result only holds under a certain non degeneracy condition. The alternative approach developed in this chapter was very useful here. Using this setup the question was solved completely.

In Chapter 8 in [29] one can also find some theorems which handle with questions related to isomorphisms between Moufang sets (e.g. Lemma 8.18). Nevertheless one verifies that the heart of the problem cannot be solved using this theory. It is in fact translated in another setup.

Moreover as the classification program of $\tilde{B}_2$ buildings requires a comparison of Moufang sets of different nature we followed a more elementary strategy. In this way we could compare indifferent Moufang sets with other ones, solve a lot of question even if the characteristic of the ground fields is 2 and the groups are not algebraic groups and ultimately give an theorem which characterizes classical Moufang sets in some sense
Moreover most of the results proved in this chapter will find application in Chapter 4 dealing with existence and non-existence of certain $\tilde{B}_2$ Moufang buildings.

3.2 Projective Moufang sets

We start the discussion on Moufang sets by considering the family of projective ones. Implicitly these sets were studied before under other names and with other terminology. One of the first to investigate isomorphisms between two such Moufang sets was L.K. Hua. (cfr. [12]). A version of his description of all possible isomorphisms can be found in section 8.12.3 on pp147-149 in [29]. Moreover the techniques used in loc. cit. will enable us to compute in this section all Moufang subsets in a special case. We give a formal description of what is meant by projective Moufang set and prove a first proposition.
Let k be a division ring, E a 2-dimensional right k-vector space and X be the set of all vector lines in E. To simplify the calculations we use a coordinate system. Choose a base $\{e_1, e_2\}$ of E. Denote $\langle e_1x + e_2\rangle$ as (x) and $\langle e_1\rangle$ as (∞). This means we can write $X = \{(x)|x \in k\} \cup \{(\infty)\}$.
Choose as point set the set X.
As to the root group structure we start by giving descriptions of $U_{(\infty)}$ and $U_{(0)}$. In classical terms $U_{(\infty)}$ and $U_{(0)}$ coïncide with transvection groups with centers e_1, e_2 respectively. A typical example of a root elation $u((\infty);(0),(x))$ $\in U_{(\infty)}$ has matrix representation with respect to the ordered base $\{e_1, e_2\}$:

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$$

whereas an element of the form $u((0);(\infty),(x)) \in U_{(0)}$ has matrix representation with respect to the ordered base $\{e_1, e_2\}$:

$$\begin{pmatrix} 1 & 0 \\ x^{-1} & 1 \end{pmatrix}.$$

All other root groups are conjugates of $U_{(\infty)}$ under appropriate elements of $U_{(0)}$. Namely if $(x) \in X$, we define $U_x = gU_\infty g^{-1}$, where $g = u((0);(\infty),(x))$. We check that $(X, (U_x)_{x\in X})$ defines a Moufang set.

1. Condition **MoS1**.
We first prove that $U_{(\infty)}$ acts regularly on $X \setminus\{(\infty)\}$. Let (x), $(y) \in X\setminus \{(\infty)\}$. Then the root elation with matrix representation with respect to the ordered base $\{e_1, e_2\}$:

$$\begin{pmatrix} 1 & y-x \\ 0 & 1 \end{pmatrix}$$

is the unique element of $U_{(\infty)}$ sending (x) to (y).
By similar arguments one checks that also $U_{(0)}$ acts regularly on $X \setminus \{(0)\}$.
As the other root groups are defined as conjugates of $U_{(\infty)}$, condition **MoS1** is clearly satisfied.

2. Condition **MoS2**.
As $U_{(\infty)}$ and $U_{(0)}$ coïncide with the transvection groups with centers e_1 and e_2, a root group U_x corresponds to the group of transvections of center x. Hence the set $\{U_x | \ x \in X\}$ is stabilized by $GL_2(k)$ and condition **MoS2** holds.
In this way we obtain a Moufang set $(X, (U_x)_{x\in X})$ which is denoted by $\mathcal{P}(k)$ and is called *a projective Moufang set defined over the division ring* k. As already mentioned we prove the following proposition.

Proposition 84 *Consider a projective Moufang set $\mathcal{P}(k)$ defined over the field k.*
(i) If $char(k) \neq 2$ every Moufang subset Y of $\mathcal{P}(k)$ corresponds to a subfield of k.
(ii) If $char(k) = 2$ every Moufang subset Y of $\mathcal{P}(k)$ corresponds after a right choice of coordinate system to a subset l of k
satisfying : ($\bar{k}$ is the field generated by l)
(i) $l = l^{-1}$
(ii) $1 \in l$
(iii) *The set l is a vectorspace over a subfield k' of k containing $\bar{k}^2$.*

proof :

Suppose as above that the point set of $\mathcal{P}(k)$ is defined as the set of all vectorlines of a 2 dimensional right k-vectorspace E. Choose an ordered base $\{e_1, e_2\}$ of E such that with notations as above $\mathcal{P}(k) = (X = (\{(x)|x \in k\} \cup \{\infty\}, (U_x)_{x\in X})$.
Let Y be a Moufang subset of $\mathcal{P}(k)$. Set $l = \{t \in k|(t) \in Y\}$. Without loss of generality we can assume that (0), $(\infty) \in Y$ and $(1) \in Y$ as Y contains by definition at least 3 elements.
We show that $(l, +)$ is a subgroup of $(k, +)$.
Let $s, t \in l$. As $(Y, (Stab_{U_y}(Y))_{y\in Y})$ is a Moufang set, $u((\infty); (0), (t))$ stabilizes Y. This means that $u(\infty; 0, t)(s) = (s + t) \in Y$ and hence $s + t \in l$. By similar arguments one deduces that if $t \in l$ also $-t \in l$. Hence $(l, +)$ is a

subgroup of $(k,+)$.
To proceed we restate a formula used on p148 in [29].
It is based on the following general observation. Let (Δ, W, S, d) be a Moufang building with root groups $(U_\beta)_{\beta\in\Phi}$. Then there exist for every $u_\alpha \in U_\alpha$ unique $u_{-\alpha}$ and $u'_{-\alpha} \in U_{-\alpha}$ such that $u_{-\alpha}u_\alpha u'_{-\alpha}$ interchanges α and $-\alpha$ in the standard apartment.
As Moufang sets are 1 dimensional Moufang buildings we can specialize this property. Chambers are points of the Moufang sets and apartments are pairs of chambers. The above observation means that every $u((\infty);(0),(a)) \in U_{(\infty)}$, determines unique $\theta_1, \theta_2 \in U_{(0)}$ with :

$$\begin{aligned} \theta_1 u((\infty);(0),(a))\theta_2((0)) &= (\infty) \\ \theta_1 u((\infty);(0),(a))\theta_2((\infty)) &= (0). \end{aligned}$$

The only possible choice for θ_1 and θ_2 is $\theta_1 = \theta_2 = u((0);\,(\infty),(-a))$ and $\theta_1 u((\infty);(0),(a))\theta_2$ has a matrix representation with respect to the ordered base $\{e_1,\, e_2\}$:

$$\begin{pmatrix} 1 & 0 \\ -a^{-1} & 1 \end{pmatrix}\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -a^{-1} & 1 \end{pmatrix} = \begin{pmatrix} 0 & a \\ -a^{-1} & 0 \end{pmatrix}.$$

Hence for $(x) \in X\setminus \{(0),\,(\infty)\}$ we find :

$$\theta_1 u((\infty);(0),(a))\theta_2((x)) = (-ax^{-1}a).$$

As $u((0);(\infty),(-a)) \in Stab_{U_{(0)}}(Y)$, we find $\theta_1\; u((\infty);(0),(a))\; \theta_2(Y) = Y$. This means in particular that if $a,\, b \in l,\, b \neq 0$ also $ab^{-1}a \in l$. Setting $a = 1$ or $b = 1$ this shows that if $y \in l$ also y^{-1} and $y^2 \in l$.
According to the characteristic we distinguish two cases.
(1) $Char(k) \neq 2$.
Let $a, b \in l$, then $(a+b)^2 \in l$. But $(a+b)^2 = a^2 + 2ab + b^2$ with $a^2, b^2 \in l$. This implies $2ab \in l$. Hence also $((2ab)^{-1} + (2ab)^{-1})^{-1} = ab \in l$.
This proves that l is a subfield of k.
(2) $Char(k) = 2$.
Denote the field generated by l as $\bar{k}$. We see that l has the following properties :
$(i) l^{-1} = l$
$(ii) 1 \in l$

(iii) The set l is a vector space over $\bar{k}^2$.
Remains to check the converse.
Let $\mathcal{P}(k)$ be as in the beginning of the proof. This means $\mathcal{P}(k)$ is defined using the a two dimensional right k-vector space E with base $\{e_1, e_2\}$ used to coordinatize the Moufang set. Suppose l is a subset of k such that l is a subfield of k if $char(k) \neq 2$ and l satisfies conditions (i), (ii) and (iii) of the proposition if $char(k) = 2$. Let $Y = \{(x)|x \in l\} \cup \{(\infty)\}$. We show that Y is a Moufang subset of $\mathcal{P}(k)$. This will be done if we prove that for any three points (a), (b), $(c) \in Y$, the element $u((a);(b),(c))$ stabilizes Y. If $(a),(b)$ and (c) are not mutually different we have $u(a;b,c) = 1$, so we can suppose $a \neq b, b \neq c$ and $a \neq c$. One calculates that $u((a);(b),(c))$ has matrix representation with respect to the ordered base $\{e_1,\, e_2\}$:

$$\begin{pmatrix} \frac{a^2+bc}{a^4+a^2b^2+a^2c^2+b^2c^2} & \frac{a^2(b+c)}{a^4+a^2b^2+a^2c^2+b^2c^2} \\ \frac{b+c}{a^4+a^2b^2+a^2c^2+b^2c^2} & \frac{a^2+bc}{a^4+a^2b^2+a^2c^2+b^2c^2} \end{pmatrix}$$

Let $(x) \neq (\infty) \in Y$. Then $u((a);(b),(c))\ (x) = (((a^2+bc)x + a^2b + a^2c)(bx + cx + a^2 + bc)^{-1})$.
If $(x) = (\infty)$ we find $u((a);(b),(c))\ (\infty) = (a^2+bc)(b+c)^{-1} = (a^2(b+c) + bc^2 + b^2c)$.
Let $char(k) \neq 2$.
Then the condition on l implies $u((a);(b),(c))(x) \in Y$ if $(x) \in Y$.
Let $char(k) = 2$.
Conditions (i) and (iii) on l yield that $(a^2+bc)(b+c)^{-1} = a^2(b+c)^{-1}$ $+(b^{-1}+c^{-1})^{-1} \in Y$.
By the same conditions the element $((a^2+bc)x+a^2b+a^2c)(bx+cx+a^2+bc)^{-1}$ belongs to l if and only if $((a^2+bc)x + a^2b + a^2c)(bx + cx + a^2 + bc)$ belongs to l. Now $((a^2+bc)x + a^2b + a^2c)(bx + cx + a^2 + bc) = (a^2b + b^2c + a^2c + bc^2)x^2 + (a^2b^2 + a^2c^2 + a^4 + b^2c^2)x + (a^4b + a^4c + a^2b^2c + a^2bc^2)$. By Property (iii) one finds $((a^2 + bc)x + a^2b + a^2c)\ (bx + cx + a^2 + bc) = b(a^4 + a^2c^2 + a^2x^2 + c^2x^2) + c(a^4 + a^2b^2 + a^2x^2 + b^2x^2) + x(a^4 + a^2b^2 + a^2c^2 + b^2c^2) \in l$. This proves $u((a);(b),(c))(x) \in Y$, $\forall (x) \in Y$ and Y is a Moufang subset of $\mathcal{P}(k)$. □

Motivated by this proposition we give the following definition.

Definition 85 Let k be a field of characteristic 2, k' a subfield of k containing k^2 and l a subset of k satisfying :

(i) l is an additive subgroup of k,
(ii) $l^{-1} = l$,
(iii) $1 \in l$,
(iv) l generates k as a ring,
(v) l is a vectorspace over k'.
By an *indifferent* Moufang set $\mathcal{P}(k', l; k)$ we mean a Moufang subset Y of a projective Moufang set $\mathcal{P}(\bar{k})$ such that after the right choice of coordinate system $Y = \{(x) | x \in l\} \cup \{(\infty)\}$. If for an indifferent Moufang set $\mathcal{P}(k', l; k)$, $k' = k^2$ we will denote it shortly as $\mathcal{P}(l; k)$.

Remark that if $\mathcal{P}(k', l; k)$ is an indifferent Moufang set then identity map from points of $\mathcal{P}(k', l; k)$ to points of $\mathcal{P}(k^2, l; k)$ defines a Moufang set isomorphism. Hence $\mathcal{P}(k', l; k) \cong \mathcal{P}(k^2, l; k)$.

3.3 Induced Moufang sets in generalized polygons.

In this section we explain the well known procedure to construct Moufang sets given a Moufang polygon. It is not hard to see how this procedure can be generalized to the tree case.
Let $\Gamma = (\mathcal{P}, \mathcal{L}, I)$ be a generalized Moufang n-gon such that $n < \infty$. Consider $x \in \Gamma$. If $z \in \Gamma(x)$, choose a root α_z with $z \in Int(\alpha)$ and $x \in \partial\alpha$. The group U_{α_z} induces on $\Gamma(x) \setminus \{z\}$ a regular permutation group. It can be shown that the action of U_{α_z} is independent of the initial choice of α_z. Therefore we can identify U_{α_z} with a permutation group $U_z \leq Sym(\Gamma(x))$. Repeating this procedure for every $y \in \Gamma(x)$ we get a pair $(\Gamma(x), (U_z)_{z \in \Gamma(x)})$. As for every $z \in \Gamma(x)$, U_z acts regularly on $\Gamma(x) \setminus \{z\}$, condition **MoS1** is satisfied for $(\Gamma_x, (U_z)_{z \in \Gamma(x)})$.
By assumption on Γ, $u_{\alpha_y} U_{\alpha_z} u_{\alpha_y}^{-1} = U_{u_{\alpha_y}(\alpha_z)}$. As $u_{\alpha_y}(\alpha_z)$ is a root such that x is contained in $\partial u_{\alpha_y}(z)$ and $u_{\alpha_y}(z) \in \Gamma(x) \cap Int(u_{\alpha_y}(\alpha_z))$, $u_{\alpha_y} U_{\alpha_z} u_{\alpha_y}^{-1} \in \{U_z | z \in \Gamma(x)\}$. This means condition **MoS2** is also satisfied and $(\Gamma(x), (U_z)_{z \in \Gamma(x)})$ is a Moufang set. We call it an *induced Moufang set* on $\Gamma(x)$ in Γ, and denote it by $\mathcal{M}_{\Gamma(x)}(\Gamma)$. Using the transitive action of the little projective group on points and lines (for a proof of this fact we refer to Theorem 64 of Chapter 2 taking into account that the group N acts transitively on the chambers of the standard apartment Σ_0) it is not hard to show that $\mathcal{M}_{\Gamma(x)}(\Gamma)$ depends up to isomorphism only on the type of x. In other words if x and x' are two

points in Γ, $\mathcal{M}_{\Gamma(x)}(\Gamma)$ and $\mathcal{M}_{\Gamma(x')}(\Gamma)$ will be isomorphic. Hence we can talk about the isomorphism class of induced Moufang sets on a line pencil or a point row of Γ. The isomorphism class of induced Moufang sets on a line pencil is denoted by $\mathcal{M}_l(\Gamma)$. Similarly $\mathcal{M}_p(\Gamma)$ stands for the isomorphism class of induced Moufang sets on a point row in Γ.

3.4 Desarguesian projective planes and Moufang sets

In this section we give a description of the induced Moufang sets on the point rows and line pencils of a Desarguesian projective plane. The terminology and notation will be used in Chapter 4 dealing with existence and non-existence of certain Moufang buildings.
Throughout this section Π denotes a Desarguesian projective plane. Using classical theory (cfr [13]) we know that there exists a division ring k and a 3-dimensional right k-vector space E such that $\Pi \cong PG(E)$. For the sequel we will identify in most cases Π and $PG(E)$. We state the following Lemma which will be used in Chapter 4.

Lemma 86 *If Π be a Desarguesian projective plane defined over a division ring k then $\mathcal{M}_p(\Pi) \cong \mathcal{P}(k)$ and $\mathcal{M}_l(\Pi) \cong \mathcal{P}(k^{opp})$.*

proof :

By assumption we have $\Pi = PG(E)$, where E is a 3-dimensional right k-vector space. Choose a base $B = \{e_1, e_2, e_2\}$ of E. Let E^* be the dual vector space of E and denote the dual base of B as $B^* = \{e_1^*, e_2^*, e_3^*\}$ with $e_i^*(e_j) = \delta_{ij}$. As usual we will consider E^* as a right k^{opp}-vector space.
Every point of Π corresponds then to a vector line of the form $\langle e_1x_1 + e_2x_2 + e_3x_3\rangle$ and every line of Π corresponds to a vector line in E^* of the form $\langle e_1^*y_1 + e_2^*y_2 + e_3^*y_3 \rangle$.
Denote by Σ_0 be the standard apartment $\{\langle e_1\rangle, \langle e_2\rangle, \langle e_3\rangle\}$.
As a generic point row to calculate $\mathcal{M}_p(\Pi)$ we choose $\langle e_1, e_2\rangle$. Write the point set of Γ $(\langle e_1, e_2\rangle)$ as $\{\langle e_1v + e_2\rangle | v \in k\}, \cup\{\langle e_1\rangle\}$. A typical root elation of the induced Moufang set $\mathcal{M}_{\Gamma(\langle e_1,e_2\rangle)}$ with fixed point $\langle e_1\rangle$, which sends $\langle e_2\rangle$ to $\langle e_1t + e_2\rangle$ has as matrix representation with respect to the base ordered

B :

$$\begin{pmatrix} 1 & t & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

whereas a root elation fixing $\langle e_2 \rangle$ and sending $\langle e_1 \rangle$ to $\langle e_1 t + e_2 \rangle$ has as matrix representation with respect to the base ordered base B :

$$\begin{pmatrix} 1 & 0 & 0 \\ t^{-1} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Consider the projective Moufang set $\mathcal{P}(k)$ coordinatized in a canonical way. Define the map β from $\mathcal{P}(k)$ to $\mathcal{M}_{\Gamma(\langle e_1, e_2 \rangle)}$ by :

$$\begin{aligned} \beta(v) &= \langle e_1 v + e_2 \rangle \\ \beta(\infty) &= \langle e_1 \rangle. \end{aligned}$$

We check that β defines a Moufang set isomorphism. Using Lemma 41 of Chapter 1 this will be done if we check that the maps $\beta_{(\infty)}$ from $U_{(\infty)}$ and $U_{\langle e_1 \rangle}$ and $\beta_{(0)}$ from $U_{(0)}$ to $U_{\langle e_2 \rangle}$ with :

$$\begin{aligned} \beta_{(\infty)}(u((\infty);(0),(t))) &= \beta \circ u((\infty);(0),(t)) \circ \beta^{-1} \\ \beta_{(0)}(u((0);(\infty),(t)) &= \beta \circ u((0);(\infty),(t)) \circ \beta^{-1} \end{aligned}$$

define bijections.
Let $(v) \in \mathcal{P}(k)$.
We have :

$$\begin{aligned} \beta u((\infty);(0),(t)\beta^{-1}\beta((v)) &= \beta((v+t)) \\ &= \langle e_1(v+t) + e_2 \rangle \\ &= u(\langle e_1 \rangle; \langle e_2 \rangle, \langle e_1 t + e_2 \rangle)(\langle e_1 v + e_2 \rangle). \end{aligned}$$

This shows $\beta_{(\infty)}(u((\infty);(0),(t)) = u(\langle e_1 \rangle; \langle e_2 \rangle, \langle e_1 t + e_2 \rangle)$. In a similar way one checks that $\beta_{(0)}$ defines a bijection from $U_{(0)}$ to $U_{\langle e_2 \rangle}$. Therefore β defines a Moufang set isomorphism from $\mathcal{P}(k)$ to $\mathcal{M}_{\Gamma(\langle e_1, e_2 \rangle)}$.
The Moufang set $\mathcal{M}_l(\Pi)$ can be calculated in a similar way as $\mathcal{M}_p(\Pi)$ using the dual projective plane $PG(E^*)$. For sake of completeness and for application in Chapter 4 we give the explicit calculations. As generic line pencil to calculate $\mathcal{M}_l(\Pi)$ we choose $\Gamma(\langle e_2 \rangle)$. Using the dual base B^* the elements of

$\Gamma(\langle e_2\rangle)$ can be written as $\{\langle e_1^* v^* + e_3^*\rangle \, | v^* \in k^{opp}\} \cup \{\langle e_1^*\rangle\}$. Let $t^* \in k^{opp}$. A typical root elation of $\mathcal{M}_{\Gamma(\langle e_2\rangle)}$ (Π) fixing $\langle e_1^*\rangle$ and sending $\langle e_3^*\rangle$ to $\langle e_1^* t^* + e_3^*\rangle$ has as matrix representation with respect to the ordered base $\{ e_1^*, e_2^*, e_3^*\}$:

$$\begin{pmatrix} 1 & 0 & t^* \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Similarly the root elation fixing $\langle e_3^*\rangle$ sending $\langle e_1^*\rangle$ to $\langle e_1^* t^* + e_3^*\rangle$ has as matrix representation with respect to the base $\{ e_1^*, e_2^*, e_3^*\}$:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ t^{*-1} & 0 & 1 \end{pmatrix}.$$

Let $\mathcal{P}(k^{opp})$ be the projective Moufang set defined over k^{opp} coordinatized in a canonical way. Define the map β^* from $\mathcal{P}(k^{opp})$ to $\mathcal{M}_{\Gamma(\langle e_2\rangle)}$ by :

$$\begin{aligned} \beta^*(v^*) &= \langle e_1^* v^* + e_3^*\rangle \\ \beta(\infty) &= \langle e_1^*\rangle. \end{aligned}$$

Using similar arguments as for $\mathcal{M}_{\Gamma(\langle e_1, e_2\rangle)}$ one easily checks that β^* defines a Moufang set isomorphism. This completes the proof. □

3.5 Classical generalized quadrangles

In this section we introduce two classes of generalized quadrangles which will be called *classical*. In order to make calculations on the quadrangles we will use a coordinatization similar to the one introduced in [37]. For more information on the quadrangles and coordinate systems we refer the reader to chapter 2 and 3 of loc. cit.

3.5.1 Symplectic quadrangles

Let k be a field, E a k-vector space and f a non-degenerate symplectic bilinear form on E i.e. f is a function from $E \times E$ to k satisfying :

$$f(x\lambda + y\mu, z) = \lambda f(x, z) + \mu f(y, z)$$

$$\begin{aligned} f(x,x) &= 0 \\ Rad(f) &= 0 \end{aligned}$$

where $x, y, z \in E$, $\lambda, \mu, \nu \in k$ arbitrary and $Rad(f) = \{v \in E \mid f(v,u) = 0, \forall u \in E\}$. Remark that the equality $f(x+y, x+y) = 0$ implies $f(x,y) = -f(y,x)$, allowing us to define orthogonality (denoted by $\perp$) according to the formula :

$$x \perp y \Leftrightarrow f(x,y) = 0.$$

Orthogonality is clearly a symmetric binary relation on $E \times E$.
For a subspace X of E one defines :

$$X^{\perp} = \{y \in E \mid f(x,y) = 0, \forall x \in X\}.$$

A subspace X is called *isotropic* if $X \cap X^{\perp} \neq \{0\}$ and *totally isotropic* if $X \subset X^{\perp}$. Amongst the totally isotropic subspaces there are maximal ones all having the same dimension. This number is called the Witt index of f and denoted by $\nu(f)$. If the form f is non-degenerate and $dim(E) < \infty$ then necessarily $dim(E) = 2m$ with $\nu(f) = m$. Under these conditions one can choose a base $\{e_i\}_{1 \leq i \leq 2m}$ satisfying $f(e_i, e_{i+m}) = 1$ and $f(e_i, e_j) = 0$ if $j \neq i + m$. Such a base is also called a *symplectic base.* (For exact proofs of these facts we refer to the classical theory for example [6], [7] or Chapter 8 in [29].)
To construct a generalized quadrangle we start with a 4 dimensional right k-vector space E and a non-degenerate symplectic form f of Witt index 2. With respect to a symplectic base $\{e_i\}_{1 \leq i \leq 4}$, f is represented by the form :

$$x_1 y_3 - x_3 y_1 + x_2 y_4 - x_4 y_2.$$

Call points $\mathcal{P}$ all totally isotropic 1 spaces i.e. all projective points of $PG(E)$. Lines $\mathcal{L}$ are all totally isotropic 2-spaces in E i.e. projective lines of $PG(E)$ on which f vanishes. An easy calculation shows that $\langle x \rangle$ is collinear with $\langle y \rangle$ if and only if $f(x,y) = 0$. We leave it to the reader to check that the rank 2 geometry $(\mathcal{P}, \mathcal{L}, I)$ with I the natural incidence defines a generalized quadrangle. This quadrangle is denoted by $W(k)$ and is called a *symplectic quadrangle.*

3.5.2 Coordinatization of $W(k)$

Choose $\{e_i\}_{1\leq i\leq 4}$ such that the ordered set $\{e_1, e_3, e_2, e_4\}$ is a symplectic base i.e. :

$$f(e_1, e_2) = 1 \qquad f(e_1, e_j) = 0,\ j \neq 2$$
$$f(e_3, e_4) = 1 \qquad f(e_3, e_j) = 0,\ j \neq 4.$$

A straightforward check shows that the following table written down with respect to this base exhausts all points and lines of $W(k)$. Hence it provides a coordinate system for the generalized quadrangle. Round brackets denote points and square brackets indicate lines.

Points	
Coordinates in $W(k)$	elements in $PG_3(k)$
(∞)	$(1, 0, 0, 0)$
(x)	$(x, 0, 1, 0)$
(v, y)	$(-y, 0, v, 1)$
(x, w, x')	$(w - xx', 1, -x', -x)$
Lines	
Coordinates in $W(k)$	elements in $PG_3(k)$
$[\infty]$	$\langle(1, 0, 0, 0), (0, 0, 1, 0)\rangle$
$[v]$	$\langle(1, 0, 0, 0), (0, 0, v, 1)\rangle$
$[x, w]$	$\langle(x, 0, 1, 0), (w, 1, 0, -x)\rangle$
$[v, y, v']$	$\langle(-v, 0, y, 1), (v', 1, -y, 0)\rangle$

3.5.3 Generalized quadrangles defined by (σ, ϵ)-quadratic forms

In this section we will use notations and definitions concerning division rings with involutions as introduced in Chapter 1. For more information we refer consequently to this chapter. The following definitions and notations are mainly based on Chapter 8 in [29], section 2.3. in [37] and Chapter 10 in [4]. To construct quadrangles we first discuss some definitions and basic properties. Throughout this paragraph we will use the definitions and notations introduced in section 1.3 of Chapter1.

Definition 87 Let k be a division ring with involution σ and E a right k-vectorspace. A function f going from $E \times E$ to k is a *σ-sequilinear form* (or shortly *sesquilinear* form if σ is clear from the context) if it is biadditive and :

$$f(x\lambda, y\mu) = \lambda^\sigma f(x,y)\mu,\ \forall x, y \in E, \forall \lambda, \mu \in k.$$

Definition 88 A σ-sesquilinear form f is called *reflexive* if there exists a constant $\epsilon \in k$ such that :

$$f(x,y) = f(y,x)^\sigma \epsilon,\ \forall x, y \in E$$

A form satisfying this equation is also called (σ, ϵ)-*hermitian* form. In particular a $(\sigma, 1)$-hermitian form is indicated as hermitian and a $(\sigma, -1)$-form as anti-hermitian.

Definition 89 A reflexive (σ, ϵ)-hermitian form f is called *trace valued* if there exists a σ-sesquilinear form g such that :

$$f(x,y) = g(x,y) + (g(y,x))^\sigma \epsilon,\ \forall x, y \in E.$$

Definition 90 A function q from E to $k^{(\sigma,\epsilon)}$ is called (σ, ϵ)-*quadratic* if the following conditions hold :
(i) $q(x\lambda) = \lambda q(x) \lambda^\sigma,\ \forall \lambda \in k, x \in E$.
(ii) There exists a trace valued (σ, ϵ)-hermitian form f on $E \times E$ such that :

$$q(x+y) = q(x) + q(y) + f(x,y),\ \forall x, y \in E.$$

or equivalently to (i) and (ii)
$(i)'$ There exists a σ-sesquilinear form g with :

$$q(x) = g(x,x) + k_{(\sigma,\epsilon)},\ \forall x \in E.$$

For the proof of the equivalence of $(i), (ii)$ and $(i)'$ we refer to 8.2.1. on pp121-122 in [29]. To simplify notation in most of the cases the coset $k_{(\sigma,\epsilon)}$ will be omitted. This means that we write for example $q(x) = g(x,x)$ instead of $q(x) = g(x,x) + k_{(\sigma,\epsilon)}$. In the sequel we will call in a lot of cases a (σ, ϵ)-quadratic form a *pseudo-quadratic form* if σ and ϵ are of no importance and a $(\sigma, 1)$-quadratic form simply a σ-*quadratic form*.
Given any (σ, ϵ)-sesquilinear f on a space E we can introduce orthogonality on $E \times E$, (denoted with the symbol $\perp$) defined by :

$$x \perp y \Leftrightarrow f(x,y) = 0.$$

The conditions on f ensure that $\perp$ is well defined i.e. $\perp$ is a symmetric binary relation on E.
Given any subspace $X \subseteq E$ we set :

$$X^{\perp} = \{x \in E \mid f(x,y) = 0, \forall y \in X\}.$$

In particular $E^{\perp}$ is denoted by $Rad(f)$, and the form f is called *non-degenerate* if $Rad(f) = 0$. A subspace $X \subseteq E$ is called *isotropic* of $X \cap X^{\perp} \neq \{0\}$, *non-isotropic* if $X \cap X^{\perp} = \{0\}$ and *totally isotropic* if $X \subseteq X^{\perp}$. Using Zorns Lemma it can be shown that amongst the totally isotropic subspaces there are maximal ones sharing the same dimension called the *Witt index* of f, denoted by $\nu(f)$.
Due to the properties of q, one easily checks $q^{-1}(0)$ is a union of 1-dimensional subspaces of E. A subspace X of E is called *totally singular* if $X \subset q^{-1}(0)$. Similarly as above one can show that amongst the totally singular subspaces there are maximal one's all having the same dimension called the *Witt index of* q, denoted by $\nu(q)$. If $q^{-1}(0) = 0$, q is called *anisotropic*. A straightforward calculation (see 8.2.3. on p123 in [29]) shows that any (σ, ϵ)-quadratic form q, determines uniquely the (σ, ϵ)-hermitian form f as above. The set $q^{-1}(0) \cap Rad(f)$ is thus a well defined subspace of E. If $q^{-1}(0) \cap Rad(f) = \{0\}$, q will be called *non-degenerate*.
In order to construct a generalized quadrangle we start with a division ring k endowed with an involution σ, E a right k-vector space and q a (σ, ϵ)-quadratic form of Witt index 2. Define the following incidence structure $(\mathcal{P}, \mathcal{L}, I)$, where $\mathcal{P}$ is the point set, $\mathcal{L}$ the line set and I the incidence relation. Points are all totally singular vector lines in E, while lines are the totally singular planes in E. When working in the projective space $PG(E)$ associated to E this means that points and lines correspond to projective points and projective lines on which q vanishes. Incidence is the one induced by $PG(E)$. A straightforward check shows that if $s = \langle x \rangle$ and $t = \langle y \rangle$ in P, s and t are collinear if and only if $f(x,y) = 0$. We leave it as an exercise to the reader to check that $(\mathcal{P}, \mathcal{L}, I)$ is a generalized quadrangle. In the following we will denote it by $Q(E, q, k, \sigma)$. To end this section we mention the following useful observations as explained in 8.2.1 and 8.2.2 in [29].
Suppose q is a (σ, ϵ)-quadratic form on a vector space E, and let $c \in k^*$. Then the form cq defined by $cq(v) = c.q(v)$ defines a (σ', ϵ') quadratic form where $t^{\sigma'} = ct^{\sigma}c^{-1}$ and $\epsilon' = c(c^{\sigma})^{-1}\epsilon$. Under these conditions the forms q and cq are said to be *proportional* to one another.
Important for further calculations is the following lemma (Lemma in section

8.2.2 in [29]).
Let $Q(E, q, k, \sigma)$ be generalized quadrangle defined by the (σ, ϵ)-quadratic form q and $c \in k$. Then clearly cq defines a non-degenerate (σ', ϵ')-quadratic form on E of Witt index 2 with $t^{\sigma'} = ct^{\sigma}\ c^{-1}$ and $\epsilon' = c(c^{\sigma})^{-1}\epsilon$. Hence we can consider the quadrangle $Q(E, cq, k, \sigma')$. We have the following lemma concerning $Q(E, cq, k, \sigma')$

Lemma 91 *Let $Q(E, q, k, \sigma)$ be a generalized quadrangle defined by a (σ, ϵ)-quadratic form q. Then for $c \in k$ the quadrangle $Q(E, q, k, \sigma)$ is isomorphic to $Q(E, cq, k, \sigma')$ with $t^{\sigma'} = ct^{\sigma}c^{-1}$, $\forall\ t \in k$.*

proof :

Let $Q(E, q, k, \sigma)$ and $Q(E, cq, k, \sigma')$ be as in the Lemma. Define the bijection β from $Q(E, q, k, \sigma)$ to $Q(E, cq, k, \sigma')$ as :

$$\beta(\langle c \rangle) = \langle c \rangle.$$

One easily checks then that β defines an isomorphism from $Q(E, q, k, \sigma)$ to $Q(E, cq, k, \sigma')$. □

Lemma 92 *Every pseudo-quadratic form is proportional to a σ'-quadratic form, for suitable σ'. Every pseudo-quadratic form which is not quadratic is proportional to a $(\sigma, -1)$-quadratic form, where σ can be chosen in such a way that $1 \in k_{\sigma,-1} = Tr(\sigma)$.*

proof :

See 8.2.2. on p123 in [29]. □

3.5.4 Coordinatization of $Q(E, q, k, \sigma)$

In this section we will introduce a coordinate system for quadrangles of the form $Q(E, q, k, \sigma)$ based on the coordinatization described in Chapter 3 in [37]. The following proposition, which is analogous to the Proposition 2.3.4 in [37], is of crucial importance.

Proposition 93 *Let q be a non-degenerate (σ, ϵ)-quadratic form of Witt index 2 on E a right k-vector space. Then there exist four vectors e_i, $i \in \{-2, -1, 1, 2\}$, a direct sum decomposition :*

$$E = e_{-2}k \oplus e_{-1}k \oplus E_0 \oplus e_1k \oplus e_2k$$

with $f(e_{-2}, e_2) = \epsilon$, $f(e_{-1}, e_1) = 1$, $f(e_i, e_j) = 0$, if $i + j \neq 0$ and a non-degenerate anisotropic form q_0 on E_0 such that for $v = e_{-2}x_{-2} + e_{-1}x_{-1} + e_0 + e_1x_1 + e_2x_2$ with $x_i \in k$ and $e_0 \in E_0$:

$$q(v) = x_{-2}^{\sigma}\epsilon x_2 + x_{-1}^{\sigma}x_1 + q_0(e_0).$$

proof :

The proof is similar to the proof of Proposition 2.3.4. in [37]. □

As to the coordinates we explain how to handle points. For the lines analogous calculations hold.
Choose a base $\{e_{-2}, e_{-1}, e_1, e_2\}$ as in Proposition 93.
Consider a arbitrary point $\langle x\rangle$ of $Q(E, q, k, \sigma)$. Then $x = e_{-2}x_{-2} + e_{-1}x_{-1} + v_0 + e_1x_1 + e_2x_2$, $v_0 \in E_0$ and $x_i \in k$.
Two cases occur :

First case : $x_2 \neq 0$.
After a possible multiplication we can assume $x_2 = 1$. Expressing $q(x) = 0$ gives

$$x_{-2}^{\sigma}\epsilon + q_0(v_0) + x_{-1}^{\sigma}x_1 = 0.$$

Thus $x_{-2} = v_1 - x_{-1}^{\sigma}x_1$, with $v_1 \in k$ such that $q(v_0) + v_1 = 0$. This point is coordinatized as $(-x_{-1}^{\sigma}, (v_0, v_1), x_1^{\sigma})$.

Second case : $x_2 = 0$.
If $x_{-1} \neq 0$, we can assume without loss of generality $x_{-1} = 1$. Expressing that $\langle x\rangle$ belongs to $Q(E, q, k, \sigma)$ gives $x_1 + q_0(v_0) = 0$. This point $\langle x\rangle$ is coordinatized as $((v_0, x_1), -x_{-2})$.
If $x_{-1} = 0$, the conditions on q_0 imply that v_0 has to be 0.
If in this case $x_1 = 0$ then $\langle x\rangle$ is the point $\langle e_{-2}\rangle$. This point is labelled by (∞).
On the other hand if $x_1 \neq 0$ after an eventual multiplication $x_1 = 1$. In

coordinates $\langle x\rangle$ is denoted by (x_{-2}). To recapitulate we have the following table :

Points	
Coordinates in $Q(E,q,k,\sigma)$	Points in $PG(E)$
(∞)	$(1,0,0,0,0)$
(x)	$(x,0,0,1,0)$
$((v_0,v_1),y)$	$(-y,1,v_0,v_1,0)$
$(x,(w_0,w_1),x')$	$(w_1+xx'^{\sigma},-x^{\sigma},w_0,x'^{\sigma},1)$

A similar reasoning for lines leads to the following table :

Lines	
Coordinates in $Q(E,q,k,\sigma)$	Lines in $PG(E)$
$[\infty]$	$\langle(1,0,0,0,0),(0,0,0,1,0)\rangle$
$[(v_0,v_1)]$	$\langle(1,0,0,0,0),(0,1,v_0,v_1,0)\rangle$
$[x,(w_0,w_1)]$	$\langle(x,0,0,1,0),(w_1,-x^{\sigma},w_0,0,1)\rangle$
$[(v_0,v_1),y,(v'_0,v'_1)]$	$\langle(-y,1,v_0,v_1,0),(v'_1,0,v'_0,y^{\sigma}\epsilon-f(v_0,v'_0),1)\rangle$

There are two labelling sets used for the coordinatization. One is the field k and the other one is the $R_{0,1}=\{(v_0,v_1)\in E_0\times k\,|\,q_0(v_0)+v_1=0\}$. Denote $R_1=\{t\in k|\,\exists v_0\in V_0\,|\,(v_0,t)\in R_{0,1}\}$ and $R_0=E_0$. As $R_{0,1}\subset R_0\times R_1$ we define projections (denoted by subscripts 0 and 1) by :

$$\begin{aligned}(v_0,v_1)_0 &= v_0\in R_0,\\ (v_0,v_1)_1 &= v_1\in R_1.\end{aligned}$$

Given a labelling set of the form $R_{0,1}$ we define the following operation $\oplus$ by :

$$(v_0,v_1)\oplus(w_0,w_1)=(v_0+w_0,v_1+w_1-f(v_0,w_0))$$

for (v_0,v_1), $(w_0,w_1)\in R_{0,1}$. One easily checks that $(v_0,v_1)\oplus(w_0,w_1)\in R_{0,1}$. As :

$$\begin{aligned}(v_0,v_1)\oplus(0,0) &= (v_0,v_1)\\ (v_0,v_1)\oplus(-v_0,-v_1-f(v_0,v_0)) &= (0,0)\\ (u_0,u_1)\oplus((v_0,v_1)\oplus(w_0,w_1)) &= ((u_0,u_1)\oplus(v_0,v_1))\oplus(w_0,w_1)\end{aligned}$$

$\forall$ (u_0,u_1), (v_0,v_1), $(w_0,w_1)\in R_{0,1}$ we see that $\oplus$ defines a group structure on $R_{0,1}$ which we will denote by $(R_{0,1},\oplus)$.

Definition 94 A Moufang set $(X, (U_x)_{x\in X})$ which is isomorphic to an induced Moufang set $\mathcal{M}_p(Q)$ or $\mathcal{M}_l(Q)$, where Q is a classical generalized quadrangle will be called a *classical Moufang set.*

3.6 Quadrangles of indifferent type

Let k be a field of characteristic 2 with subfield k' containing k^2, $l \subset k$ a vector space over k' and l' a vector space over k^2 i.e. :

$$k^2 \subset l' \subset k' \subset l \subset k.$$

Suppose l and l' meet the following conditions :
(1) $l^{-1} = l$, $l'^{-1} = l'$, $1 \in l \cap l'$
(2) l generates k as a ring, l' generates k' as a ring
Consider the geometry obtained by choosing all point rows and line pencils of $W(k)$ coordinatized over l, l' respectively i.e. we restrict in the coordinatization table of $W(k)$ as given in section 3.5.2, x, x', y to l and v, v', w to l'. Incidence is the one induced by $W(k)$. Denote this incidence structure by $Q(k, k'; l, l')$. A straightforward check shows it is a generalized quadrangle, called a *quadrangle of indifferent type.*

3.7 Coordinatization of $Q(k, k'; l, l')$

This is the coordinatization inherited from $W(k)$.

3.8 Moufang structures

As for the coordinatization we follow for to the description of the Moufang structure of the quadrangles under consideration the approach described in chapters 4 and 5 in [37].

3.8.1 Moufang structure of $W(k)$

Consider $W(k)$ with its coordinatization as described in section 3.5.2. Let Σ_0 be the standard apartment $\{(\infty), [\infty], (0), [0,(0,0)], (0,(0,0),0), [(0,0)$, 0 , $(0,0)]$, $((0,0),0)$,$[(0,0)]\}$. As already mentioned there are two isomorphism classes of Moufang sets associated to $W(k)$, namely $\mathcal{M}_p(W(k))$ and

$\mathcal{M}_l(W(k))$.
We start with the description of a representative of $\mathcal{M}_p(W(k))$.
Consider as point row $\Gamma([0]) = \{(0,x)|x \in k\} \cup \{(\infty)\}$. To describe the root group $U_{(\infty)}$ we choose the root $\alpha_{(\infty)} = \{[0,0], (0), [\infty], (\infty), [0]\}$. As explained in section 3.3 the root elations with respect to $\alpha_{(\infty)}$ induce the root group $U_{(\infty)}$. The action of a typical element $u((\infty);(0,0),(0,t)) \in U_{(\infty)}$ is given by:

Elements of $W(k)$	Image under $u((\infty);(0,0),(0,t))$
(∞)	(∞)
(x)	(x)
(v,y)	$(v,y+t)$
(x,w,x')	$(x,w+2tx,x'+t)$
$[\infty]$	$[\infty]$
$[v]$	$[v]$
$[x,w]$	$[x,w+xt]$
$[v,y,v']$	$[v,y+t,v']$

The formula $u((\infty);(0,0),(0,t_1))\, u((\infty);(0,0),(0,t_2)) = u((\infty);(0,0),(0,t_1+t_2))$ implies that all root groups are isomorphic to the additive group on k. Other root groups are calculated by conjugating the group $U_{(\infty)}$ with appropriate elements of the little projective group. This representation of $(X = \{(x)|x \in k\} \cup \{(\infty)\}, (U_z)_{z\in X})$ shows that it is isomorphic to $\mathcal{P}(k)$. Namely consider $\mathcal{P}(k)$ defined in a 2-dimensional k-vector space V. Choose a coordinate system of $\mathcal{P}(k)$. Then a concrete Moufang set isomorphism from $\mathcal{P}(k)$ to $\mathcal{M}_{\Gamma((0,0))}(W(k))$ is given by β with :

$$\begin{aligned}\beta(\infty) &= (\infty)\\ \beta(x) &= (0,x),\ \forall x \in k\end{aligned}$$

Leaves us with the description of the induced Moufang set on a line pencil. Consider as pencil $\Gamma((0)) = \{[0,t]|t \in k\} \cup \{[\infty]\}$. In order to calculate $U_{[\infty]}$ we consider the root $\alpha_{[\infty]} = \{(0),[\infty],(\infty),[0],(0,0)\}$ in Σ_0. A typical element of $U_{[\infty]}$, lets say $u([\infty];[0,0],[0,t])$, acts in the following way :

Elements of $W(k)$	Image under $u([\infty];[0,0],[0,t])$
(∞)	(∞)
(x)	(x)
(v,y)	(v,y)
(x,w,x')	$(x,w+t,x')$
$[\infty]$	$[\infty]$
$[v]$	$[v]$
$[x,w]$	$[x,w+t]$
$[v,y,v']$	$[v,y,v'+t]$

As before $u([\infty];[0,0],[0,t])\,u([\infty];[0,0],[0,s]) = u([\infty];[0,0],[0,s+t])$, $\forall s,t \in k$, yielding that $U_{[\infty]}$ is isomorphic to the additive group on k. Other root groups are computed using the little projective group of $W(k)$. We thus obtain a Moufang set $(X = \{[0,t] | t \in k\} \cup \{(\infty)\}, (U_z)_{z\in X})$. As in the case of $\mathcal{M}_{\Gamma([0])}(W(k))$ one easily shows that $\mathcal{M}_{\Gamma((0))}(W(k)) \cong \mathcal{P}(k)$.

3.8.2 Moufang structure of $Q(E,q,k,\sigma)$.

Similar as for the symplectic quadrangle $W(k)$ we calculate two classes of induced Moufang sets, namely $\mathcal{M}_p(Q(E,q,k,\sigma))$ and $\mathcal{M}_l(Q(E,q,k,\sigma))$. Choose a fixed coordinate system for $Q(E,q,k,\sigma))$ associated to the decomposition $E = e_{-2}k \oplus e_{-1}k \oplus E_0 \oplus e_1k \oplus e_2k$ and suppose B_0 is an ordered base of E_0. Consider the standard apartment $\Sigma_0 = \{(\infty), [\infty], (0), [0,(0,0)], (0,(0,0),0), [(0,0),0,(0,0)]\ ,([0,0],0)\ ,\ [(0,0)]\}$. As generic point row we choose $\Gamma([(0,0)]) = \{((0,0),x)\ |x \in k\} \cup \{(\infty)\}$. In order to calculate the root group $U_{(\infty)}$, we use the root $\alpha_{(\infty)} = \{[0,(0,0)], (0), [\infty], (\infty), [(0,0)]\}$. The action of a typical element $u((\infty);((0,0),0),((0,0),t)) \in U_{\alpha_{(\infty)}}$ is given by :

Elements of $Q(E,q,k,\sigma)$	Image under $u((\infty);((0,0),0),((0,0),t))$
(∞)	(∞)
(x)	(x)
$((v_0,v_1),y)$	$((v_0,v_1),y+t)$
$(x,(w_0,w_1),x')$	$(x,(w_0,w_1)\oplus(0,tx^\sigma - xt^\sigma),x')$
$[\infty]$	$[\infty]$
$[(v_0,v_1)]$	$[(v_0,v_1)]$
$[x,(w_0,w_1)]$	$[x,(w_0,w_1)\oplus(0,tx^\sigma - xt^\sigma)]$
$[(v_0,v_1),y,(v_0',v_1')$	$[(v_0,v_1),y+t,(v_0',v_1')]$

This means that $u((\infty);((0,0),0),((0,0),t))$ has matrix representation with respect to the ordered base $B = \{e_{-1},\, B_0,\, e_1\}$:

$$\begin{pmatrix} 1 & -t & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & I_{|B_0|} & 0 & 0 \\ 0 & 0 & 0 & 1 & t^\sigma \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

The group $U_{\alpha_{(\infty)}}$ induces by construction the root group $U_{(\infty)}$ of $\mathcal{M}_{\Gamma([(0,0)])}(Q(E,q,k,\sigma)$ acting on $\Gamma([(0,0)])$. Other root groups can be found by the conjugating $U_{\alpha_{(\infty)}}$ with appropriate elements of the little projective group and restricting the action to $\Gamma([(0,0)])$.
This defines the Moufang set $\mathcal{M}_{\Gamma([(0,0)])}(Q(E,q,k,\sigma)) = (\{((0,0),x)|x \in k\} \cup\{(\infty)\},\, (U_z)_{z\in X})$. As for the symplectic quadrangle one easily shows that $\mathcal{M}_{\Gamma([(0,0)])} \cong \mathcal{P}(k)$.
Remains to describe $\mathcal{M}_l(Q(E,q,\sigma,K)$.
As line pencil we choose $\Gamma((0)) = \{[0,(x_0,x_1)]\ |(x_0,x_1) \in R_{0,1}\} \cup [\infty]$. To calculate the root group $U_{[\infty]}$ we use the root $\alpha_{[\infty]} = \{(0),\, [\infty],\, (\infty),\, [(0,0)],\, ((0,0),0)\}$. The action of a typical element of $U_{\alpha_{[\infty]}}$ say $u([\infty];\ [0,(0,0)],\ [0,(t_0,t_1)])$ is given by :

Elements of $Q(E,q,k,\sigma)$	Image under $u([\infty];[0,(0,0)],[0,(t_0,t_1)])$
(∞)	(∞)
(x)	(x)
$((v_0,v_1),y)$	$((v_0,v_1,y-f(t_0,v_0)))$
$(x,(w_0,w_1),x')$	$(x,(t_0,t_1)\oplus(w_0,w_1),x')$
$[\infty]$	$[\infty]$
$[(v_0,v_1)]$	$[(v_0,v_1)]$
$[x,(w_0,w_1)]$	$[x,(t_0,t_1)\oplus(w_0,w_1)]$
$[(v_0,v_1),y,(v_0',v_1')]$	$[(v_0,v_1),y-f(t_0,v_0),(t_0,t_1\oplus(v_0',v_1')]$

Thus $u([\infty];[0,(0,0)],[0,(t_0,t_1)])$ has matrix representation with respect to the ordered base B :

$$\begin{pmatrix} 1 & 0 & f(t_0,B_0) & 0 & t_1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & I_{|B|_0|} & 0 & t_0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

By construction $U_{\alpha_{[\infty]}}$ induces the root group $U_{[\infty]}$ acting on $\Gamma((0))$. As usual one calculates other root groups after conjugating $U_{\alpha_{[\infty]}}$ with appropriate elements of the little projective group.
An easy calculation shows that :

$$\begin{aligned} &u([\infty];[0,(0,0)],[0,(x_0,x_1)])u([\infty];[0,(0,0)],[0,(y_0,y_1)]) \\ &= u([\infty];[0,(0,0)],[0,(x_0,x_1)\oplus(y_0,y_1)]). \end{aligned}$$

One easily deduces from this equation that $U_{[\infty]}$ and hence all root groups of $\mathcal{M}_{\Gamma((0))}(Q(E,q,k,\sigma))$ are isomorphic to $(R_{0,1},\oplus)$.

3.8.3 Moufang structure of $Q(k,k';l,l')$

Root groups are induced by the root groups of $W(k)$. As an example we look at the action of $u([\infty];[0,0],[0,t])$ on $\Gamma((0))$, with $t\in l'$.

Elements of $Q(k,k';l,l')$	Image under $u([\infty];[0,0],[0,t])$
(∞)	(∞)
(x)	(x)
(v,y)	(v,y)
(x,w,x')	$(x,w+t,x')$
$[\infty]$	$[\infty]$
$[v]$	$[v]$
$[x,w]$	$[x,w+t]$
$[v,y,v']$	$[v,y,v'+t]$

Remark that similar calculations as for $W(k)$ yield that $\mathcal{M}_p(Q(k,k';l,l')\cong\mathcal{P}(l;k)$ and $\mathcal{M}_l(Q(k,k';l,l')\cong\mathcal{P}(l';k')$. In the next two sections we will give alternative descriptions for the induced Moufang sets so far considered. This will simplify and clarify in a lot of cases notations and calculations. In the sequel we will work in almost all cases with these alternative descriptions.

3.9 Different types of Moufang quadrangles

In this section we make divide the quadrangles we so far considered in different classes. In the list which we present there will of course be some overlaps.

Symplectic quandrangles These are quadrangles of the form $W(k)$.

Orthogonal quadrangles By this we mean quadrangles of the form $Q(E,q,k,\sigma)$ for which $\sigma = 1$. In the sequel we will denote them also by $QO(E,q,k)$.

Hermitian quadrangles By this we mean quadrangles of the form $Q(E,q,k,\sigma)$ for which $Z(k) = k$ and $\sigma \neq 1$. In the sequel we will denote them also by $QH(E,q,k,\sigma)$.

Unitary quadrangles These are quadrangles of the form $Q(E,q,k,\sigma)$ for which $Z(k) \neq k$. In the sequel we will also denote them by $QU(E,q,k,\sigma)$.

Indifferent quadrangles Quadrangles of the form $Q(k,k';l,l')$ as described in section 131.

3.10 An alternative description of induced Moufang sets of $W(k)$ and $\mathcal{M}_p(Q(E,q,k,\sigma))$

In sections 3.8.1 and 3.8.2 we saw that $\mathcal{M}_p(W(k)) \cong \mathcal{M}_l(W(k)) \cong \mathcal{P}(k)$ and also $\mathcal{M}(Q(E,q,k,\sigma)) \cong \mathcal{P}(k)$. Hence the alternative description of these Moufang sets is provided by the description of $\mathcal{P}(k)$ given in section 3.2.

3.11 An alternative description of induced Moufang sets of $Q(k,k';l,l')$

In section 3.8.3 we already mentioned that $\mathcal{M}_p(Q(k,k';l,l')) \cong \mathcal{P}(l;k)$ and $\mathcal{M}_l(Q(k,k';l,l')) \cong \mathcal{P}(l';k')$. Hence the alternative description we will use for these Moufang sets is the one induced by the descriptions of $\mathcal{P}(l;k)$ and $\mathcal{P}(l';k')$ as used in section 3.2.

3.12 An alternative description of $\mathcal{M}_l(Q(E,q,k,\sigma))$

We start by giving a general construction of a family of Moufang sets $\{(X, (U_x)_{x\in X}\}$. Subsequently we show that every $\mathcal{M}_l(Q(E,q,k,\sigma))$ corresponds to such a Moufang set and conversely that every element of this family belongs to the class $\mathcal{M}_l(Q(E,q,k,\sigma))$ for some generalized quadrangle $Q(E,q,k,\sigma)$.

3.12.1 General setup and coordinatization.

Let k be a division ring with involution σ and V a right k-vector space. Suppose q is a non-degenerate (σ,ϵ)-quadratic form of Witt index 1. Denote the set of all totally singular vector lines in V by X. Inspired by the coordinatization of generalized quadrangles we introduce the following coordinate system.
Using techniques similar as those for proving Proposition 93 it is not hard to check that V can be decomposed as :

$$V = e_{-1}k \oplus V_0 \oplus e_1 k,$$

such that $q(e_i) = 0$, $i = 1, -1$, $e_{-1}^{\perp} \cap e_1^{\perp} = V_0$, $f(e_1, e_{-1}) = 1$ and $q|_{V_0}$ is anisotropic. Fix such a decomposition and denote $R_{0,1} = \{(v_0, v_1) \in V_0 \times k | q(v_0) + v_1 = 0\}$, $R_0 = V_0$, $R_1 = \{t \in k | \exists v_0 \in R_0| \ (v_0, t) \in R_{0,1}\}$. One checks that $X = \{\langle e_{-1}\rangle\} \cup \{\langle e_{-1}v_1 + v_0 + e_1\rangle \ |(v_0, v_1) \in R_{0,1} \ \}$. In the sequel we label $\langle e_{-1}v_1 + v_0 + e_1\rangle$ as (v_0, v_1) and $\langle e_{-1}\rangle$ as (∞). In this way we obtain a coordinatization. Remark that this coordinatization depends as in the quadrangle case on the initial decomposition of V. Therefore a label of the form (v_0, v_1) will only have meaning if this decomposition is known.

3.12.2 Description of root groups and switching of coordinates.

Consider the set X as in the foregoing section. We define a root group structure on X. We start by giving a general procedure to calculate root group elements and give concrete descriptions of $U_{(\infty)}$ and $U_{(0,0)}$.
Let x, y and $z \in X$. In order to calculate $u(x; y, z)$ we consider a decomposition $V = \bar{e}_{-1}k \oplus \bar{V}_0 \oplus \bar{e}_1 k$ such that $\langle \bar{e}_{-1}\rangle = x$ and $\langle \bar{e}_1\rangle = y$. Suppose that with respect to the coordinate system associated with this decomposition

$z = (t_0, t_1)$. Choose an ordered base $\bar{B}_0$ of $\bar{V}_0$. Then we define $u(x; y, z)$ as the linear transformation on V with matrix representation :
(with respect to the ordered base $\bar{B} = \{\bar{e}_{-1}, \bar{B}_0, \bar{e}_1\}$)

$$\begin{pmatrix} 1 & f(t_0, \bar{B}_0) & t_1 \\ 0 & I_{|\bar{B}_0|} & t_0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{3.1}$$

Choose a fixed decomposition of $V = e_{-1}k \oplus V_0 \oplus e_1 k$. Denote the coordinate system associated to this decomposition by superscript 1 i.e. $(v_0, v_1)^1 = \langle e_{-1}v_1 + v_0 + e_1\rangle$ and $(\infty)^1 = \langle e_{-1}\rangle$. Let B_0 be an ordered base of V_0.
Using the recipe described above we calculate the actions of $U_{(\infty)}$ and $U_{(0,0)}$. By formula (3.1) a typical element $u((\infty); (0,0), (t_0, t_1))$ has matrix representation with respect to the ordered base $\{e_{-1}, B_0, e_1\}$:

$$\begin{pmatrix} 1 & f(t_0, B_0) & t_1 \\ 0 & I_{|B_0|} & t_0 \\ 0 & 0 & 1 \end{pmatrix}.$$

In order to calculate a matrix representation of an typical element $u((0,0);(\infty), (v_0, v_1))$, $(v_0 \neq 0)$, of $U_{(0,0)}$ we decompose V as $V = (e_1\epsilon^{-1})k \oplus V_0 \oplus e_{-1}k$. Coordinates associated with this decomposition will be denoted with a superscript 2 i.e. $(v_0, v_1)^2 = \langle (e_1\epsilon^{-1})v_1 + v_0 + e_{-1}\rangle$ and $(\infty)^2 = \langle e_1\rangle$. Remark that the following equalities hold

$$\begin{aligned} (v_0, v_1)^1 &= (v_0 v_1^{-1}, \epsilon v_1^{-1})^2, v_0 \neq 0 \\ (\infty)^1 &= (0,0)^2 \\ (0,0)^2 &= (\infty)^1. \end{aligned}$$

In particular $(t_0, t_1)^1 = (t_0 t_1^{-1}, \epsilon t_1^{-1})^2$. Formula (3.1)implies that $u((0,0);(\infty), (t_0, t_1))$ has matrix representation :

$$\begin{pmatrix} 1 & f(t_0 t_1^{-1}, B_0) & \epsilon t_1^{-1} \\ 0 & I_{|B_0|} & t_0 t_1^{-1} \\ 0 & 0 & 1 \end{pmatrix}$$

with respect to the ordered base $\{-e_1, B_0, e_{-1}\}$. Thus with respect to the ordered base $\{e_{-1}, B_0, e_1\}$ has matrix representation :

$$\begin{pmatrix} 1 & 0 & 0 \\ t_0 t_1^{-1} & I_{|B_0|} & 0 \\ -\epsilon t_1^{-1} & -f(t_0 t_1^{-1}, B_0) & 1 \end{pmatrix}.$$

If in the sequel we use two coordinate systems to describe X such that the first is associated to the decomposition $V = e_{-1}k \oplus V_0 \oplus e_1 k$ and the second to $V = (e_1\epsilon^{-1})k \oplus V_0 \oplus e_{-1}k$ we say that we use a *switch of coordinates.*
We check that $(X, (U_x)_{x\in X})$ is a Moufang set. Choose a coordinate system associated to the decomposition $Ve_{-1}k \oplus V_0 \oplus e_1 k$. Let B_0 be an ordered base of V_0.
1. Condition **MoS1** :
By the matrix description of $U_{(\infty)}$ it is clear that it acts regularly on $X \setminus \{(\infty)\}$. As all root groups have the same matrix representation with respect to different coordinate systems condition **MoS1** is satisfied.
2. Condition **MoS2** :
Let $u((\infty);(0,0),(t_0,t_1)) \in U_{(\infty)}$ and $v \in U_r$, $r \in X$.
The element $vu((\infty);(0,0),(t_0,t_1))v^{-1}$ is a linear transformation of V which sends $v((0,0))$ to $v((t_0,t_1))$ and has matrix representation of the form (1) with respect to the ordered base $\{v(e_{-1}), v(B_0), v(e_1)\}$. Hence $v\ u((\infty);(0,0),(t_0,t_1))\ v^{-1}$ belongs to $U_{v(\infty)}$ by the description of the root groups and condition **MoS2** is satisfied.
Thus $(X, (U_x)_{x\in X})$ is a Moufang set. In the sequel we will denote it by $\mathcal{M}(V, q, k, \sigma)$.
Remark that by similar arguments to prove condition **MoS2** we see that any linear transformation g that satisfies $q(x) = q(g(x))$, $\forall x \in V$, defines a permutation of the points of $\mathcal{M}(V, q, k, \sigma)$ such that $g\ u((\infty);(0,0),(x_0,x_1))\ g^{-1} \in U_{g(\infty)}$. This implies that the transvection group of $\mathcal{M}(V, q, k, \sigma)$ is normalized by the group of linear transformations of V preserving the form q.
To end this section we mention a special class of Moufang sets of the form $\mathcal{M}(V, q, \sigma, k)$.

Definition 95 A *polar line* is a Moufang set $\mathcal{M}(V, q, \sigma, k)$ such that $dim(V) = 2$ and $Z(k) \neq k$. If q is a (σ, ϵ)-quadratic form we denote it by $Pol(k, \sigma, \epsilon)$. If in particular $\epsilon = -1$ a polar line $Pol(k, \sigma, \epsilon)$ will be shortly denoted as $Pol(k, \sigma)$.

Definition 96 An *extended polar line* is a Moufang set $\mathcal{M}(V, q, k, \sigma)$ with abelian root groups such that $Z(k) \neq k$.

Concerning polar lines we have the following lemma.

Lemma 97 *A polar line of the form $Pol(k,\sigma)$ defined by a $(\sigma,-1)$-quadratic form, where k is a generalized quaternion algebra and σ its standard involution, is isomorphic to the projective Moufang set $\mathcal{P}(Z(k))$.*

proof :

As the polar line is defined by a $(\sigma,-1)$-quadratic form it follows that the points set of $Pol(k,\sigma)$ equals $\{(0,\theta)|\theta \in Tr(\sigma)\} \cup (\infty)$. The assumptions on k and σ imply that $Tr(\sigma) = Z(k)$. Consider the projective Moufang set $\mathcal{P}(Z(k))$ with certain coordinatization. Then one easily shows that the bijection from $Pol(k,\sigma)$ to $\mathcal{P}(Z(k))$given by :

$$\begin{aligned} \beta((0,\theta)) &= (\theta) \\ \beta((\infty)) &= (\infty) \end{aligned}$$

defines a Moufang set isomorphism. □

3.12.3 Proportional Moufang sets

Consider a Moufang set of the form $\mathcal{M}(V,q,k,\sigma)$ and $c \in k$, $c \neq 0$. As mentioned in section 3.5.3, the form cq is a (σ',ϵ')-quadratic form where $t^{\sigma'} = ct^{\sigma}c^{-1}$ and $\epsilon' = c(c^{\sigma^{-1}})\epsilon$. Moreover cq is non-degenerate on V and has Witt index 2. By this we can consider the Moufang set $\mathcal{M}(V,cq,k,\sigma')$ which is isomorphic to the original $\mathcal{M}(V,q,k,\sigma)$. In order to construct an isomorphism we consider a decomposition $V = e_{-1}k \oplus V_0 \oplus e_1 k$ with associated coordinatization using the labelling set $R_{0,1} = \{(v_0,v_1) \in V_0 \times k | q(v_0) + v_1 = 0\}$. As the (σ',ϵ')-sesquilinear form associated to cq is given by cf, a coordinatization of $\mathcal{M}(V,cq,k,\sigma')$ can be obtained using the decomposition $V = \bar{e}_{-1}k \oplus V_0 \oplus \bar{e}_1 k$ with $\bar{e}_{-1} = e_{-1}c^{-1}$ and $\bar{e}_1 = e_1$. The labelling set is given by $\bar{R}_{0,1} = \{(\bar{v}_0,\bar{v}_1) \in V_0 \times k|\ cq(\bar{v}_0) + \bar{v}_1 = 0\}$. We will denote this coordinate system with superscript c.
We find :

$$\begin{aligned} (\bar{v}_0,\bar{v}_1)^c &= \langle \bar{e}_{-1}\bar{v}_1 + \bar{v}_0 + \bar{e}_1 \rangle, \forall (\bar{v}_0,\bar{v}_1) \in \bar{R}_{0,1} \\ &= \langle e_{-1}c^{-1}\bar{v}_1 + \bar{v}_0 + e_1 \rangle \\ &= (\bar{v}_0, c^{-1}\bar{v}_1) \\ (\infty)^c &= (\infty). \end{aligned}$$

Using these equations one easily check that the bijection β from $\mathcal{M}(V,q,k,\sigma)$ to $\mathcal{M}(V,cq,k,\sigma')$ given by :

$$\begin{aligned}\beta(v_0,v_1) &= (v_0,cv_1)^c, \forall (v_0,v_1)\in R_{0,1}\\ \beta(\infty) &= (\infty)^c\end{aligned}$$

defines a Moufang set isomorphism.

Definition 98 Given a Moufang set of the form $\mathcal{M}(V,q,k,\sigma)$ and $c\in k$, we call the Moufang set $\mathcal{M}(V,cq,k,\sigma^c)$ with $t^{\sigma^c}=ct^{\sigma}c^{-1}$, $\forall t\in k$ *proportional to* $\mathcal{M}(V,q,k,\sigma)$ with factor c. The isomorphism β constructed above will be denoted in the sequel as ψ_c. Moreover suppose we consider a coordinatization of $\mathcal{M}(V,q,k,\sigma)$ associated to the decomposition $V=e_{-1}k\oplus V_0+e_1k$. As explained above we can consider the coordinate system of $\mathcal{M}(V,cq,k,\sigma)$ associated to the decomposition $V=\bar{e}_{-1}k\oplus V_0\oplus\bar{e}_1k$ with $\bar{e}_{-1}=e_{-1}c^{-1}$, $\bar{e}_1=e_1$. Under these conditions both coordinate systems will be called *proportional*.

Using Lemma 92 we see that given a Moufang set of the form $\mathcal{M}(V,q,k,\sigma)$, with $\sigma\neq 1$ there always exists a constant $c\in k$ such that cq is a $(\sigma^c,-1)$-quadratic form with $t^{\sigma^c}=ct^{\sigma}c^{-1}$. Therefore we will assume in most cases that for every Moufang set of the form $\mathcal{M}(V,q,k,\sigma)$ with $\sigma\neq 1$, q is a $(\sigma,-1)$-quadratic form. This assumption will simplify in a lot of cases the notation and calculations.

3.12.4 $\mathcal{M}(V,q,k,\sigma)$ and induced Moufang sets.

We prove the following lemma.

Lemma 99 *Every Moufang set $\mathcal{M}(V,q,k,\sigma)$ is isomorphic to a Moufang set $\mathcal{M}_{\Gamma(x)}(Q(E,q,k,\sigma))$, where x is an arbitrary point in the generalized quadrangle of the form $Q(E,q,k,\sigma)$, and conversely every $\mathcal{M}_{\Gamma(x)}(Q(E,q,k,\sigma))$ where $Q(E,q,k,\sigma)$ is a generalized quadrangle defined by a (σ,ϵ)-quadratic form and x an arbitrary point in $Q(E,q,k,\sigma)$, is isomorphic to a Moufang set $\mathcal{M}(V,q,k,\sigma)$.*

proof :

Consider a Moufang set $\mathcal{M}(V,q,k,\sigma)$. Choose a decomposition $V=e_{-2}k\oplus V_0\oplus e_2k$ with associated coordinate system using the label set $R_{0,1}$. Let q be a (σ,ϵ)-quadratic form and suppose the (σ,ϵ)-hermitian form associated to

q is given by f. Let $\bar{E} = e_{-1}k \oplus V \oplus e_1k$, where e_{-1} and e_1 are free vectors independant of V. Define $\bar{f}$ and $\bar{q}$ by :

$$\begin{aligned}\bar{f}|_V &= f\\ \bar{f}(e_{-1},V_0) &= 0\\ \bar{f}(e_1,V_0) &= 0\\ \bar{f}(e_i,e_j) &= \delta_{i,-j},\ i,j \in \{-2,-1,1,2\}\\ \bar{q}|_V &= q\\ \bar{q}(e_{-1}) &= 0\\ \bar{q}(e_1) &= 0.\end{aligned}$$

Extend $\bar{f}$ and $\bar{q}$ such that they define a (σ,ϵ)-quadratic and (σ,ϵ)-hermitian form on $\bar{E}$. Using a coordinatization induced by the decomposition $\bar{E} = e_{-2}k \oplus e_{-1}k \oplus V_0 \oplus e_1 \oplus e_2$ as described in section 3.5.4 one easily checks that $\mathcal{M}_{\Gamma((0))}(Q(\bar{E},\ \bar{q},k,\sigma))$ is isomorphic to $\mathcal{M}(V,q,k,\sigma)$ under the bijection β given by :

$$\begin{aligned}\beta([0,(x_0,x_1)]) &= (x_0,x_1),\ \forall (x_0,x_1) \in R_{0,1}\\ \beta((\infty)) &= (\infty).\end{aligned}$$

As to the converse we consider a generalized quadrangle of the form $Q(E,\,q,$ $k,\,\sigma)$. Choose a coordinatization with associated decomposition $E = e_{-2}k \oplus e_{-1}k \oplus E_0 \oplus e_1k \oplus e_2k$ and labelling set $R_{0,1} = \{(v_0,v_1) \in E_0 \times k |\ q(v_0)+v_1 = 0\}$. Set $V = e_{-2}k \oplus E_0 \oplus e_2k$. Then q is a non-degenerate (σ,ϵ)-quadratic form of Witt index 1 on V and we can consider $\mathcal{M}(V,q,k,\sigma)$. Consider the coordinatization of $\mathcal{M}(V,q,k,\sigma)$ associated to the decomposition $V = e_{-2}k \oplus E_0 \oplus e_2k$, i.e. :

$$\begin{aligned}(v_0,v_1) &= e_{-2}v_1 + v_0 + e_2,\ \forall (v_0,v_1) \in R_{0,1}\\ (\infty) &= (e_{-2}).\end{aligned}$$

Then one checks that the map β defined by :

$$\begin{aligned}\beta([0,(v_0,v_1)]) &= (v_0,v_1),\ \forall (v_0,v_1) \in R_{0,1}\\ \alpha([\infty]) &= (\infty)\end{aligned}$$

defines a Moufang set isomorphism from $\mathcal{M}_{\Gamma((0))}(Q(E,q,k,\sigma))$ to $\mathcal{M}(V,q,k,\sigma)$.
$\square$

3.13 The automorphisms s_x and r_x.

In the calculations that follow an important role is played by the special automorphisms s_x and r_x of the Moufang sets under consideration. We start by giving a short motivation and calculate in some special cases the exact action of these automorphisms.

The automorphisms s_x and r_x are a special case of the following lemma.

Lemma 100 *Given a Moufang set $(X, (U_x)_{x\in X})$. If $u_a \in U_a$, $u_a \neq 1$ and $a \neq b$, there exist unique elements $u_b, u'_b \in U_b$ such that $u_b u_a u'_b$ interchanges a and b.*

proof :

Consider the equations

$$\begin{aligned} u_b u_a u'_b(a) &= b \\ u_b u_a u'_b(b) &= a. \end{aligned}$$

Using the conditions of Moufang sets one easily checks that this has unique solutions u_b and u'_b. □
Using Lemma 100 we give definitions of s_x and r_x.
Let $\mathcal{P}(k)$ be projective Moufang set. Choose a fixed coordinatization. By Lemma 100 every $u((\infty);(0),(v))$, $v \neq 0$, determines unique $w, w' \in U_{(0)}$ such that $wu((\infty);(0),(v))w'$ interchanges (0) and (∞). In the sequel we will denote this element by $s_{(v)}$.
One easily checks that $w = w' = u((0);(\infty),(-v))$ and hence :

$$s_{(v)} = u((0);(\infty),(-v))u((\infty);(0),(v))u((0);(\infty),(-v)).$$

Let $\mathcal{M}(V, q, k, \sigma)$ be a Moufang set as in section 3.12. Choose a fixed coordinatization of the set, associated to a decomposition $V = e_{-1}k \oplus V_0 \oplus e_1 k$. Then Lemma 100 implies that every $u((\infty);(0,0), (v_0, v_1))$, $v_0 \neq 0$, determines unique $w, w' \in U_{(\infty)}$ such that $wu((\infty);(0,0),(v_0,v_1))w'$ interchanges $(0,0)$ and (∞). In the sequel we will denote this element by $s_{(v_0,v_1)}$.
One easily checks that $w = w' = u((0,0);(\infty), (-v_0, -v_1 + f(v_0, v_0))$.
Hence :

$$\begin{aligned} s_{(v_0,v_1)} &= u((0,0);(\infty),(-v_0,-v_1+f(v_0,v_0))u((\infty);(0,0),(v_0,v_1)) \\ &\quad u((0,0);(\infty),(-v_0,-v_1+f(v_0,v_0)). \end{aligned}$$

Let $Q(E,q,k,\sigma)$ be a quadrangle as in section 3.5.3. Choos a fixed coordinatization and let $\mathcal{M}_{\Gamma((0))}(Q(E,q,k,\sigma))$ be an induced Moufang set in this quadrangle. Given $u([\infty];[0,(0,0)],[0,(v_0,v_1)])$ there exist by Lemma 100 unique elements $z,\ z' \in U_{[0,(0,0)]}$ such that $z\ u([\infty];[0,(0,0)],[0,(v_0,v_1)])\ z'$ interchanges $[\infty]$ and $[0,(0,0)]$. In the sequel we denote this element by $s_{[0,(v_0,v_1)]}$. As for $\mathcal{M}(V,q,k,\sigma)$ one checks that $z = z' = u([0,(0,0)];[\infty],[0,(-v_0,-v_1 + f(v_0,v_0)])$. Hence :

$$\begin{aligned}&s_{[0,(v_0,v_1)]}\\ &= u([0,(0,0)];[\infty],[0,(-v_0,-v_1+f(v_0,v_0)])u([\infty];[0,(0,0)],[0,(v_0,v_1)])\\ &\ u([0,(0,0)];[\infty],[0,(-v_0,-v_1+f(v_0,v_0)])\end{aligned}$$

As to the definition of r_x we make the following conventions.
Let $\mathcal{P}(k)$ be a projective Moufang set. Choose a fixed coordinate system as above. Using Lemma 100 every $u((0);(\infty),(v))$ determines unique elements $y, y' \in U_{(\infty)}$ such that $yu((0);(\infty),(v))y'$ interchanges (0) and (∞). In the sequel we will denote this element by $r_{(v)}$. One easily checks that in this case $y = y' = u((\infty);(0),(-v))$ and hence :

$$\begin{aligned}&r_{(v)}\\ &= u((\infty);(0),(-v))u((0);(\infty),(v))u((\infty);(0),(-v)).\end{aligned}$$

If $\mathcal{M}(V,q,k,\sigma)$ is a Moufang set as in section 3.12 we choose a fixed coordinate system of this set. By Lemma 100 every $u((0,0);(\infty),(v_0,v_1))$ determines a unique elements y and y' such that $yu((0,0);(\infty),(0,0))y'$ interchanges $(0,0)$ and (∞). For the sequel we will denote this element by $r_{(v_0,v_1)}$. One easily checks that $y = y' = u((\infty);(0,0),(-v_0,-v_1+f(v_0,v_0))$ and hence :

$$\begin{aligned}r_{(v_0,v_1)} &= u((\infty);(0,0),(-v_0,-v_1+f(v_0,v_0))u((0,0);(\infty),(v_0,v_1))\\ &\quad u((\infty);(0,0),(-v_0,-v_1+f(v_0,v_0)).\end{aligned}$$

Let $\mathcal{M}_{\Gamma((0))}(Q(E,q,k,\sigma))$ be an induced Moufang set in $Q(E,q,k,\sigma)$ as above. Lemma 100 shows that every $u([0,(0,0)];[\infty],[0,(v_0,v_1)])$ determines unique elements $t,\ t' \in U_{[\infty]}$ such that $t\ u([0,(0,0)];[\infty],[0,(v_0,v_1)])\ t'$ interchanges $[\infty]$ and $[0,(0,0)]$. For the sequel we will denote this element by $r_{[0,(v_0,v_1)]}$. One easily checks that $t = t' = u([\infty];[0,(0,0)],[0,(-v_0,-v_1 + f(v_0,v_0))])$.

Hence :

$$\begin{aligned}&r_{[0,(v_0,v_1]}\\ &= u([\infty];[0,(0,0)],[0,(-v_0,-v_1+f(v_0,v_0))])u([0,(0,0)];[\infty],(v_0,v_1)])\\ &\ u([\infty];[0,(0,0)],[0,(-v_0,-v_1+f(v_0,v_0))]).\end{aligned}$$

We calculate in some special cases the exact action of s_x and r_x.

<u>First case</u> : the projective Moufang set $\mathcal{P}(k)$.
Choose a coordinatization of the set such that $\langle e_1\rangle = (\infty)$ and $\langle e_2\rangle = (0,0)$. Given $u((\infty);(0),(v))$ we saw that $w = w' = u((0);(\infty),(-v))$ and the matrix representation of $s_{(v)}$ with respect to the ordered base $\{e_1, e_2\}$ becomes :

$$\begin{aligned} s_{(v)} &= \begin{pmatrix} 1 & 0 \\ 0 & -v^{-1}\end{pmatrix}\begin{pmatrix} 1 & v \\ 0 & 0\end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -v^{-1}\end{pmatrix}\\ &= \begin{pmatrix} 0 & v \\ -v^{-1} & 0\end{pmatrix}.\end{aligned}$$

In a complete similar way one finds :

$$r_{(v)} = \begin{pmatrix} 0 & v \\ -v^{-1} & 0\end{pmatrix}.$$

<u>Second case</u> : $s_{(v_0,v_1)}$ with $(v_0,v_1) \in \mathcal{M}(V,q,k,\sigma)$ with q a $(\sigma,-1)$-quadratic form and $v_0 \in Rad(f)$, where f is the form associated to q.
Choose a fixed coordinate system associated to a decomposition $V = e_{-1}k \oplus V_0 \oplus e_1 k$. Let B_0 be an ordered base of V_0. Given $u((\infty);(0,0),(v_0,v_1)$ we saw the elements w and w' are given by $w = w' = u((0,0);(\infty),(-v_0,-v_1))$. Thus we find in matrix notation with respect to the ordered base $\{e_{-1},B_0,e_1\}$:

$$\begin{aligned} s_{(v_0,v_1)} &= \begin{pmatrix} 1 & 0 & 0 \\ v_0v_1^{-1} & I_{|B_0|} & 0 \\ -v_1^{-1} & 0 & 1\end{pmatrix}\begin{pmatrix} 1 & 0 & v_1 \\ 0 & I_{|B_0|} & v_0 \\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ v_0v_1^{-1} & I_{|B_0|} & 0 \\ -v_1^{-1} & 0 & 1\end{pmatrix}\\ &= \begin{pmatrix} 0 & 0 & v_1 \\ 0 & I_{|B_0|} & 0 \\ -v_1^{-1} & 0 & 1\end{pmatrix}\end{aligned}$$

In a similar way one calculates with respect to the ordered base $\{e_{-1},B_0,e_1\}$:

$$r_{(v_0,v_1)} = s_{(v_0,v_1)} = \begin{pmatrix} 0 & 0 & v_1 \\ 0 & I_{|B_0|} & 0 \\ -v_1^{-1} & 0 & 0\end{pmatrix}.$$

Third case : $s_{[0,(v_0,v_1)]}$ where $[0,(v_0,v_1)]$ is a line a generalized quadrangle $Q(E,q,k,\sigma)$ defined by a $(\sigma,-1)$-quadratic form q with associated form f and $v_0 \in Rad(f)$. Suppose $Q(E,q,k,\sigma)$ is coordinatized using the decomposition $E = e_{-1}k \oplus e_{-1}k \oplus E_0 \oplus e_1k \oplus e_2k$. Let B_0 be an ordered base of E_0. Similar calculations as for $\mathcal{M}(V,q,k,\sigma)$ one shows that $s_{[0,(v_0,v_1)]}$ has as matrix representation with respect to the ordered base $B = \{e_{-2},\, e_{-1},\, B_0,\, e_1,\, e_2\}$:

$$\begin{pmatrix} 0 & 0 & 0 & 0 & v_1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & I_{|B_0|} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -v_1^{-1} & 0 & 0 & 0 & 0 \end{pmatrix}$$

In a complete similar way one finds that $r_{[0,(v_0,v_1)]}$ has as matrix representation with respect to the ordered base B :

$$\begin{pmatrix} 0 & 0 & 0 & 0 & v_1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & I_{|B_0|} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -v_1^{-1} & 0 & 0 & 0 & 0 \end{pmatrix}.$$

3.14 Different types of Moufang sets

In this section we make a division of the Moufang sets under consideration. Amongst the classical Moufang sets we distinguish 4 classes. Motivation will become clear when calculating the isomorphism classes. (In this list k stands for a division ring with involution σ, V is a right k-vector space and q is a (σ,ϵ)-quadratic form.)

Moufang sets of type 1 :
Projective Moufang sets $\mathcal{P}(k)$.

Moufang sets of type 2 :
These are Moufang sets of the form $Q(V,q,k,\sigma)$ with $\sigma = 1$. We denote them by $\mathcal{MO}(V,q,k)$ and call them *orthogonal Moufang sets.*

Moufang sets of type 3 :
By this we mean Moufang sets of the form $\mathcal{M}(V, q, k, \sigma)$ with $Z(k) = k$ and $\sigma \neq 1$. We denote them by $\mathcal{M}H(V, q, k, \sigma)$ and call them *hermitian Moufang sets.*

Moufang sets of type 4 :
These are Moufang sets of the form $\mathcal{M}(V, q, k, \sigma)$ with $Z(k) \neq k$. We call these Moufang sets *unitary Moufang sets* and denote them by $\mathcal{M}U(V, q, k, \sigma)$).

Moufang sets of type 5 :
These are the indifferent Moufang sets of the form $\mathcal{P}(k', l; k')$ as described in section 3.2.

Important to notice as concerns this division is that there is overlap in this list. As will be seen in the rest of this chapter several Moufang sets belong to different classes. Furthermore we introduce the following notation. For a Moufang set of the form $\mathcal{M}(V, q, k, \sigma)$ we denote its transvection group by $T\mathcal{M}(V, q, k, \sigma)$. For an orthogonal Moufang set $\mathcal{M}O(V, q, k)$, $TO(V, q, k)$ stands for its transvection group. In a similar way we will denote for Moufang sets $\mathcal{M}H(V, q, k, \sigma)$, $\mathcal{M}U(V, q, k, \sigma)$, $\mathcal{P}(k)$ and $\mathcal{P}(l; k)$ the transvection groups by $TH(V, q, k, \sigma)$, $TU(V, q, k, \sigma)$, $T\mathcal{P}(k)$ and $T\mathcal{P}(l; k)$. Finally we remark that for any orthogonal Moufang set $\mathcal{M}O(V, q, k)$ with $char(k) = 2$ the equation $q(v + v) = 0 = q(v) + q(v)$ leads to $f(v, v) = 0$, $\forall v \in V$.

3.15 Isomorphism problems

3.15.1 General theory

Lemma 101 *Let $(X, U_x)_{x \in X})$ and $(X', U_{x'})_{x' \in X'})$ be two isomorphic classical or mixed Moufang sets defined over division rings k and k'. Then $char(k) = char(k')$.*

proof :

One easily checks that every root elation u_x satisfies :

$$ord(u_x) = char(k).$$

As every Moufang set isomorphism induces an isomorphism between root groups the lemma follows. □

Lemma 102 *Consider two Moufang set of the form* $\mathcal{M}(V,q,k,\sigma)$ *and* $\mathcal{M}$ (V', q', k', σ') *where* q *is a* (σ,ϵ)*-quadratic form and* q' *a* (σ',ϵ')*-quadratic form. Suppose* φ *is a bijective semi-linear transformation from* V *to* V' *with associated field isomorphism* α *such that : for some constant* $c' \in k'$ *:*

$$\begin{aligned} c'(q(x))^{\alpha} &= q'(\varphi(x)), \forall x \in V \\ c'(f(x,y))^{\alpha} &= f'(\varphi(x),\varphi(y)), \forall x,y \in V \end{aligned}$$

where c' *satisfies :*

$$\begin{aligned} c'\lambda^{\sigma\alpha}c'^{-1} &= \lambda^{\alpha\sigma'}, \forall \lambda \in k \\ c'\epsilon^{\alpha} &= c'^{\sigma'}\epsilon' \end{aligned}$$

Then φ *induces a Moufang set isomorphism* β *from* $\mathcal{M}(V,q,k,\sigma)$ *to* $\mathcal{M}(V',q',k',\sigma')$ *defined by :*

$$\beta(\langle x\rangle) = \langle\varphi(x)\rangle, \forall\langle x\rangle \in \mathcal{M}(V,q,k,\sigma).$$

proof :

Remark firstly that the conditions on c' imply that $(c'k_{\sigma,\epsilon})^{\alpha} = k_{\sigma',\epsilon'}$. This follows from the equation :

$$\begin{aligned} c'(t-t^{\sigma}\epsilon)^{\alpha} &= c't^{\alpha} - c't^{\sigma\alpha}\epsilon^{\alpha} \\ &= c't^{\alpha} - t^{\alpha\sigma'}c'\epsilon^{\alpha} \\ &= c't^{\alpha} - t^{\alpha\sigma'}c'^{\sigma'}\epsilon' \\ &= (c't^{\alpha}) - (c't^{\alpha})^{\sigma'}\epsilon'. \end{aligned}$$

This implies that the map β is a well defined bijection from points of $\mathcal{M}(V, q, k, \sigma)$ to $\mathcal{M}(V',q',k',\sigma')$. We prove that for any $\langle x\rangle, \in \mathcal{M}(V,q,k,\sigma)$ the map $\beta_{\langle x\rangle}$ defined by :

$$\beta_{\langle x\rangle}u(\langle x\rangle;\langle y\rangle,\langle z\rangle) = \beta\circ u(\langle x\rangle;\langle y\rangle,\langle z\rangle)\circ\beta^{-1}$$

defines a bijection from $U_{\langle x\rangle}$ to $U_{\langle\varphi(x)\rangle}$. Lemma 41 then implies that β defines a Moufang set isomorphism.
We use the description of the root groups in $\mathcal{M}(V,q,k,\sigma)$ as described in section 3.12.2. Let $\langle e_{-1}\rangle \in \mathcal{M}(V,q,k,\sigma)$. Choose a coordinatization of $\mathcal{M}(V,q,k,\sigma)$ associated to the decomposition $V = e_{-1}k \oplus V_0 \oplus e_1 k$ with labelling set $R_{0,1} = \{(v_0,v_1) \mid q(v_0)+v_1 = 0\}$. Choose a coordinatization of $\mathcal{M}(V',q',k',\sigma')$ associated to the decomposition $V' = e'_{-1}k' \oplus V'_0 \oplus e'_1 k'$ such that $\varphi(e_{-1}) = e'_{-1}c$ and $\varphi(e_1) = e'_1$. Remark that $(\infty) = \langle e_{-1}\rangle$ for $\mathcal{M}(V,q,k,\sigma)$ and $(\infty) = \langle e'_{-1}\rangle = \beta((\infty))$ for $\mathcal{M}(V',q',k',\sigma')$. Moreover for $(v_0,v_1) \in \mathcal{M}(V,q,k,\sigma)$ we find :

$$\begin{aligned}
\beta((v_0,v_1)) &= \langle \varphi(e_{-1}v_1 + v_0 + e_1)\rangle \\
&= \langle \varphi(e_{-1})v_1^\alpha + \varphi(v_0) + \varphi(e_1)\rangle \\
&= \langle e'_{-1}(c'v_1^\alpha) + \varphi(v_0) + e'_1\rangle \\
&= (\varphi(v_0), c'v_1^\alpha)
\end{aligned}$$

Let $(v_0,v_1) \in R_{0,1}$ and $(w'_0,w'_1) \in R'_{0,1}$.
Then we have :

$$\begin{aligned}
&\beta u((\infty);(0,0),(v_0,v_1))\beta^{-1}((w'_0,w'_1)) \\
&= \beta u((\infty);(0,0),(v_0,v_1))((\varphi^{-1}(w'_0),(c'^{-1})^{\alpha^{-1}}{w'_1}^{\alpha^{-1}})) \\
&= \beta((\varphi^{-1}(w'_0)+v_0, v_1 + (c'^{-1})^{\alpha^{-1}}{w'_1}^{\alpha^{-1}} - f(v_0,\varphi^{-1}(w'_0))) \\
&= (w'_0+\varphi(v_0), c'v_1^\alpha + w'_1 - c'(f(v_0,\varphi^{-1}(w'_0)))^\alpha) \\
&= (w'_0+\varphi(v_0), c'v_1^\alpha + w'_1 - f'(\varphi(v_0),w'_0)) \\
&= u((\infty);(0,0),(\varphi(v_0),c'v_1^\alpha))((w'_0,w'_1)) \\
&= u(\beta((\infty));\beta((0,0)),\beta((v_0,v_1)))((w'_0,w'_1))
\end{aligned}$$

showing that $\beta_{\langle x\rangle}$ defines a bijection from $U_{\langle x\rangle}$ to $U_{\langle\varphi(x)\rangle}$. This completes the proof. □

Lemma 103 *Let $(X,(U_x)_{x\in X})$ be a Moufang set of the form $\mathcal{M}(V,q,k,\sigma)$ which is coordinatized using the decomposition $V = e_{-1}k \oplus V_0 \oplus e_1 k$ with associated labelling set $R_{0,1} = \{(v_0,v_1) \in V_0 \times k \mid q(v_0)+v_1 = 0\}$. Then $Z((R_{0,1},\oplus))$ $= \{(v_0,v_1) \in R_{0,1} \mid f(v_0,w_0) = f(w_0,v_0),\ \forall w_0 \in V_0\}$. In particular if $\sigma \neq 1$, $Z((R_{0,1},\oplus)) = \{(v_0,v_1) \in R_{0,1} \mid v_0 \in Rad(f)\}$.*

proof :

Let $\mathcal{M}(V, q, k, \sigma)$ and $R_{0,1}$ be as in the theorem.
Suppose $(v_0, v_1) \in Z((R_{0,1}))$. Then this means :

$$(v_0, v_1) \oplus (w_0, w_1) = (w_0, w_1) \oplus (v_0, v_1), \ \forall (w_0, w_1) \in R_{0,1}.$$

Equivalently :

$$f(v_0, w_0) = f(w_0, v_0), \ \forall w_0 \in V_0.$$

Let $\sigma \neq 1$, and suppose $f(v_0, w_0) \neq 0$ for a $w_0 \in V_0$.
We find that :

$$f(v_0, w_0\lambda) = f(w_0\lambda, v_0), \ \forall \lambda \in k$$

yielding :

$$\lambda^\sigma = \lambda, \ \forall \lambda \in k,$$

a contradiction.
Hence in this case we find $Z(R_{0,1}, \oplus) = \{(v_0, v_1) \in R_{0,1} \ | v_0 \in Rad(f)\}$. □

Lemma 104 *A classical Moufang set $(X, (U_x)_{x \in X})$ has commutative root groups if and only if:*

(i) it is of type 1,

(ii) it is of type 2,

(iii) it is of type 3 and $dim(V) = 2$,

(iv) it is of type 4 and $codim(Rad(f)) = 2$.

An indifferent Moufang set always has commutative root groups.

proof :

Let $(X, (U_x)_{x \in X})$ be a projective Moufang set $\mathcal{P}(k)$. As in this case the root groups are isomorphic to the additive group on k the lemma holds.
Suppose $(X, (U_x)_{x \in X})$ is a Moufang set of the form $\mathcal{M}(V, q, k, \sigma)$. Choose a coordinatization of $\mathcal{M}(V, q, k, \sigma)$ with associated decomposition $V = e_{-1}k \oplus V_0 \oplus e_1 k$ and labelling set $R_{0,1} = \{(v_0, v_1) \in V_0 \times k | q(v_0) + v_1 = 0\}$. As we already saw the root groups are isomorphic to $(R_{0,1}, \oplus)$. By Lemma 103 we know that the root groups are commutative if and only if the form f is

symmetric on V_0.
If $\mathcal{M}(V,q,k,\sigma)$ is of type 2, this condition is clearly satisfied.
If $\mathcal{M}(V,q,k,\sigma)$ is of type 3 or 4, Lemma 103 yields $codim(Rad(f)) = 2$.
If $\mathcal{M}(V,q,k,\sigma)$ is of type 3, Theorem 8.2.4 of [29] implies that f is non-degenerate. Hence in this case the root groups are commutative if and only if $V_0 = 0$.
The statement about the indifferent Moufang sets is clear as these are Moufang subsets of projective ones. □

Corollary 105 *Let $\mathcal{MU}(V,q,k,\sigma)$ be a unitary Moufang set defined over a generalized quaternion algebra with standard involution σ in characteristic non 2. Suppose q is a $(\sigma,-1)$ quadratic form and choose a coordinatization associated to a decomposition $V = e_{-1}k \oplus V_0 \oplus e_1k$ with labelling set $R_{0,1} = \{(v_0,v_1) \in V_0 \times k\ |q(v_0)+v_1 = 0\}$. Then $Z(R_{0,1},\oplus) = \{(0,\theta)|\theta \in Tr(\sigma)\}$. Therefore $\mathcal{MU}(V,q,k,\sigma)$ has commutative root groups if and only if $dim(V) = 2$ and $\mathcal{MU}(V,q,k,\sigma) \cong \mathcal{P}(Z(k))$.*
Moreover if $\mathcal{MU}(V,q,k,\sigma)$ is a unitary Moufang set defined over a generalized quaterion algebra k with standard involution σ, and $dim(V) = 2$ we find in any case that $\mathcal{MU}(V,q,k,\sigma) \cong \mathcal{P}(Z(k))$.

proof :

Let $R_{0,1}$ be as in the theorem. By Lemma 103 we have that

$$Z((R_{0,1},\oplus)) = \{(v_0,v_1) \in R_{0,1}|v_0 \in Rad(f)\}.$$

But if $char(k) \neq 2$ we have $q = f/2$ showing $Rad(f) = \{v_0 \in V_0\ |q(v_0) = 0\}$. As q is anisotropic on V_0 we thus find :

$$Z((R_{0,1},\oplus)) = \{(0,\theta)|\theta \in Tr(\sigma)\}.$$

This means that if $char(k) \neq 2$, $Z((R_{0,1},\oplus)) = R_{0,1}$ if and only if $dim(V) = 2$. But then the point set of $\mathcal{MU}(V,q,k,\sigma)$ consists of $\{(0,\theta)\ |\theta \in Fix(\sigma) = Z(k)\} \cup \{(\infty)\}$.
Let $\mathcal{MU}(V,q,k,\sigma)$ a unitary Moufang set defined by a $(\sigma,-1)$-quadratic form such that $dim(V) = 2$ defined over generalized quaternion algebra k with standard involution σ. Lemma 97 implies that $\mathcal{MU}(V,q,k,\sigma) \cong \mathcal{P}(Z(k))$.

This implies in particular that if $char(k) \neq 2$ $\mathcal{M}U(V,q,k,\sigma)$ has commutative root groups if and only if it is isomorphic to $\mathcal{P}(Z(k))$. $\square$

Lemma 106 *Let $\mathcal{M}(O(V,q,k))$ be an orthogonal Moufang set coordinatized with respect to a decomposition $V = e_{-1}k \oplus V_0 \oplus e_1k$ using the labelling set $R_{0,1} = \{(v_0,v_1) \in V_0 \times k | q(v_0) + v_1 = 0\}$ and suppose B_0 is an ordered base of V_0.*
Then for (t_0,t_1), (v_0,v_1) :

$$s_{(t_0,t_1)}(v_0,v_1) = (t_0 f(t_0,v_0)v_1^{-1} + v_0v_1^{-1}t_1, t_1v_1^{-1}t_1).$$

Thus $s_{(t_0,t_1)}$ has matrix representation with respect to the ordered base $\{e_{-1}, B_0, e_1\}$:

$$\begin{pmatrix} 0 & 0 & t_1 \\ 0 & I_{|B_0|} & t_0t_1^{-1}f(B_0,t_0) \\ t_1^{-1} & 0 & 0 \end{pmatrix}.$$

<u>proof</u> :

Consider a decomposition of V as $V = e_{-1}k \oplus V_0 \oplus e_1k$ with associated coordinatization with labelling set $R_{0,1} = \{(v_0,v_1) \in V_0 \times k | q(v_0) + v_1 = 0\}$. Denote this coordinate system with superscript 1 i.e.

$$\begin{aligned} (v_0,v_1)^1 &= \langle e_{-1}v_1 + v_0 + e_1 \rangle, \forall (v_0,v_1) \in R_{0,1} \\ (\infty)^1 &= \langle e_{-1} \rangle. \end{aligned}$$

Remember that

$$\begin{aligned} s_{(t_0,t_1)^1} &= u((0,0)^1;(\infty)^1,(-t_0,t_1)^1)u((\infty)^1;(0,0)^1,(t_0,t_1)^1) \\ &\quad u((0,0)^1;(\infty)^1,(-t_0,t_1)^1). \end{aligned}$$

In order to calculate the action of $u((0,0)^1;(\infty)^1,(-t_0,t_1)^1)$ on the Moufang set we will make use of a switch of coordinates as explained in section 3.12.2. This means that besides the first coordinate system we consider a second system associated to the decomposition $V = e_1k \oplus V_0 \oplus e_{-1}k$. Coordinates with respect to this second system will be denoted by superscript 2 i.e.

$$\begin{aligned} (v_0,v_1)^2 &= \langle e_1v_1 + v_0 + e_{-1} \rangle, \forall (v_0,v_1) \in R_{0,1)} \\ (\infty)^2 &= \langle e_1 \rangle. \end{aligned}$$

Remark that the following equalities hold :

$$\begin{aligned}(v_0, v_1)^1 &= (v_0 v_1^{-1}, v_1^{-1})^2,\ \forall (v_0, v_1) \in R_{0,1} \setminus \{(0,0)\}\\ (0,0)^1 &= (\infty)^2\\ (\infty)^2 &= (0,0)^1\end{aligned}$$

and if $\mathcal{MO}(V, q, k)$ is orthogonal then $f(v_0, v_0) = -2q(v_0)$, $\forall v_0 \in V_0$.
Let $(v_0, v_1)^1$ be any element of $\mathcal{MO}(V, q, k, \sigma)$ with $v_0 \neq 0$.
We calculate :

$$u((0,0)^1; \infty^1, (-t_0, t_1))(v_0, v_1)^1 = (-t_0 + v_0, t_1 + v_1 + f(t_0, v_0))^1$$

Set $A = t_1 + v_1 + f(t_0, v_0)$.
We have :

$$\begin{aligned}& u((0,0)^1; (\infty)^1, (t_0, t_1)^1)(-t_0 + v_0, A)^1\\ = \ & u((\infty)^2; (0,0)^2, (t_0 t_1^{-1}, t_1^{-1})^2)((-t_0 + v_0)A^{-1}, A^{-1})^2\\ = \ & (t_0 t_1^{-1} + (-t_0 + v_0)A^{-1}, A^{-1} + t_1^{-1} - f(t_0 t_1^{-1}, (-t_0 + v_0)A^{-1})^2\\ = \ & (t_0 t_1^{-1} + (-t_0 + v_0)A^{-1}, A^{-1} + t_1^{-1} - 2A^{-1} - A^{-1} t_1^{-1} f(t_0, v_0))^2\\ = \ & (t_0 t_1^{-1} + (-t_0 + v_0)A^{-1}, -A^{-1} + t_1^{-1} - A^{-1} t_1^{-1} f(t_0, v_0))^2\\ = \ & (t_0 v_1^{-1}(A - t_1) + v_0 t_1 v_1^{-1}, A t_1 v_1^{-1})^1\end{aligned}$$

Moreover:

$$\begin{aligned}& u((\infty)^1; (0,0)^1, (-t_0, t_1)^1)(t_0 v_1^{-1}(A - t_1) + v_0 t_1 v_1^{-1}, A t_1 v_1^{-1})^1\\ & = u((\infty)^1; (0,0)^1, (-t_0, t_1)^1)(t_0(1 + v_1^{-1} f(t_0, v_0) + v_0 t_1 v_1^{-1}, t_1 v_1^{-1} t_1\\ & + t_1 + t_1 v_1^{-1} f(t_0, v_0))^1\\ & = (t_0 v_1^{-1} f(t_0, v_0) + v_0 t_1 v_1^{-1}, t_1 v_1^{-1} t_1 + t_1 + t_1 v_1^{-1} f(t_0, v_0) + f(t_0, t_0)\\ & + v_1^{-1} f(t_0, t_0) f(t_0, v_0) + t_1 v_1^{-1} f(t_0, v_0) + t_1)^1\\ & = (t_0 v_1^{-1} f(t_0, v_0) + v_0 t_1 v_1^{-1}, t_1 v_1^{-1} t_1\\ & + 2t_1(1 + v_1^{-1} f(t_0, v_0)) + f(t_0, t_0)(1 + v_1^{-1} f(t_0, v_0)))^1\\ & = (t_0 v_1^{-1} f(t_0, v_0) + v_0 t_1 v_1^{-1}, t_1 v_1^{-1} t_1)^1\end{aligned}$$

□

Lemma 107 *Let $\mathcal{MO}(V, q, k)$ be an orthogonal Moufang set coordinatized over a labelling set $R_{0,1}$. Then :*

$$s_{(v_0\lambda, v_1\lambda^2)} s_{(v_0, v_1)}((w_0, w_1)) = (w_0 \lambda^2, w_1 \lambda^4),\ \forall (v_0, v_1), (w_0, w_1) \in R_{0,1}, \lambda \in k.$$

Hence

$$s_{(v_0,v_1)}s_{(v_0\lambda,v_1\lambda^2)} \in Z(Fix_{T\mathcal{M}O(V,q,k)}(\{(\infty),(0,0)\}),\ \forall (v_0,v_1)\in R_{0,1},\lambda\in k.$$

proof :

Suppose $R_{0,1}$ is the labelling set of a coordinatization of $\mathcal{M}O(V,q,k)$ associated to the decomposition $V = e_{-1}k \oplus V_0 \oplus e_1 k$. Let (v_0,v_1), (w_0,w_1) $\in R_{0,1}$, $\lambda \in k$. Then we have using Lemma 106 :

$$\begin{array}{rl} & s_{(v_0\lambda,v_1)\lambda^2}s_{(v_0,v_1)}((w_0,w_1)) \\ = & s_{(v_0\lambda,v_1\lambda^2)}((v_0 f(v_0,w_0){w_1}^{-1}+w_0{w_1}^{-1}v_1,{v_1}^2{w_1}^{-1})) \\ = & (v_0\lambda(f(v_0\lambda,v_0 f(v_0,w_0){w_1}^{-1}+w_0{w_1}^{-1}v_1))w_1{v_1}^{-2} \\ & +(v_0 f(v_0,w_0){w_1}^{-1}+w_0{w_1}^{-1}v_1){v_1}^{-2}w_1v_1\lambda^2, w_1{v_1}^{-2}{v_1}^2\lambda^4) \\ = & (v_0\lambda^2 f(v_0,v_0)f(v_0,w_0){v_1}^{-2}+v_0\lambda^2 f(v_0,w_0)v_1^{-1} \\ & +v_0 f(v_0,w_0)\lambda^2 v_1^{-1}+w_0\lambda^2, w_1\lambda^4 \\ = & (v_0\lambda^2 f(v_0,w_0)v_1^{-1}(f(v_0,v_0)v_1^{-1}+2)+w_0\lambda^2,w_1\lambda^4) \\ = & (w_0\lambda^2,w_1\lambda^4) \end{array}$$

As an arbitrary element of $Fix_{T\mathcal{M}O(V,q,k)}$ $\{(\infty),\ (0,0)\})$ has a matrix representation with respect to the ordered base $\{e_{-1},\ B_0,\ e_1\}$ of the form :

$$\begin{pmatrix} \mu & 0 & 0 \\ 0 & A_0 & 0 \\ 0 & 0 & \mu^{-1} \end{pmatrix}$$

we see that $s_{(v_0\lambda,v_1\lambda^2)}\ s_{(v_0,v_1)} \in Z(Fix_{T\mathcal{M}O(V,q,k)}\ \{(\infty),\ (0,0)\})$. □

Lemma 108 *Let $\mathcal{M}U(V,q,k,\sigma)$ be a unitary Moufang set defined by a $(\sigma,1)$-quadratic form q where $char(k)=2$. Assume that the form associated to q is given by f. Let (t_0,t_1), $(v_0,v_1)\in R_{0,1}$. If $f(t_0,t_0)=0$ we have for (v_0,v_1) $\in R_{0,1}$:*

$$s_{(t_0,t_1)}((v_0,v_1)) = (t_0t_1^{-1}f(t_0,v_0)v_1^{-1}t_1+v_0v_1^{-1}t_1,t_1v_1^{-1}t_1).$$

proof :

Remark that $f(t_0,t_0)$ is equivalent to the condition $t_1^\sigma = t_1$. The Lemma then follows by the calculations made in Lemma 106 taking into account that $char(k) = 2$ and $f(t_0, t_0) = 0$. □

Lemma 109 *Let $\mathcal{M}U(V, q, k, \sigma)$ be a type 4 Moufang set with q a $(\sigma, -1)$ quadratic form. Suppose that g is the σ-sesquilinear form such that $q(v) = g(v) + k_{\sigma,\epsilon}$. Then the set $\{g(w)|\ w \in Rad(f)\}$ is contained in $Fix(\sigma)$. Moreover any coordinatization of $\mathcal{M}U(V, q, k, \sigma)$ with associated labelling set $R_{0,1}$ satisfies :*

$$\{(v_0, v_1)_1 | (v_0, v_1) \in Z(R_{0,1}, \oplus)\} \subset Fix(\sigma).$$

and for $(t_0, t_1), (v_0, v_1) \in R_{0,1} \setminus \{(0, 0\}$ with $(t_0, t_1) \in Z(R_{0,1}, \oplus)$ we have :

$$s_{(t_0,t_1)}(v_0, v_1) = (v_0 v_1^{-1} t_1, t_1 v_1^{-1} t_1),\ \forall (t_0, t_1), (v_0, v_1) \in R_{0,1} \setminus \{(0, 0)\},$$

and

$$r_{(t_0,t_1)}(v_0, v_1) = (v_0 v_1^{-1} t_1, t_1 v_1^{-1} t_1),\ \forall (t_0, t_1), (v_0, v_1) \in R_{0,1} \setminus \{(0, 0)\}.$$

proof :

Let $v \in Rad(f)$ with $v \neq 0$. (Remark that this is only possible if $char(k) = 2$).
The equation :

$$\begin{aligned} q(v(\lambda + \mu)) &= (\lambda + \mu)^\sigma g(v)(\lambda + \mu) \\ &= g(v\lambda) + \lambda^\sigma g(v)\mu + \mu^\sigma g(v)\lambda + g(v\mu) + k_{\sigma,-1} \\ &= g(v\lambda) + g(v\mu) + f(v\lambda, v\mu) + k_{\sigma,-1} \\ &= q(v\lambda) + q(v\mu),\ \forall \lambda, \mu \in k \end{aligned}$$

implies :

$$\lambda^\sigma g(v)\mu + \mu^\sigma g(v)\lambda \in k_\sigma,\ \forall \lambda, \mu \in k.$$

Equivalently :

$$\lambda^\sigma (g(v) + g(v)^\sigma)\mu \in k_\sigma,\ \forall \lambda, \mu \in k.$$

If $g(v) \neq g(v)^\sigma$ this means $k = k_\sigma$ a contradiction as $k_\sigma \subset Fix(\sigma)$ and $\sigma \neq 1$. Hence $g(v)^\sigma = g(v)$. Choose a coordinatization of $\mathcal{M}U(V, q, k, \sigma)$

associated to the decomposition with labelling set $R_{0,1} = \{(v_0, v_1) \in V_0 \times k | q(v_0) + v_1 = 0\}$. Let B_0 be an ordered base of V_0. If $(v_0, v_1) \in Z((R_{0,1}, \oplus))$ Lemma 103 shows that $v_0 \in Rad(f)$, but then $v_1 = g(v_0) + r$ for a $r \in Tr(\sigma)$. Hence v_1 belongs to $Fix(\sigma)$ as $g(v_0) \in Fix(\sigma)$.
Let $(t_0, t_1) \in Z((R_{0,1}, \oplus))$, then $t_0 \in Rad(f)$. Using matrix representation of root elations with respect to the ordered base $\{e_{-1}, B_0, e_1\}$ as explained in section 3.12.2 we have :

$$\begin{aligned} s_{(t_0,t_1)} &= u((0,0);(\infty),(t_0,t_1))u((\infty);(0,0),(t_0,t_1))u((0,0);(\infty),(t_0,t_1)) \\ &= \begin{pmatrix} 1 & 0 & 0 \\ t_0 t_1^{-1} & I_{|B_0|} & 0 \\ t_1^{-1} & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & t_1 \\ 0 & I_{|B_0|} & t_0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ t_0 t_1^{-1} & I_{|B_0|} & 0 \\ t_1^{-1} & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & t_1 \\ 0 & I_{|B_0|} & 0 \\ t_1^{-1} & 0 & 0 \end{pmatrix}. \end{aligned}$$

Hence

$$s_{(t_0,t_1)}(v_0, v_1) = (v_0 v_1^{-1} t_1, t_1 v_1^{-1} t_1),\ \forall (v_0, v_1) \in R_{0,1}.$$

The statement for $r_{(t_0,t_1)}$ follows by similar arguments. □

The following Lemmas are translations of well known isomorphism theorems to the language of Moufang set.

Lemma 110 *Let k be a generalized quaternion algebra. Then $\mathcal{P}(k)$ is isomorphic to a non-commutative orthogonal Moufang set $\mathcal{MO}(V', q', Z(k))$ such that $dim(V') = 6$.*

proof :

We use the theory on generalized quaternion algebras as briefly exposed on p73 and 74 in [6]. This means that we can choose in k elements i and j such that $k = Z(k) \oplus iZ(k) \oplus jZ(k) \oplus jiZ(k)$.
Moreover if $char(k) \neq 2$ we can assume

$$\begin{aligned} i^2 &= \alpha_0 \\ j^2 &= \beta_0 \\ ij &= -ji \end{aligned}$$

where α_0 and β_0 are non squares in $\in Z(k)$.
If $char(k) = 2$ these elements can be assumed to satisfy :

$$\begin{aligned} i^2 &= i + \alpha_0 \\ j^2 &= \beta_0 \\ ij &= ji + i \end{aligned}$$

where α_0 and β_0 are non squares in $Z(k)$. Let σ be the standard involution in k. Then we denote the norm with respect to σ as N, i.e. $N(x) = x^\sigma x$. Define the orthogonal Moufang set $\mathcal{M}O(V', q', Z(k)))$ in the following way. Let $V' = e'_{-1}Z(k) \oplus V'_0 \oplus e'_1 Z(k)$ with $V'_0 = {e'_0}^1 Z(k) \oplus {e'_0}^2 Z(k) \oplus {e'_0}^3 Z(k) \oplus {e'_0}^4 Z(k)$ and define the forms q' and f' as follows. Let $x' = e'_{-1}x'_{-1} + {e'_0}^1 z_1 + {e'_0}^2 z_2 + {e'_0}^3 z_3 + {e'_0}^4 z_4 + e'_1 x'_1$ and $y' = e'_{-1}y'_{-1} + {e'_0}^1 u_1 + {e'_0}^2 u_2 + {e'_0}^3 u_3 + {e'_0}^4 u_4 + e'_1 y'_1$. Set $\lambda = z_1 + iz_2 + jz_3 + jiz_4$ and $\mu = u_1 + iu_2 + ju_3 + jiu_4$, with x'_{-1}, x'_1, y'_{-1}, y'_1, z_i, $u_i \in Z(k)$, $1 \leq i \leq 4$.

$$\begin{aligned} q'(x', x') &= (x'_{-1})x'_1 + N(\lambda) \\ f'(x', y') &= x'_{-1}y'_1 + x'_1 y'_{-1} + \lambda^\sigma \mu + \mu^\sigma \lambda \end{aligned}$$

One easily checks that f' is a trace valued quadratic form and q' is a quadratic form such that $q'(x' + y') = q'(x') + q'(y') + f(x', y')$, $\forall x', y' \in V'$. As $q'(e'_1) = q'(e'_{-1}) = 0$ and $q'|_{V'_0}$ is anisotropic since it is the norm function N, q' is a quadratic form on V' of Witt index 1. This means that we can consider the Moufang set $\mathcal{M}O(V', q', Z(k))$. In the sequel we will use the coordinatization of this set associated to the decomposition $V' = e'_{-1}Z(k) \oplus V'_0 \oplus e'_1 Z(k)$ with labelling set $R'_{0,1}$. Consider the projective Moufang set $\mathcal{P}(k)$ with canonical coordinatization as explained in section 3.2. Define the bijection β from to $\mathcal{M}O(V', q', Z(k))$ to $\mathcal{P}\ (k)$ in the following way :

$$\begin{aligned} \beta((\infty)) &= (\infty) \\ \beta(({e'_0}^1 z_1 + {e'_0}^2 z_2 + {e'_0}^3 z_3 + {e'_0}^4 z_4, -N(\lambda)) &= (\lambda). \end{aligned}$$

with $\lambda = z_1 + iz_2 + jz_3 + jiz_4$. Using Lemma 41 we check that β defines a Moufang set isomorphism. It will thus enough to show that the map $\beta_{(\infty)}$ defines as :

$$\beta_{(\infty)}(u_\infty) = \beta \circ u_\infty \circ \beta^{-1}$$

defines a map from $U_{(\infty)}$ to $U_{(\infty)}$ and similarly show that the map $\beta_{(0,0)}$ defined by :

$$\beta_{(0,0)}(u_0) = \beta \circ u_0 \circ \beta^{-1}$$

defines a map from $U_{(0,0)}$ to $U_{(0)}$.
Let (t'_0, t'_1), $(v'_0, v'_1) \in R'_{0,1}$ with $v'_0 = {e'_0}^1 z_1 + {e'_0}^2 z_2 + {e'_0}^3 z_3 + {e'_0}^4 z_4$, $v'_1 = -N(\lambda)$ where $\lambda = z_1 + iz_2 + jz_3 + jiz_4$, $t'_0 = e'0^1 u_1 + {e'_0}^2 u_2 + {e'_0}^3 u_3 + {e'_0}^4 u_4$, $t'_1 = -N(\mu)$ if $\mu = u_1 + iu_2 + ju_3 + jiu_4$.
We find :

$$\begin{aligned}
&\beta_{(\infty)}(u((\infty);(0,0),(t'_0,t'_1)))((\lambda))\\
&= \beta(u((\infty);(0,0),(t'_0,t'_1))(v'_0,v'_1))\\
&= \beta((t'_0 + v'_0, -N(\lambda) - N(\mu) - \lambda^\sigma \mu - \mu^\sigma \lambda))\\
&= \beta((t'_0 + v'_0, -N(\lambda + \mu))\\
&= (\mu + \lambda)\\
&= u((\infty);(0),(\mu))(\lambda)
\end{aligned}$$

showing that $\beta_{(\infty)}$ $(u((\infty);(0,0),(t'_0,t'_1)) = u((\infty);(0),\beta((t'_0,t'_1)))$.
As to the map $\beta_{(0,0)}$ we reason as follows. In $\mathcal{MO}(V', q', k')$ we find that $s_{(e_0,-1)} U_{(\infty)}$ $s^{-1}_{(e_0,-1)} = U_{(0,0)}$ and in $\mathcal{P}(k)$ we have $s_{(1)}$ $U_{(\infty)}$ $s^{-1}_{(1)} = U_{(0)}$. By construction of β we have $\beta((e_0,-1)) = (1)$. Therefore it will be enough if we show that $\beta \circ s_{({e'_0}^1,-1)} \circ \beta^{-1} = s_{(1)}$ in order to show that $\beta_{(0,0)}$ defines a map from $U_{(0,0)}$ to $U_{(0)}$. Let $(v'_0, v'_1) \in R'_{0,1}$ with $v'_0 = {e'_0}^1 z_1 + {e'_0}^2 z_2 + {e'_0}^3 z_3 + {e'_0}^4 z_4$, v'_1 and $v'_1 = -N(\lambda)$ where we put $\lambda = z_1 + iz_2 + jz_3 + jiz_4$.
If $char(k) \neq 2$ we have :

$$\begin{aligned}
&\beta(s_{(e_0,-1)}((v'_0,v'_1)))\\
&= \beta(-{e'_0}^1 f'({e'_0}^1, v'_0)(N(\lambda))^{-1} + v'_0(N(\lambda))^{-1}, -(N(\lambda))^{-1})\\
&= \beta(-({e'_0}^1 z_1 - {e'_0}^2 z_2 - {e'_0}^3 z_3 - {e'_0}^4 z_4)(N(\lambda))^{-1}, -(N(\lambda))^{-1})\\
&= (-\lambda^\sigma (N(\lambda))^{-1}, -(N(\lambda))^{-1})\\
&= (-\lambda^{-1})\\
&= s_{(1)}((\lambda))
\end{aligned}$$

and in this case we find thus that $\beta \circ s_{({e'_0}^1,-1)} \circ \beta^{-1} = s_{(1)}$.
If $char(k) = 2$ we have :

$$\begin{aligned}
&\beta(s_{({e'_0}^1,1)}((v'_0, v'_1))) \\
&= \beta((f'({e'_0}^1, v'_0)(N(\lambda))^{-1} + v'_0(N(\lambda))^{-1}, (N(\lambda))^{-1})) \\
&= \beta({e'_0}^1(z_1 + z_2) + {e'_0}^2 z_2 + {e'_0}^3 z_3 + {e'_0}^4 z_4)(N(\lambda))^{-1}, (N(\lambda))^{-1})) \\
&= (\lambda^\sigma (N(\lambda))^{-1}) \\
&= (\lambda^{-1}) \\
&= s_{(1)}((\lambda))
\end{aligned}$$

and thus we find that also in this case $\beta \circ s_{({e'_0}^1,1)} \circ \beta^{-1} = s_{(1)}$. The non commutativity of $\mathcal{MO}(V', q', Z(k))$ follows from the non commutativity of $\mathcal{P}(k)$. This completes the proof.

□

Lemma 111 *Let $\mathcal{MO}(V, q, k, \sigma)$ be an orthogonal Moufang set such that $dim(V) = 3$. Then $\mathcal{MO}(V, q, k) \cong \mathcal{P}(k)$.*

proof :

Choose a coordinatization of $\mathcal{MO}(V, q, k)$ associated to the decomposition $V = e_{-1}k \oplus V_0 \oplus e_1 k$ with labelling set $R_{0,1} = \{(v_0, v_1) \mid q(v_0) + v_1 = 0\}$. Using the results of 3.12.3 we can assume without loss of generality that there exists a vector $e_0 \in V_0$, with $(e_0, -1) \in R_{0,1}$. Indeed if this is not the case we choose a $(v_0, v_1) \in R_{0,1} \setminus \{(0, 0)\}$. Consider the proportional Moufang set $\mathcal{MO}(V, -v_1^{-1} q)$ coordinatized using the decomposition $V = \bar{e}_{-1}k \oplus V_0 \oplus \bar{e}_1 k$ with $\bar{e}_{-1} = -e_{-1}v_1$ and $\bar{e}_1 = e_1$ and labelling set $\bar{R}_{0,1}$. By construction we find then that $(v_0, -1) \in \bar{R}_{0,1}$.
Consider the projective Moufang set $\mathcal{P}(k)$ coordinatized in the canonical way as explained in section 3.2. Define the bijection β from $\mathcal{P}(k)$ to $\mathcal{MO}(V, q, k)$ by :

$$\begin{aligned}
\beta((v)) &= (e_0 v, -v^2) \\
\beta((\infty)) &= (\infty).
\end{aligned}$$

We show that β defines Moufang set isomorphism. By Lemma 41 it suffices to show that the two maps $\beta_{(0)}$ and $\beta_{(\infty)}$ with :

$$\beta_{(\infty)}(u((\infty); (0), (t)) \quad = \quad \beta \circ u((\infty); (0), (t)) \circ \beta^{-1}$$

$$\beta_{(0)}(u((0);(\infty),(t)) \;=\; \beta\circ u((0);(\infty),(t))\circ\beta^{-1}$$

define bijections from $U_{(\infty)}$ to $U_{(\infty)}$ and from $U_{(0)}$ to $U_{(0,0)}$.
Let (s), $(t)\in\mathcal{P}(k)$, with $\beta(s)=(s_0,s_1)$, $\beta(t)=(t_0,t_1)$.
Then we find for (v_0,v_1) with $\beta^{-1}((v_0,v_1))=(v)$:

$$\begin{aligned}\beta(u((\infty);(0),(s))\beta^{-1}(v_0,v_1) &= \beta((s+v))\\ &= \beta(s)\oplus\beta(v)\\ &= (s_0,s_1)\oplus(v_0,v_1)\\ &= u((\infty);(0,0),(s_0,s_1))((v_0,v_1))\end{aligned}$$

Thus :

$$\beta_{(\infty)}(u((\infty);(0),(s)))=u((\infty);(0,0),\beta(s)).$$

Remains to show that β defines a bijection from $U_{(0)}$ to $U_{(0,0)}$. As $U_{(0)}=s_{(1)}U_{(\infty)}s_{(1)}$, $U_{(0,0)}=s_{(e_0,-1)}U_{(\infty)}s_{(e_0,-1)}$ and $\beta(1)=(e_0,-1)$ it will be enough to show that :

$$\beta\circ s_{(1)}\circ\beta^{-1}=s_{(e_0,-1)}.$$

Let $v\in k$, with $v\neq 0$.
We have :

$$\begin{aligned}\beta s_{(1)}\beta^{-1}((e_0v,v^2)) &= \beta((v^{-1}))\\ &= (e_0v^{-1},v^{-2})\\ &= (e_0(f(e_0,e_0)-1)v^{-1},v^{-2})\\ &= (e_0f(e_0,e_0v)v^{-2}-e_0v^{-1},v^{-2})\\ &= s_{(e_0,-1)}((e_0v,v^2))\\ \beta s_{(1)}\beta^{-1}(\infty) &= (0,0)\\ &= s_{(e_0,-1)}(\infty)\\ \beta s_{(1)}\beta^{-1}(0,0) &= (\infty)\\ &= s_{(e_0,-1)}(0,0)\end{aligned}$$

showing that $\beta\circ s_{(1)}\circ\beta^{-1}=s_{(e_0,-1)}$. This completes the proof. □

Lemma 112 *Let $\mathcal{M}O(V,q,k)$ be a orthogonal Moufang set with $dim(V)=4$ and $codim(Rad(f))\neq 2$. Then $\mathcal{M}O(V,q,k)\cong\mathcal{P}(\bar{k})$ where $\bar{k}$ is a quadratic Galois extension of k.*

proof :

Choose a coordinatization of $\mathcal{M}O(V,q,k)$ associated to the decomposition $V=e_{-1}k\oplus V_0\oplus e_1k$ with labelling set $R_{0,1}$. Similar arguments as in the proof of Lemma 111 show we can assume without loss of generality that there exists a $e_0\in V_0$ such that $(e_0,-1)\in R_{0,1}$. In particular $q(e_0)=1$ and $f(e_0,e_0)=2$. Let a_0 be a second vector such that $\langle e_0,a_0\rangle=V_0$. Consider the quadratic polynomial $p(X)=X^2+f(e_0,a_0)X+q(a_0)$ in $k[X]$. Let γ be a root of $p(X)$ in an algebraic closure of k. Remark that the other root of $p(X)$ is given by $f(e_0,a_0)-\gamma$. As $f|_{V_0}\neq 0$, $p(X)$ is a quadratic and separable polynomial and $k(\gamma)$ is thus a quadratic Galois extension of k. Let α be the non trivial automorphism of $k(\gamma)$ fixing k which sends γ to $\gamma+f(e_0,a_0)$. Denote for $x\in k$, $N(x)=xx^\alpha$, $Tr(x)=x+x^\alpha$. Every $x\in k(\gamma)$ can thus be written as $\lambda+\gamma\mu$, λ, $\mu\in k$. Consider $\mathcal{P}(k(\gamma))$ with canonical coordinatization as explained in section 3.2. Define the bijection β from $\mathcal{P}(k(\gamma))$ to $\mathcal{M}O(V,q,k)$ by :

$$\begin{aligned}\beta(\lambda+\gamma\mu) &= (e_0\lambda+a_0\mu,\lambda^2+2\mu\lambda f(e_0,a_0)\gamma+q(a_0)\mu^2)\\ &= (e_0\lambda+a_0\mu,N(\lambda+\gamma\mu))\\ \beta(\infty) &= (\infty).\end{aligned}$$

We show that β defines Moufang set isomorphism. By Lemma 41 it suffices to show that the two maps $\beta_{(0)}$ and $\beta_{(\infty)}$ defined by :

$$\begin{aligned}\beta_{(\infty)}(u((\infty);(0),(t)) &= \beta\circ u((\infty);(0),(t))\circ\beta^{-1}\\ \beta_{(0)}(u((0);(\infty),(t)) &= \beta\circ u((0);(\infty),(t))\circ\beta^{-1}\end{aligned}$$

determine bijections from $U_{(\infty)}$ to $U_{(\infty)}$ and from $U_{(0)}$ to $U_{(0,0)}$.
Let (s), $(t)\in\mathcal{P}(k(\gamma))$, with $\beta(s)=(s_0,s_1)$, $\beta(t)=(t_0,t_1)$.
Then we find for (v_0,v_1) with $\beta^{-1}((v_0,v_1))=(v)$:

$$\begin{aligned}\beta(u((\infty);(0),(s))\beta^{-1}(v_0,v_1) &= \beta((s+v))\\ &= \beta(s)\oplus\beta(v)\end{aligned}$$

$$\begin{aligned} &= (s_0, s_1) \oplus (v_0, v_1) \\ &= u((\infty); (0,0), (s_0, s_1))((v_0, v_1)) \end{aligned}$$

Thus :

$$\beta_{(\infty)}(u((\infty); (0), (s))) = u((\infty); (0,0), \beta(s)).$$

Remains to show that the map $\beta_{(0)}$ defines a bijection from $U_{(0)}$ to $U_{(0,0)}$. As $U_{(0)} = s_{(1)} U_{(\infty)} s_{(1)}$ and $U_{(0,0)} = s_{(e_0,-1)} U_{(\infty)} s_{(e_0,-1)}$ it will be enough to show that :

$$\beta \circ s_{(1)} \circ \beta^{-1} = s_{(e_0,-1)}.$$

Let λ, $\mu \in k$ such that $\lambda + \gamma\mu \neq 0$.
We find :

$$\begin{aligned} & \beta s_{(1)} \beta^{-1}((e_0\lambda + a_0\mu, -N(\lambda + \gamma\mu))) \\ = \; & \beta s_{(1)}((\lambda + \gamma\mu)) \\ = \; & \beta(-(\lambda + \gamma\mu)^{-1}) \\ = \; & \beta(-(\lambda + \gamma^{\alpha}\mu)(N(\lambda + \gamma\mu))^{-1})) \\ = \; & \beta(((-(f(e_0, a_0\mu + \lambda)) + \gamma\mu)(N(\lambda + \gamma\mu))^{-1})) \\ = \; & (-e_0(\lambda + f(e_0, a_0)\mu)(N(\lambda + \gamma\mu))^{-1} + a_0\mu(N(\lambda + \gamma\mu))^{-1}, -(N(\lambda + \gamma\mu))^{-1}). \end{aligned}$$

Using Lemma 109 we find :

$$\begin{aligned} & s_{(e_0,-1)}(e_0\lambda + a_0\mu, -N(\lambda + \gamma\mu)) \\ = \; & (-e_0 f(e_0, e_0\lambda + a_0\mu)(N(\lambda + \gamma\mu))^{-1} \\ & +(e_0\lambda + a_0\mu)(N(\lambda + \gamma\mu))^{-1}, (N(\lambda + \gamma\mu))^{-1}) \\ = \; & (-e_0(f(e_0, e_0) - 1)\lambda(N(\lambda + \gamma\mu))^{-1} - e_0 f(e_0, a_0)\mu(N(\lambda + \gamma\mu))^{-1} \\ & +a_0\mu(N(\lambda + \gamma\mu))^{-1}, (-N(\lambda + \gamma\mu))^{-1}) \\ = \; & (-e_0(\lambda + f(e_0, a_0)\mu)(N(\lambda + \gamma\mu))^{-1} + a_0\mu(N(\lambda + \gamma\mu))^{-1}, -(N(\lambda + \gamma\mu))^{-1})) \\ = \; & \beta s_{(1)} \beta^{-1}((e_0\lambda + a_0\mu, N(\lambda + \gamma\mu)). \end{aligned}$$

As also :

$$\begin{aligned} \beta s_{(1)} \beta^{-1}(\infty) &= (0,0) \\ &= s_{(e_0,-1)}(\infty) \\ \beta s_{(1)} \beta^{-1}(0,0) &= (\infty) \\ &= s_{(e_0,-1)}(0,0) \end{aligned}$$

we find that $\beta s_{(1)} \beta^{-1} = s_{(e_0,-1)}$. This closes the proof. □

Lemma 113 *Let $\mathcal{M}O(V,q,k)$ be an orthogonal Moufang set such that $codim(Rad(f)) = 2$. Then $\mathcal{M}O(V,q,k)$ is isomorphic to an indifferent Moufang set of the form $\mathcal{P}(l;k')$.*

proof :

Suppose $\mathcal{M}O(V,q,k)$ is as in the lemma. Choose a coordinatization of the set associated to a decomposition $V = e_{-1}k \oplus V_0 \oplus e_1 k$ with labelling set $R_{0,1}$. Remark that the assumption on f implies $Rad(f) = V_0$ and hence $\{q(w) \,|w \in Rad(f)\} = \{q(w_0) \,|w_0 \in V_0\}$. Let $(e_0, c^{-1}) \in R_{0,1}$. Then the set $l = \{cq(w)|w \in Rad(f)\}$ clearly satisfies :
(i) l is an additive subgroup of k,
(ii) $l^{-1} = l$ as $c^{-1}q(w)^{-1} = cq(w(q(w)c)^{-1}), \;\; \forall\, w \in Rad(f)$,
(iii) $1 \in l$.
(iv) l is a vectorspace over k^2.
Therefore we can consider the indifferent Moufang set $\mathcal{P}(l;k')$ where k' is the subfield of k generated as a ring by l. We prove that $\mathcal{M}O(V,q,k)$ is isomorphic to $\mathcal{P}(l;k')$. Define the bijection β from $\mathcal{M}O(V,q,k)$ to $\mathcal{P}(l;k')$ as follows :

$$\begin{aligned} \beta((\infty)) &= (\infty) \\ \beta((v_0, v_1)) &= (cv_1). \end{aligned}$$

We use Lemma 41 to show that β defines a Moufang set isomorphism. Let (t_0, t_1), $(v_0, v_1) \in R_{0,1}$. Then we find :

$$\begin{aligned} \beta(u((\infty);(0,0),(t_0,t_1))((v_0,v_1))) &= \beta((t_0+v_0, t_1+v_1)) \\ &= (c(t_1+v_1)) \\ &= u((\infty);(0),(ct_1))((cv_1)) \\ &= u((\infty);(0),\beta((t_0,t_1))((cv_1)) \end{aligned}$$

hence $\beta \circ u((\infty);(0,0),(t_0,t_1)) \circ \beta^{-1} = u((\infty);(0),\beta((t_0,t_1)))$. As (t_0,t_1) was chosen arbitrarily this shows that $\beta\, U_{(\infty)}\, \beta^{-1} = U_{(\infty)}$.
Remains to show that the map $\beta_{(0,0)}$ defined by :

$$\beta_{(0,0)}(u_0) = \beta \circ u_0 \circ \beta^{-1}$$

determines a bijection from $U_{(0,0)}$ to $U_{(0)}$. As before we use the fact that $U_{(0,0)} = s_{(e_0,c^{-1})}\, U_{(\infty)}\, s^{-1}_{(e_0,c^{-1})}$, $U_{(0)} = s_{(1)}\, U_{(\infty)}\, s_{(1)}$ and $\beta((e_0,c^{-1})) = (1)$.

This means that if we show $\beta \circ s_{(e_0,c^{-1})} \circ \beta^{-1} = s_{(1)}$ then the statement about $\beta_{(0,0)}$ holds.
Let $(v_0, v_1) \in R_{0,1}$ the we find :

$$\begin{aligned} \beta(s_{(e_0,c^{-1})}((v_0,v_1))) &= \beta((v_0 v_1^{-1} c^{-1}, c^{-1} v_1^{-1} c^{-1})) \\ &= s_{(1)}(cv_1) \\ &= s_{(1)}\beta((v_0,v_1)). \end{aligned}$$

Hence $\beta \circ s_{(e_0,c^{-1})} \circ \beta^{-1} = s_{(1)}$ and β defines a Moufang set isomorphism.

□

Lemma 114 *Every unitary Moufang set $\mathcal{M}U(V,q,k,\sigma)$ with non-commutative root groups where k is a generalized quaternion algebra with standard involution σ and $dim(V) = 3$ is isomorphic to a hermitian Moufang set $\mathcal{M}H(V',q',k',\sigma')$ with $dim(V') = 4$ and k' isomorphic to a quadratic Galois extension of $Z(k)$. Conversely every hermitian Moufang set $\mathcal{M}H(V,q,k,\sigma)$ with $dim(V) = 4$ is isomorphic to a unitary Moufang set $\mathcal{M}U(V',q',k',\sigma')$ with non-commutative root groups, where $dim(V') = 3$, k' a generalized quaternion algebra with standard involution σ', and k isomorphic to a quadratic Galois extension of $Z(k')$.*

proof :

Let $\mathcal{M}U(V,q,\sigma)$ be unitary Moufang set defined over a generalized quaternion algebra with standard involution σ such that $dim(V) = 3$. Without loss of generality we can assume that q is a $(\sigma,-1)$-quadratic form. Assume $q(v) = g(v,v) + Tr(\sigma)$ and $q(v+w) = q(v) + q(w) + f(v,w)$ with f a $(\sigma,-1)$-hermitian form and g a σ-sesquilinear form. Choose a coordinatization of $\mathcal{M}U(V,q,k,\sigma)$ associated to a decomposition $V = e_{-1}k \oplus V_0 \oplus e_1 k$ with labelling set $R_{0,1}$. As k is a generalized quaternion algebra there exist (cfr [6] p73) a ω, $\theta \in k$, with $k = Z(k) \oplus wZ(k) \oplus \theta Z(k) \oplus \omega\theta Z(k)$ such that :

$char(k) = 2$ and :
$\omega^2 = \omega + \alpha_0$, $\theta^2 = \beta_0$, α_0, $\beta_0 \in Z(k) \setminus Z(k)^2$ and $\theta\omega = \omega\theta + \theta$, $\theta^\sigma = \theta$ and $\omega^\sigma = \omega + 1$

$char(k) \neq 2$ and :
$\omega^2 = \alpha_0$, $\theta^2 = \beta_0$, α_0, $\beta_0 \in Z(k) \setminus Z(k)^2$, $\omega\theta = -\theta\omega$, $\omega^\sigma = -\omega$ and $\theta^\sigma = -\theta$.

Denote the norm function for k with N i.e. for $\lambda = z_1 + wz_2 + \theta z_3 + \theta w z_4$ we have $N(\lambda) = \lambda^\sigma\lambda$. Without loss of generality we can assume that $V_0 = \langle v_0 \rangle$ with $g(v_0, v_0) = -v_1 = -w$ (use Lemma 3.12.3 and section 3.12.3). Let $L_{v_1} = Z(k(v_1) = Z(k)w$. Then L_{v_1} is a separable quadratic Galois extension of $Z(k)$ on which σ acts non trivially with $Fix|_{L_{v_1}}(\sigma) = Z(k)$. Remark that $V_0 = v_0 L_{v_1} \oplus (v_0\theta)\ L_{v_1}$. Therefore we can define a hermitian Moufang set $\mathcal{M}H(V', q', k', \sigma)$ in the following way. Set $V' = e'_{-1}L_{v_1} \oplus V_0 \oplus e'_1 L_{v_1}$. Let $x' = e'_{-1}x'_{-1} + x'_0 + e'_1 x'_1$ and $y' = e'_{-1}y'_{-1} + y'_0 + e'_1 y'_1$ with $x'_0 = v_0(z_1 + \omega z_2) + (v_0\theta)(z_3 + \omega z_4)$ and $y'_0 = v_0(l_1 + l_2\omega) + (v_0\theta)\ (l_3 + \omega l_4)$, z_i, $l_i \in Z(k)$. Call $\lambda = z_1 + \omega z_2 + \theta z_3 + \theta\omega z_4$. and $\mu = l_1 + \omega l_2 + \theta l_3 + \theta\omega l_4$.
Define the forms q' and f' on V' in the following way :

$$\begin{aligned} g'(x', x') &= -(x'_{-1})^\sigma x'_1 + N(\lambda)v_1 + Tr(\sigma') \\ f'(x', y') &= \lambda^\sigma \mu v_1 - v_1^\sigma \lambda^\sigma \mu. \end{aligned}$$

One easily checks that f' defines a trace valued $(\sigma, 1)$-hermitian form on V' and q' a $(\sigma, 1)$-quadratic form on V' such that $q'(x' + y') = q'(x') + q'(y') + f(x', y')$, $\forall x'$, $y' \in V'$. Moreover by construction one checks that q' is anisotropic on V_0. As $q'(e'_{-1}) = q'(e'_1) = 0$, q' defines a $(\sigma', -1)$-quadratic form on V' of Witt index 1. Put $k' = L_{v_1}$. We can thus consider the hermitian Moufang set $\mathcal{M}H(V', q', k', \sigma')$. When working with $\mathcal{M}H(V', q', k', \sigma')$ we will use in the sequel its coordinatization associated to the decomposition $V' = e'_{-1}L_{v_1} \oplus V_0 \oplus e'_1\ L_{v_1}$. The labelling set is denoted as $R'_{0,1}$. We show that $\mathcal{M}H(V', q', k', \sigma')$ is isomorphic to $\mathcal{M}U(V, -q, k, \sigma)$, which is a Moufang set proportional and hence isomorhphic to $\mathcal{M}U(V, q, k, \sigma)$. When working with $\mathcal{M}U(V, -q, k, \sigma)$ we will use the coordinatization associated to the decomposition $V = (-e_{-1})k \oplus V_0 \oplus e_1 k$. Define the map β from $\mathcal{M}H(V', q', k', \sigma')$ to $\mathcal{M}U(V, -q, k, \sigma)$ in the following way :

$$\begin{aligned} &\beta((v_0(z_1 + \omega z_2) + v_0\theta(z_3 + \omega z_4), N(\lambda)v_1 + u)) \\ &= (v_0\lambda^\sigma, \lambda v_1^\sigma \lambda^\sigma + u) \\ &\beta((\infty)) \\ &= (\infty). \end{aligned}$$

where $u \in Z(k)$ and $\lambda = z_1 + z_2\omega + z_3\theta + z_4\theta\omega$. The construction of $\mathcal{M}H(V', q', k', \sigma')$ implies that β defines a bijection from $\mathcal{M}H(V', q', k', \sigma')$ to $\mathcal{M}U(V, q, k, \sigma)$. We check that β defines a Moufang set isomorphism using Lemma 41. Therefore it will be enough if we show that the map $\beta_{(\infty)}$ determined by :

$$\beta_{(\infty)}(u_\infty) = \beta \circ u_\infty \circ \beta^{-1},\ \forall u_\infty \in U_{(\infty)}$$

defines a map from $U_{(\infty)}$ to $U_{(\infty)}$ and similarly that the map $\beta_{(0,0)}$ determined by :

$$\beta_{(0,0)}(u_0) = \beta \circ u_0 \circ \beta^{-1},\ \forall u_0 \in U_{(0,0)}$$

defines a map from $U_{(0,0)}$ to $U_{(0,0)}$. Firstly we show the claim for $\beta_{(\infty)}$.
Let u, z_i, $l_i \in Z(k)$, $1 \le i \le 4$.
We calculate :

$$\begin{aligned}
&\beta((v_0(z_1+\omega z_2), N(z_1+\omega z_2)v_1) \oplus (v_0(l_1+\omega l_2), N(l_1+\omega l_2)v_1))\\
&= \beta((v_0((z_1+l_1)+\omega(z_2+l_2), N(z_1+\omega z_2)+N(l_1+\omega l_2)\\
&-f'(v_0(z_1+\omega z_2), v_0(l_1+\omega l_2)))\\
&= \beta((v_0((z_1+l_1)+\omega(z_2+l_2)), N((z_1+l_1)+\omega(z_2+l_2))\\
&-(l_1+\omega l_2)^\sigma(z_1+\omega z_2)v_1 - (z_1+\omega z_2)^\sigma(l_1+\omega l_2)v_1^\sigma)\\
&= (v_0((z_1+l_1)+\omega(z_2+l_2)^\sigma, N((z_1+l_1)+\omega(z_2+l_2))v_1^\sigma\\
&-(l_1+\omega l_2)^\sigma(z_1+\omega z_2)v_1 - (z_1+\omega z_2)^\sigma(l_1+\omega l_2)v_1^\sigma))\\
&= (v_0((z_1+l_1)+\omega(z_2+l_2))^\sigma, N(z_1+\omega z_2)v_1^\sigma\\
&+N(l_1+\omega l_2)v_1^\sigma + f(v_0(z_1+\omega z_2)^\sigma, v_0(l_1+\omega l_2)^\sigma))\\
&= (v_0(z_1+\omega z_2)^\sigma, N(z_1+\omega z_2)v_1^\sigma) \oplus (v_0(l_1+\omega l_2)^\sigma, N(l_1+\omega l_2)v_1)\\
&= \beta((v_0(z_1+\omega z_2), N(z_1+\omega z_2)v_1)) \oplus \beta((v_0(l_1+\omega l_2), N(l_1+\omega l_2)v_1))
\end{aligned}$$

where we used the fact that $v_1 = w$ and thus $\beta(v_0(z_1+\omega z_2), N(z_1+\omega z_2)v_1)$ $(v_0(z_1+\omega z_2)^\sigma, N(z_1+\omega z_2)v_1)$ and similarly $\beta(v_0(l_1+\omega l_2), N(l_1+\omega l_2)v_1)$ $= (v_0(l_1+\omega l_2)^\sigma, N(l_1+\omega l_2)v_1)$.
By similar calculations one checks that :

$$\begin{aligned}
&\beta((v_0\theta(z_3+\omega z_4), N(z_3+\omega z_4)v_1) \oplus (v_0\theta(l_3+\omega l_4), N(l_3+\omega l_4)v_1))\\
&= \beta((v_0\theta(z_3+\omega z_4), N(z_3+\omega z_4)v_1)) \oplus \beta((v_0\theta(l_3+\omega l_4), N(l_3+\omega l_4)v_1)).
\end{aligned}$$

Call $\lambda = z_1 + \omega z_2 + \theta z_3 + \theta\omega z_4$.
We find :

$$\begin{aligned}
&\beta((v_0(z_1+\omega z_2), N(z_1+\omega z_2)v_1) \oplus (v_0\theta(z_3+\omega z_4), N(\theta(z_3+\omega z_4)v_1))\\
&= \beta((v_0(z_1+\omega z_2) + v_0\theta(z_3+\omega z_4), N(z_1+\omega z_2) + N(\theta(z_3+\omega z_4))))\\
&= \beta((v_0(z_1+\omega z_2) + v_0\theta(z_3+\omega z_4)), N(z_1+\omega z_2+\theta z_3+\omega\theta z_4))\\
&= (v_0(z_1+\omega z_2+\theta z_3+\theta\omega z_4), \lambda v_1^\sigma \lambda^\sigma)
\end{aligned}$$

and :

$$\beta((v_0(z_1+\omega z_2), N(z_1+\omega z_2)v_1)) \oplus \beta((v_0\theta(z_3+\omega z_4), N(\theta(z_3+\omega z_4))v_1))$$
$$= (v_0(z_1+\omega z_2)^\sigma, (z_1+\omega z_2)v_1^\sigma(z_1+\omega z_2)^\sigma) \oplus (v_0(\theta(z_3+\omega z_4)^\sigma,$$
$$N(\theta(z_3+\omega z_4)v_1^\sigma)$$
$$= (v_0(z_1+\omega z_2+\theta z_3+\theta\omega z_4)^\sigma, (z_1+\omega z_2)v_1^\sigma(z_1+\omega z_2)^\sigma$$
$$+(\theta(z_3+\omega z_4))v_1^\sigma(\theta(z_3+\omega z_4))^\sigma + f(v_0(z_1+\omega z_2)^\sigma, v_0(\theta(z_3+\omega z_4))^\sigma).$$

Let $char(k)=2$ then we find :

$$\lambda v_1^\sigma\lambda^\sigma + (z_1+\omega z_2)v_1^\sigma(z_1+\omega z_2)^\sigma + (\theta(z_3+\omega z_4))v_1^\sigma(\theta(z_3+\omega z_4))^\sigma$$
$$+f(v_0(z_1+\omega z_2)^\sigma, v_0(\theta(z_3+\omega z_4)^\sigma)$$
$$= (z_1+\omega z_2)v_1^\sigma(\theta(z_3+\omega z_4))^\sigma + (\theta(z_3+\omega z_4))v_1^\sigma(z_1+\omega z_2)^\sigma$$
$$+f(v_0(z_1+\omega z_2)^\sigma, v_0(\theta(z_3+\omega z_4))^\sigma)$$
$$= (z_1+\omega z_2)(w+1)(\theta(z_3+\omega z_4))^\sigma + (\theta(z_3+\omega z_4))(w+1)(z_1+\omega z_2)^\sigma$$
$$+(z_1+\omega z_2)(\theta(z_3+z_4\omega))^\sigma$$
$$= (z_1+\omega z_2)\omega(z_3+z_4+\omega z_4)\theta + \theta(z_3+\omega z_4)(w+1)(z_1+z_2+\omega z_2)$$
$$= (z_1+\omega z_2)(z_3\omega+z_4\alpha_0) + \theta(z_3+z_3\omega+z_4\alpha_0)(z_1+z_2+\omega z_2)$$
$$= (z_1+\omega z_2)\theta(z_3\omega+z_3+z_4\alpha_0) + \theta(z_3+\omega z_3+z_4\alpha_0)(z_1+z_2+\omega z_2)$$
$$= 0$$

This means that $\lambda\ v_1^\sigma\ \lambda^\sigma = (z_1+\omega z_2)v_1^\sigma(z_1+\omega z_2)^\sigma\ +(\theta(z_3+\omega z_4))v_1^\sigma$ $(\theta(z_3+\omega z_4))^\sigma\ +f(v_0(z_1+\omega z_2)^\sigma, v_0(\theta(z_3+\omega z_4))^\sigma)$.
If $char(k)\neq 2$ we have for $\lambda = z_1+\omega z_2+\theta(z_3+\omega z_4)$ that the equation :

$$\lambda v_1^\sigma\lambda^\sigma$$
$$= (z_1+\omega z_2)v_1^\sigma(z_1+\omega z_2)^\sigma$$
$$+(\theta(z_3+\omega z_4))v_1^\sigma(\theta(z_3+\omega z_4))^\sigma$$
$$+f(v_0(z_1+\omega z_2)^\sigma, v_0(z_3+\omega z_4)^\sigma)$$

is equivalent to the equation :

$$-(z_1+\omega z_2)f(v_0,v_0)(\theta(z_3+\omega z_4))$$
$$= (z_1+\omega z_2)^\sigma v_1^\sigma(\theta(z_3+\omega z_4))$$
$$+(\theta(z_3+\omega z_4))^\sigma v_1^\sigma(z_1+\omega z_2).$$

We find :

$$(z_1+\omega z_2)^\sigma v_1^\sigma(\theta(z_3+\omega z_4)) + (\theta(z_3+\omega z_4))^\sigma v_1(z_1+\omega z_2)$$
$$= (-z_1+\omega z_2)v_1(\theta(z_3+\omega z_4) + \theta(z_1+\omega z_2)v_1(z_3+\omega z_4)$$

and :

$$\begin{aligned}&-(z_1+\omega z_2)f(v_0,v_0)(\theta(z_3+\omega z_4))\\&=(z_1+\omega z_2)v_1(\theta(z_3+\omega z_4))+\theta(-z_1+\omega z_2)(z_3+\omega z_4),\end{aligned}$$

showing that also in this case $\lambda v_1^\sigma\lambda^\sigma = (z_1+\omega z_2)v_1^\sigma(z_1+\omega z_2) + (\theta(z_3+\omega z_4)v_1^\sigma(\theta(z_3+\omega z_4))^\sigma + f(v_0(z_1+\omega z_2)^\sigma, v_0(\theta(z_3+\omega z_4))^\sigma)$. We thus find that in any case :

$$\begin{aligned}&\beta((v_0(z_1+\omega z_2),N(z_1+\omega z_2)v_1)\oplus(v_0\theta(z_3+\omega z_4),N(\theta(z_3+\omega z_4))v_1))\\&=\beta((v_0(z_1+\omega z_2),N(z_1+\omega z_2)v_1))\oplus((v_0\theta(z_3+\omega z_4),N(\theta(z_3+\omega z_4))v_1)).\end{aligned}$$

Moreover one easily checks that :

$$\begin{aligned}&\beta((v_0(z_1+\omega z_2)+v_0\theta(z_3+\omega z_4),N(\lambda)v_1)\oplus(0,u))\\&=\beta((v_0(z_1+\omega z_2)+v_0\theta(z_3+\omega z_4),N(\lambda)v_1))\oplus\beta((0,u)).\end{aligned}$$

As every element $(w_0', w_1') \in R_{0,1}'$ can be written in a form $(v_0(z_1+\omega z_2),N(z_1+\omega z_2)v_1) \oplus (v_0\theta(z_3+\omega z_4),N(z_3+\omega z_4)v_1) \oplus (0,u)$, with u, $z_i \in Z(k)$ the above equations show that for (w_0', w_1'), $(u_0', u_1') \in R_{0,1}'$, $\beta((w_0', w_1') \oplus (u_0', u_1') = \beta((w_0', w_1')) \oplus \beta((u_0', u_1'))$. For (w_0', w_1'), $(u_0', u_1') \in R_{0,1}'$ the element $u((\infty);(0,0),(u_0', u_1'))$ acts in the following way :

$$u((\infty);(0,0),(u_0',u_1'))((w_0',w_1'))=(u_0',u_1')\oplus(w_0',w_1').$$

It therefore follows that $\beta_{(\infty)}$ defines a mapping from $U_{(\infty)}$ to $U_{(\infty)}$.
We finally prove that $\beta_{(0,0)}$ defines a map from $U_{(0,0)}$ to $U_{(0,0)}$. Suppose that we show that $\beta \circ s_{(0,1)}\ \beta^{-1} = s_{(0,1)}$. As $s_{(0,1)}U_{(\infty)}\ s_{(0,1)} = U_{(0,0)}$ we find for $u_0 \in U_{(0,0)}$ a u_∞ with $s_{(0,1)}\ u_\infty\ s_{(0,1)} = u_0$ and we have :

$$\begin{aligned}\beta\circ u_0\circ\beta^{-1}&=\beta\circ s_{(0,1)}u_\infty s_{(0,1)}\circ\beta^{-1}\\&=s_{(0,1)}\beta\circ u_\infty\circ\beta^{-1}s_{(0,1)}\\&=s_{(0,1)}\beta_{(\infty)}(u_\infty)s_{(0,1)}\\&\in U_{(0,0)}\end{aligned}$$

and hence the proof that β defines an isomorphism is complete.
Remains to show that $\beta\ s_{(0,1)}\ \beta^{-1} = s_{(0,1)}$. Let u, $z_i \in Z(k)$, $1 \le i \le 4$. Call $\lambda = z_1 + \omega z_2 + \theta z_3 + \theta\omega z_4$.

We find :

$\beta(s_{(0,1)}((v_0(z_1+\omega z_2)+v_0\theta(z_3+\omega z_4), N(\lambda)v_1+u))$
$=\beta((v_0(z_1+\omega z_2)(-(N(\lambda)v_1+u))^{-1}+v_0\theta(z_3+\omega z_4)(-(N(\lambda)v_1+u))^{-1},$
$(-(N(\lambda)v_1+u))^{-1}))$
$=\beta((v_0(z_1+\omega z_2)(-(N(\lambda)v_1+u))^{-1}+v_0\theta(z_3+\omega z_4)(-(N(\lambda)v_1+u))^{-1},$
$N(\lambda(N(\lambda)v_1+u))^{-1}-N(\lambda)(v_1+v_1^\sigma)N(N(\lambda)v_1+u))^{-1}-u(N(\lambda)v_1+u))^{-1}))$
$=(-v_0(\lambda(N(\lambda)v_1+u)^{-1})^\sigma, \lambda(N(\lambda)v_1+u))^{-1}v_1^\sigma(\lambda(N(\lambda)v_1+u))^{-1})^\sigma$
$-N(\lambda)(v_1+v_1^\sigma)N(N(\lambda)v_1+u))^{-1}-u(N(\lambda)v_1+u))^{-1}))$

We have :

$$\begin{aligned}
(\lambda(N(\lambda)v_1+u)^{-1})^\sigma &= (N(\lambda)v_1^\sigma+u)^{-1}\lambda^\sigma \\
&= ((\lambda^\sigma)^{-1}N(\lambda)v_1^\sigma+u(\lambda^\sigma)^{-1})^{-1} \\
&= (\lambda v_1^\sigma+u(\lambda^\sigma)^{-1})^{-1} \\
&= \lambda^\sigma(\lambda v_1^\sigma\lambda^\sigma+u)^{-1}
\end{aligned}$$

implying that :

$$\begin{aligned}
-(\lambda(N(\lambda)v_1+u))^{-1} &= -\lambda^\sigma(\lambda v_1^\sigma\lambda^\sigma+u)^{-1} \\
N(N(\lambda)v_1+u)) &= N(\lambda v_1^\sigma\lambda^\sigma+u).
\end{aligned}$$

Moreover :

$$\begin{aligned}
&\lambda(N(\lambda)v_1+u))^{-1}v_1^\sigma(\lambda(N(\lambda)v_1+u)^{-1})^\sigma \\
&-N(\lambda)(v_1+v_1^\sigma)(N(N(\lambda)v_1+u))^{-1}-u(N(N(\lambda)v_1+u))^{-1} \\
&=\lambda(N(\lambda)v_1^\sigma+u)v_1^\sigma(N(\lambda)v_1+u)(N(N(\lambda)v_1+u))^{-2} \\
&-N(\lambda)(v_1+v_1^\sigma)(N(N(\lambda)v_1+u))^{-1}-u(N(N(\lambda)v_1+u))^{-1} \\
&=\lambda v_1^\sigma(N(N(\lambda)v_1+u))^{-1}-N(\lambda)(v_1+v_1^\sigma)(N(N(\lambda)v_1+u))^{-1} \\
&-u(N(N(\lambda)v_1+u))^{-1} \\
&=-(\lambda v_1\lambda^\sigma+u)(N(N(\lambda)v_1+u))^{-1} \\
&=-(\lambda v_1\lambda^\sigma+u)(N(\lambda v_1\lambda^\sigma+u))^{-1} \\
&=-(\lambda v_1^\sigma\lambda^\sigma+u)^{-1}
\end{aligned}$$

where we used the identity $\lambda v_1+v_1^\sigma\lambda^\sigma=N(\lambda)(v_1+v_1^\sigma)$ and the fact that $N(N(\lambda)v_1+u))=N(\lambda v_1\lambda^\sigma+u)$.

Thus we find that :

$$\begin{aligned}
&\beta(s_{(0,1)}((v_0(z_1+\omega z_2)+v_0\theta(\theta z_3+\theta\omega z_4), N(\lambda)v_1+u)))\\
&= (-v_0(\lambda^\sigma(\lambda v_1^\sigma\lambda^\sigma+u)^{-1}, (-\lambda v_1^\sigma\lambda^\sigma+u)^{-1})\\
&= s_{(0,1)}(v_0\lambda^\sigma, \lambda v_1^\sigma\lambda^\sigma+u)\\
&= s_{(0,1)}\beta((v_0(z_1+\omega z_2)+v_0\theta(\theta z_3+\theta\omega z_4), N(\lambda)v_1+u))).
\end{aligned}$$

This proves that $\beta\ s_{(0,1)}\ \beta^{-1} = s_{(0,1)}$. By Lemma 41 we find that β defines a Moufang set isomorphism from $\mathcal{M}H(V', q', Z(k), \sigma')$ to $\mathcal{M}U(V, -q, k, \sigma)$. As $\mathcal{M}U(V, -q, k, \sigma)$ is isomorphic to $\mathcal{M}U(V, q, k, \sigma)$ (cfr. see section 3.12.3) under ψ_{-1} with :

$$\begin{aligned}
\psi_{-1}((\infty)) &= (\infty)\\
\psi_{-1}((v_0, v_1)) &= (v_0, -v_1)
\end{aligned}$$

we find that $\beta^* = \psi_{-1}\beta$ defines an isomorphism from $\mathcal{M}H(V', q', k', \sigma')$ to $\mathcal{M}U(V, q, k, \sigma)$.

Conversely let $\mathcal{M}H(V, q, k, \sigma)$ be hermitian Moufang set with $dim(V) = 4$. As usual we assume that q is a $(\sigma, -1)$-quadratic form. Choose a coordinatization of $\mathcal{M}H(V, q, k, \sigma)$ associated to the decomposition $V = e_{-1}k \oplus V_0 \oplus e_1k$. Assume that g is a σ sesquilinear form such that $q(v) = g(v, v) + Fix(\sigma)$ and f the $(\sigma, -1)$-hermitian form satisfying $q(v+w) = q(v)+q(w)+f(v, w)$, $\forall v, w \in V$. Let v_0 be a vector of $V_0 \setminus \{0\}$ and put $g(v_0, v_0) = v_1$. As q is anisotropic on V_0 we find that $v_1 \notin Fix(\sigma)$. Hence $Fix(\sigma)(v_1) = k$. Without loss of generality we can assume that $v_1^2 = v_1 + \alpha_0$, if $char(k) = 2$ and $v_1^2 = \alpha_0$ if $char(k) \neq 2$ with $\alpha_0 \in Fix(\sigma) \setminus (Fix(\sigma))^2$. Let w_0 be a vector in V_0 such that $f(v_0, w_0) = 0$. Then we have $V_0 = v_0k \oplus w_0k$. Put $g(w_0, w_0) = w_1$. As $w_1 = v_1z_1 + z_2$ for some $z_i \in Fix(\sigma)$ and $Fix(\sigma) = Tr(\sigma)$ we can assume without loss of generality that $w_1v_1^{-1} = \beta_0 \in Fix(\sigma)$. As $v_0 \notin \langle w_0\rangle$ and $f(v_0, w_0) = 0$ we find that $\beta_0 \notin (Fix(\sigma))^2$. (Otherwise we would have $q(v_0\mu + w_0) = 0$ for a $\mu \in k$ and $v_0\mu = w_0$). Let k' be the generalized quaternion algebras with center $Fix(\sigma)$ constructed in the following way. Put $k' = Fix(\sigma) \oplus v_1Fix(\sigma) \oplus \theta' Fix(\sigma) \oplus \theta' v_1 Fix(\sigma)$ with :

$char(k') = 2$ and :
$(\theta')^2 = \beta_0$, and $\theta' v_1 = v_1\theta' + \theta'$.

$char(k) \neq 2$ and :
$\theta'^2 = \beta_0$, $v_1\theta' = -\theta' v_1$.

Denote the standard involution of k' by σ' and let N' be the norm function on k'. Remark that every element of $\mathcal{M}H(V,q,k,\sigma)$ can then be written as $((v_0(z_1+v_1z_2)+w_0(z_3+v_1z_4), N'(z_1+v_1z_2+\theta' z_3+\theta' v_1 z_4)+u)$, $u \in Fix(\sigma)$. By the construction of k' we see that k is embedded in k' and $\sigma'|_k = \sigma$. Define the unitary Moufang set $\mathcal{M}U(V',q',k',\sigma')$ in the following way. We set $V' = e'_{-1}k' \oplus v_0 k' \oplus e'_1 k'$. Let $x' = e'_{-1}x'_{-1} + v_0\lambda' + e'_1 x'_1$ and $y' = e'_{-1}y'_{-1} + v_0\mu' + e'_1 y'_1$, with x'_{-1}, x'_1, y'_{-1}, y'_1, λ', $\mu' \in k'$.
Then we set :

$$\begin{aligned} q'(x',x') &= -(x'_{-1})^{\sigma'}x'_1 - \lambda'^{\sigma}v_1^{\sigma}\lambda' + Tr(\sigma') \\ f'(x',y') &= -(x'_{-1})^{\sigma}y'_1 + (y'_{-1})^{\sigma}x'_1 + \lambda'^{\sigma'}(-v_1+v_1^{\sigma})\mu' \end{aligned}$$

Then one easily checks that q' defines a $(\sigma',-1)$-quadratic form on V' of Witt index 1 with associated trace valued $(\sigma',-1)$-hermitian form f'. Therefore we can consider the Moufang set $\mathcal{M}U(V',q',k',\sigma')$. Coordinatize this set using the decomposition $V' = e'_{-1}k' + V'_0 + e'_1 k'$ where $V'_0 = v'_0 k$.
Define the map β from $\mathcal{M}H(V,q,k,\sigma)$ to $\mathcal{M}U(V',q',k',\sigma')$ by :

$$\begin{aligned} &\beta((\infty)) \\ &= (\infty) \\ &\beta((v_0(z_1+v_1z_2)+w_0(z_2+v_1z_4), N'(z_1+v_1z_2+\theta' z_3+\theta' v_1 z_4)) \\ &= (v_0(\lambda')^{\sigma'}, \lambda' v_1^{\sigma'}(\lambda')^{\sigma'}). \end{aligned}$$

The calculations used to prove the first part of the Lemma show that β defines a Moufang set isomorphism from $\mathcal{M}H(V,q,k,\sigma)$ to $\mathcal{M}U(V',q',k',\sigma')$. This completes the proof. □

Lemma 115 *Let k be a generalized quaternion algebra in characteristic non 2 and σ a non standard involution Then every polar line $Pol(k,\sigma)$ is isomorphic to an non commutative orthogonal Moufang set $\mathcal{M}O(V',q',Z(k))$ with $dim(V') = 5$.*

Without loss of generality we can choose i, j and $\in k$ with $i^2 = \alpha_0$, $j^2 = \beta_0$, $ij = -ji$, $k = Z(k) \oplus iZ(k) \oplus jZ(k) \oplus jiZ(k)$ such that σ is given by :

$$(z_1 + iz_2 + jz_3 + jiz_4)^{\sigma} = z_1 - iz_2 + jz_3 + jiz_4, \; \forall z_i \in Z(k), 1 \le i \le 4.$$

Denote the standard involution in k by γ and the norm function in k by N. Let $Pol(k,\sigma)$ be a polar line defined by a $(\sigma,-1)$ quadratic form. Choose a coordinatization of $Pol(k,\sigma)$ associated to a decomposition $V' = e'_{-1}k \oplus e'_1k$. Then we find that the point set of $Pol(k,\sigma)$ is given by $\{(0,t)|t \in Fix(\sigma)\}$ $= \{(0, z_1{+}jz_3{+}jiz_4)\,|z_i \in Z(k)$. In order to construct an orthogonal Moufang set $\mathcal{MO}(V',q',Z(k))$ we proceed as follows. Let V' be the 5-dimensional vectorspace $e'_{-1}Z(k) \oplus V'_0 \oplus e'_1Z(k)$ with $V'_0 = {e'_0}^1Z(k) \oplus {e'_0}^2\, Z(k) \oplus {e'_0}^3Z(k)$. Define the forms g', f' and q' on V' as follows. Let $x' = e'_{-1}x'_{-1} + x'_0 + e'_1x'_1$ with $x'_0 = {e'_0}^1z'_1 + {e'_0}^2z'_2 + {e'_0}^3z'_3$ and put $\lambda' = z'_1 + jz'_2 + jiz'_3$. Let $y' = e'_{-1}y'_{-1} + y'_0 + e'_1y'_1$ with $y'_0 = {e'_0}^1u'_1 + {e'_0}^2u'_2 + {e'_0}^3u'_3$ and put $\mu' = u'_1 + ju'_2 + jiu'_3$.

$$\begin{aligned} g'(x',x') &= x'_{-1}x'_1 + N(\lambda') \\ f'(x',y') &= y'_{-1}x'_1 + x'_{-1}y'_1 + \lambda'^{\gamma}\mu' + \mu'^{\gamma}\lambda' \\ q'(x') &= g'(x',x') \end{aligned}$$

One easily checks that q' defines a quadratic form on V' of Witt index 1. Therefore we can consider the Moufang set $\mathcal{MO}(V',q',Z(k))$. Coordinatize $\mathcal{MO}(V',q',Z(k))$ using the decomposition $V' = e'_{-1}Z(k) \oplus V'_0 \oplus e'_1Z(k)$ with labelling set $R'_{0,1}$. Define the bijection β from the point set of $\mathcal{MO}(V',q',Z(k))$ to $Pol(k,\sigma)$ in the following way :

$$\begin{aligned} \beta((\infty)) &= (\infty) \\ \beta(({e'_0}^1z_1 + {e'_0}^2z_2 + {e'_0}^3z_3, -N(\lambda')) &= (0,\lambda) \end{aligned}$$

where we put $\lambda' = z'_1{+}jz'_2 + jiz'_3$. We check that β defines a Moufang set isomorphism using Lemma 41. It will thus be enough to prove that the map $\beta_{(\infty)}$ given by :

$$\beta_{(\infty)}(u_\infty) = \beta \circ u_\infty \circ \beta^{-1},\ \forall u_\infty \in U_{(\infty)}$$

defines a map from $U_{(\infty)}$ to $U_{(\infty)}$ and similarly that the map $\beta_{(0,0)}$ given by :

$$\beta_{(0,0)}(u_0) = \beta \circ u_0 \circ \beta^{-1}$$

defines a map from $U_{(0,0)}$ to $U_{(0,0)}$.
Let (v'_0, v'_1), $(t'_0, t'_1) \in R'_{0,1}$ with $v'_0 = {e'_0}^1z_1 + {e'_0}^2z_2 + {e'_0}^3z_3$ and $v'_1 = -N(\lambda')$

where we put $\lambda' = z_1 + jz_2 + jiz_3$, $t'_0 = {e'_0}^1 u_1 + {e'_0}^2 u_2 + {e'_0}^3 u_3$ and $t'_1 = -N(\mu')$ where we put $\mu' = u_1 + ju_2 + jiu_3$.
We calculate :

$$\begin{aligned}
&\beta(u((\infty);(0,0),(t'_0,t'_1))((v'_0,v'_1)))\\
&= \beta((t'_0+v'_0, -N(\lambda') - N(\mu') - \lambda'^{\gamma}\mu' - \mu'^{\gamma}\lambda'))\\
&= \beta((t'_0+v'_0, -N(\lambda'+\mu'))\\
&= (\lambda'+\mu')\\
&= u((\infty);(0,0),(0,\mu'))((0,\lambda'))
\end{aligned}$$

showing that $\beta \circ u((\infty);(0,0),(t'_0,t'_1)) \circ \beta^{-1} = u((\infty);(0,0),\beta((t'_0,t'_1)))$. As for $u_{(0,0)}$ we reason as follows. For $\mathcal{M}O(V',q',Z(k))$ we have that $s_{({e'_0}^1,-1)}$ $U_{(\infty)}$ $s_{({e'_0}^1,-1)} = U_{(0,0)}$ and similarly for $\mathcal{M}U(V',q',k',\sigma')$ we find $s_{(0,1)}$ $U_{(\infty)}$ $s_{(0,1)} = U_{(0,0)}$. Moreover by construction of β we find $\beta(({e'_0}^1,-1))) = (0,1)$. In order to show that $\beta_{(0,0)}$ defines a map from $U_{(0,0)}$ to $U_{(0,0)}$ it will therefore be enough if we show that $\beta \circ s_{({e'_0}^1,-1)} \circ \beta^{-1} = s_{(0,1)}$. Let (v'_0,v'_1) be as above i.e. $v'_0 = {e'_0}^1 z_1 + {e'_0}^2 z_2 + {e'_0}^3 z_3$ and $v'_1 = -N(\lambda')$ where we put $\lambda' = z_1 + jz_2 + jiz_3$.
We have :

$$\begin{aligned}
&\beta(s_{({e'_0}^1,-1)}((v'_0,v'_1))\\
&= \beta((-{e'_0}^1 f'({e'_0}^1, v'_0)(N(\lambda))^{-1} + v'_0(N(\lambda))^{-1}, (-N(\lambda))^{-1})\\
&= \beta((-({e'_0}^1 z_1 - {e'_0}^2 z'_2 - {e'_0}^3 z_3)(N(\lambda'))^{-1}, -(N(\lambda'))^{-1}\\
&= (-\lambda'^{\gamma}(N(\lambda'))^{-1}, -(N(\lambda'))^{-1})\\
&= (\lambda')^{-1}\\
&= s_{(0,1)}(\lambda')\\
&= s_{(0,1)}\beta((v'_0,v'_1)
\end{aligned}$$

proving that $\beta \circ s_{({e'_0}^1,-1)} \circ \beta^{-1} = s_{(0,1)}$. That $\mathcal{M}O(V',q',Z(k))$ is non commutative follows from the fact that it is isomorphic to $\mathcal{M}(V,q,k,\sigma)$ which is non commutative.

□

As to orthogonal Moufang sets over small fields we have the following lemma.

Lemma 116 *Let $\mathcal{M}O(V,q,k)$ be an orthogonal Moufang set such that $k = \mathbb{F}_2$ or $\mathbb{F}_3$, then one of the following possibilities occurs :*

(i) $k = \mathbb{F}_2$, $dim(V) = 3$ and $\mathcal{M}O(V,q,k) \cong \mathcal{P}(\mathbb{F}_2)$,

(ii) $k = \mathbb{F}_2$, $dim(V) = 4$ *and* $\mathcal{MO}(V, q, k) \cong \mathcal{P}(\mathbb{F}_4)$,

(iii) $k = \mathbb{F}_3$, $dim(V) = 3$ *and* $\mathcal{MO}(V, q, k) \cong \mathcal{P}(\mathbb{F}_3)$,

(iv) $k = \mathbb{F}_3$, $dim(V) = 4$ *and* $\mathcal{MO}(V, q, k) \cong \mathcal{P}(\mathbb{F}_9)$.

proof :

Let $\mathcal{MO}(V, q, k)$ be a in the lemma and choose a coordinatization of $\mathcal{MO}(V, q, k)$ associated to the decomposition $V = e_{-1}k \oplus V_0 \oplus e_1 k$ with labelling set $R_{0,1}$. Suppose firstly that $k = \mathbb{F}_2$.
We first show that if $dim(V) \geq 4$, $codim(Rad(f)) \neq 2$. If this where the case we would find at least two vectors v_0 and $w_0 \in V_0$ with $v_0 \neq w_0$ such that $f(v_0, w_0) = 0$ and $q(v_0) = q(w_0) = 1$. But then the equation $q(v_0 + w_0)$ $= q(v_0) + q(w_0) = 0$ implies $v_0 = w_0$ a contradiction. Suppose $dim(V) \neq 5$. Let v_0. As $dim(V_0) \geq 3$, $dim(v_0^\perp \cap V_0) \geq 2$. This means that there exists a $w_0 \neq v_0$ such that $f(v_0, w_0) = 0$. But then the equation $q(v_0 + w_0) = q(v_0)$ $+ q(w_0) = 0$ leads to $v_0 = w_0$, a contradiction. Therefore the only possiblities left are $dim(V) = 3$ and by Lemma 111, $\mathcal{MO}(V, q, k) \cong \mathcal{P}(\mathbb{F}_2)$ or $codim(Rad(f)) \neq 2$, $dim(V) = 4$ and $\mathcal{MO}(V, q, k) \cong \mathcal{P}(\mathbb{F}_4)$ by Lemma 112. Subsequently we assume that $k = \mathbb{F}_3$.
Suppose $dim(V) \geq 5$. Let v_0, $w_0 \in V_0$ such that $f(v_0, w_0) = 0$. Without loss of generality we can assume that $q(v_0) = 1$ and $q(w_0) = -1$. As $dim(V_0)$ ≥ 3, we find that $dim(v_0^\perp \cap w_0^\perp \cap V_0) = 1$. Let $t_0 \in v_0^\perp \cap w_0^\perp \cap V_0$. Then there are two possible choices for $q(t_0)$. If $q(t_0) = 1$ the equation $q(t_0 + w_0) = 0$ leads to $t_0 = w_0$, a contradiction. And if $q(t_0) = -1$, we have $q(v_0 + t_0) = 0$ and hence a $v_0 = t_0$, a contradiction. This shows that $dim(V) \leq 4$. Thus we have two possibilities. Or $dim(V) = 3$, and $\mathcal{MO}(V, q, k) \cong \mathbb{F}_3$ by Lemma 111, or $dim(V) = 4$ and $\mathcal{MO}(V, q, k) \cong \mathbb{F}_9$. □

Lemma 117 *Consider a orthogonal Moufang set* $\mathcal{M}(O(V, q, k))$. *Suppose* f *is the form associated to* q. *Assume* $5 \leq dim(V) < \infty$ *if* $Rad(f) = 0$, $codim(Rad(f)) \geq 4$ *if* $Rad(f) \neq 0$ *and* $k \neq \mathbb{F}_2$. *Then* $TO(V, q, k) = [PGO(V, q, k), PGO(V, q, k)]$.

proof :

Choose a decomposition of V such that $V = e_{-1}k \oplus V_0 \oplus e_1 k$, with associated coordinatization over the labelling set $R_{0,1} = \{(v_0, v_1) \in V_0 \times k \mid q(v_0) + v_1 = 0\}$. In particular this means that the point set of $\mathcal{M}O(V, q, k)$ can be written as $\{(v_0, v_1) | (v_0, v_1) \in R_{0,1}\} \cup \{(\infty)\}$ with :

$$\begin{aligned} (v_0, v_1) &= \langle e_{-1}v_1 + v_0 + e_1\rangle, \forall (v_0, v_1) \in R_{0,1} \\ (\infty) &= \langle e_{-1}\rangle. \end{aligned}$$

Suppose firstly that $char(k) \neq 2$.
As every $u((\infty);(0,0)\ ,(v_0,\ v_1))$ with $v_0 \neq 0$ equals $u((\infty);(0,0),\ (v_0 1/2, v_1 1/4))^2$, Proposition 10 in [7] implies $U_{(\infty)} \subset [PGO(V, q, k), PGO(V, q, k)]$. Similar calculations yield $U_{(0,0)} \subset [PGO(V, q, k), PGO(V, q, k)]$. Theorem 2 of loc. cit. states $[PGO(V, q, k), PGO(V, q, k)]$ is simple. As $TO(V, q, k)$ is generated by $U_{(\infty)}$ and $U_{(0,0)}$ and is normalized by $PGO(V, q, k)$ (cfr section 3.12.2) it follows that $[PGO(V, q, k),\ PGO(V, q, k)] = TO(V, q, k)$.
Suppose $char(k) = 2$.
Choose $(v_0, v_1) \in R_{0,1} \setminus \{(0, 0)\}$ arbitrarily.
By Lemma 107 we find that $s_{(v_0,v_1)}s_{(v_0v_1^{-1},v_1^{-1})}$ has as matrix representation with respect to the ordered base $\{e_{-1},\ B_0,\ e_1\}$:

$$\begin{pmatrix} v_1^2 & 0 & 0 \\ 0 & 1_{|B_0|} & 0 \\ 0 & 0 & v_1^{-2} \end{pmatrix}.$$

As in the non characteristic 2 case $[O(V, q, k), O(V, q, k)]$ contains all squares of linear transformations preserving the form q (see Proposition 15 in [6], for the degenerate case a similar proof holds). In particular $s_{(v_0,v_1)}s_{(v_0v_1^{-1},v_1^{-1})} \in [PGO(V, q, k), PGO(V, q, k)]$.
By the definition of s_x we find :

$$\begin{aligned} & u((\infty); (0, 0), (v_0, v_1))s_{(v_0,v_1)}s_{(v_0v_1^{-1},v_1^{-1})}u((\infty); (0, 0), (v_0v_1^{-1}, v_1^{-1})) \\ & = [u(((\infty); (0, 0), (v_0, v_1), u((0, 0); (\infty), (v_0, v_1))] \\ & [u((0, 0); (\infty), (v_0v_1^{-1}, v_1^{-1})), u((\infty); (0, 0), (v_0v_1^{-1}, v_1^{-1}))] \\ & \in [PGO(V, q, k), PGO(V, q, k)]. \end{aligned}$$

Multiplication of $u((\infty); (0, 0), (v_0, v_1))\ s_{(v_0,v_1)}\ s_{(v_0v_1^{-1},v_1^{-1})}$ $u((\infty), (0, 0); (v_0v_1^{-1},\ v_1^{-1}))$ on the right with $(s_{(v_0,v_1)}\ s_{(v_0v_1^{-1},v_1^{-1})})^{-1}$ implies :

$$\begin{aligned} & u((\infty); (0, 0), (v_0, v_1))u((\infty); (0, 0), (v_0v_1, v_1^3)) \\ = \quad & u((\infty); (0, 0), (v_0(1 + v_1), v_1(1 + v_1^2))) \qquad (3.2) \\ & \in [PGO(V, q, k), PGO(V, q, k)] \qquad (3.3) \end{aligned}$$

Two cases occur.

1. $v_1 \neq 1$.
Applying formula (3.3) for $u((\infty);(0,0),(a_0a_1,a_1^3))$ gives $u((\infty);(0,0),(v_0v_1(1+v_1^3),v_1^3(1+v_1^6))) \in [PGO(V,q,k),PGO(V,q,k)]$.
This yields :

$$u((\infty);(0,0),(v_0(1+v_1),v_1(1+v_1^2))u((\infty);(0,0),(v_0v_1(1+v_1^3),v_1^3(1+v_1^6))) \\ = u((\infty);(0,0),(v_0(1+v_1)^4,v_1(1+v_1)^8) \in [PGO(V,q,k),PGO(V,q,k)].$$

Conjugating $u((\infty);(0,0),(v_0(1+v_1)^4,v_1(1+v_1)^8)$ with the transformation with matrix representation :

$$\begin{pmatrix} (1+v_1)^{-4} & 0 & 0 \\ 0 & 1_{|B_0|} & 0 \\ 0 & 0 & (1+v_1)^4 \end{pmatrix}$$

belonging to $[PGO(V,q,k),PGO(V,q,k)]$ gives $u(\infty;(0,0),(v_0,v_1)) \in [PGO(V,q,k), PGO(V,q,k)]$.

2. $v_1 = 1$.
Granted the conditions on k there is at least one $\lambda \in k$ such that $\lambda^4 \neq 1$. Consider $(a_0\lambda^2,a_1\lambda^4) \in R_{0,1}$. By what is already proved we find $u(\infty, (0,0), (a_0\lambda^2,a_1\lambda^4)) \in [PGO(V,q,k),PGO(V,q,k)]$. Conjugating $u(\infty;(0,0), (a_0\lambda^2,a_1\lambda^4))$ with the matrix :

$$\begin{pmatrix} \lambda^2 & 0 & 0 \\ 0 & 1_{|B_0|} & 0 \\ 0 & 0 & \lambda^{-2} \end{pmatrix}$$

of $[PGO(V,q,k),PGO(V,q,k)]$ yields $u(\infty;(0,0),(a_0,a_1)) \in [PGO(V,q,k),PGO(V,q,k)]$. It follows that $U_{(\infty)} \subset [PGO(V,q,k),PGO(V,q,k)]$. Complete analogously one deduces $U_{(0,0)} \subset [PGO(V,q,k),PGO(V,q,k)]$, hence $TO(V,q,k) \subset [PGO(V,q,k),PGO(V,q,k)]$. Finally the simplicity of $[PGO(V,q,k),PGO(V,q,k)]$ (cfr Theorem 3 and Theorem 4 in [6]) shows $TO(V,q,k) = [PGO(V,q,k), PGO(V,q,k)]$. □

Lemma 118 *A hermitian Moufang set $\mathcal{M}H(V,q,k,\sigma)$ has commutative root groups if and only if $dim(V)=2$ and $\mathcal{M}H(V,q,k,\sigma)\cong\mathcal{P}(Fix(\sigma))$.*

proof :

Let $\mathcal{M}H(V,q,k,\sigma)$ be a hermitian Moufang set with commutative root groups. Choose a coordinatization of $\mathcal{M}H(V,q,k,\sigma)$ associated to the decomposition $V=e_{-1}k\oplus V_0\oplus e_1k$ with labelling set $R_{0,1}$ $\{(v_0,v_1)\in V_0\times k \mid q(v_0)+v_1=0\}$. Lemma 104 shows that in this case $dim(V)=2$. But this means that the point set of $\mathcal{M}H(V,q,k,\sigma)$ is given by $\{(0,t)\mid t\in Fix(\sigma)\}\cup\{(\infty)\}$. Consider the Moufang set $\mathcal{P}(Fix(\sigma))$ labelled in a canonical way. Define the bijection β from $\mathcal{P}(Fix(\sigma))$ to $\mathcal{M}H(V,q,k,\sigma)$ by :

$$\begin{aligned}\beta(t) &= (0,t),\ \forall t\in Fix(\sigma)\\ \beta(\infty) &= (\infty).\end{aligned}$$

Using Lemma 41 one easily checks that β defines a Moufang set isomorphism. That the converse holds i.e. if $\mathcal{M}H(V,q,k,\sigma)$ is isomorphic to $\mathcal{P}(Fix(\sigma))$ then it has commutative root groups is clear. □

Lemma 119 *Let $\mathcal{P}(k)$ be a projective Moufang set. Then $Z(k)=k$ if and only if $\mathcal{P}(k)$ is commutative.*

proof :

Coordinatize $\mathcal{P}(k)$ in a canonical way. Without loss of generality we can assume $x=(\infty)$ and $y=(0)$. Every element of $Fix_{T\mathcal{P}(k)}\{x,y\}$ has matrix representation :

$$\begin{pmatrix} t_1 & 0\\ 0 & t_2\end{pmatrix}$$

with t_1, $t_2\in Z(k)$ and $t_1t_2\in[k,k]$.
If $Z(k)=k$ then clearly $Z(Fix_{T\mathcal{P}(K)}\{x,y\})=Fix_{T\mathcal{P}(K)}\{x,y\}$.
As to the converse, the conditions of the lemma yield that $Z(Fix_{T\mathcal{P}(k)}\{x,y\})=Fix_{T\mathcal{P}(k)}\{x,y\}$. This means that for every $r,t,z\in k$, with $r\neq 0$ and $t\neq 0$:

$$\begin{aligned}(rtztr) &= (s_{(r)}s_{(1)})(s_{(t)}s_{(1)})((z))\\ &= (s_{(t)}s_{(1)})(s_{(r)}s_{(1)})((z))\\ &= (trzrt).\end{aligned}$$

Or equivalently :

$$r^{-1}t^{-1}rtztrt^{-1}r^{-1} = z, \forall z \in k.$$

If $z = 1$ this implies $trt^{-1}r^{-1} = (r^{-1}t^{-1}rt)^{-1}$ and hence $r^{-1}t^{-1}rt = [r^{-1}, t^{-1}] \in Z(k)$, $\forall\ r,\ t \in k$. Lemma 50 implies that $Z(k) = k$. □

Lemma 120 *An orthogonal Moufang set $\mathcal{M}(O(V, q, k))$ with associated bilinear form f is commutative if and only if $dim(V) = 3$, $dim(V) = 4$ or $codim(Rad(f)) = 2$.*

proof :

Choose a coordinatization of the Moufang set associated to the decomposition $V = e_{-1}k \oplus V_0 \oplus e_1k$ with labelling set $R_{0,1} = \{(v_0, v_1) \in V_0 \times k | q(v_0) + v_1 = 0\}$. Suppose $dim(V) \geq 5$ and $codim(Rad(f)) \neq 2$.
Remark that then Lemma 116 implies that $k \neq \mathbb{F}_2$ or $\mathbb{F}_3$. This implies that there exists at least one subspace $\bar{V} \subset V$ such that $dim(\bar{V}) = 5$ and $codim_{\bar{V}}(Rad(f)) \neq 2$. Consider the Moufang subset $\mathcal{M}(\bar{V}, q, k)$. As this is a Moufang subset of a commutative Moufang set it should itself be a commutative Moufang set. This means that we can reduce the situation to the case where $dim(V) = 5$, and $codim(Rad(f)) \neq 2$. We prove that the commutativity of the Moufang set leads to a contradiction.
Two cases occur.
First case : $Rad(f) \neq 0$
This means $dim(Rad(f)) = 1$ and there exist at two points (a_0, a_1) and $(b_0, b_1) \in R_{0,1}$ such that $a_0k \oplus Rad(f) \oplus b_0k = V_0$. Let $Rad(f) = \langle r_0 \rangle$. Set $e_0^1 = a_0$, $e_0^2 = b_0$ and $e_0^3 = r_0$ and denote the ordered base $\{e_0^1,\ e_0^2,\ e_0^3\}$ of V_0 as B_0. Using Lemma 106 the automorphism $s_{(a_0,a_1)}s_{(b_0,b_1)}$ has a matrix representation with respect to the ordered base $\{e_{-1}, B_0, e_1\}$ of the form :

$$\begin{pmatrix} a_1b_1^{-1} & 0 & 0 & 0 & 0 \\ 0 & z_1 & z_2 & 0 & 0 \\ 0 & z_3 & z_4 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & a_1^{-1}b_1 \end{pmatrix}$$

with $z_i \in k, 1 \leq i \leq 4$. Choose $(c_0, c_1), (d_0, d_1) \in R_{0,1}$ with $c_0 \notin Rad(f)$ and $d_0 \in Rad(f)$.

Then $s_{(c_0,c_1)}s_{(d_0,d_1)}$ has a matrix representation of the form with respect to the base $B = \{e_{-1}, B_0, e_1\}$:

$$\begin{pmatrix} c_1 d_1^{-1} & 0 & 0 & 0 & 0 \\ 0 & x_1 & y_1 & 0 & 0 \\ 0 & x_2 & y_2 & 0 & 0 \\ 0 & x_3 & y_3 & 1 & 0 \\ 0 & 0 & 0 & 0 & c_1^{-1} d_1 \end{pmatrix}$$

with $\mu, x_i, y_i \in k$, $1 \leq i \leq 3$. Expressing that $s_{(a_0,a_1)}s_{(b_0,b_1)}$ and $s_{(c_0,c_1)}s_{(d_0,d_1)}$ commute in their action on the Moufang set translates in the following set of equations :

$$x_3 z_1 + y_3 z_3 = x_3 \tag{3.4}$$

$$x_3 z_2 + y_3 z_4 = y_3 \tag{3.5}$$

Let $c_0 = \sum_{j=1}^{3} e_0^j c_0^j$. We calculate x_3 and y_3 :
Using Lemma 106 we find :

$$\begin{aligned} s_{(c_0,c_1)}s_{(d_0,d_1)}(e_0^1) &= c_0 f(e_0^1, c_0) c_1^{-1} + e_0^1 \\ &= \sum_{j=1}^{3} e_0^j f(e_0^1, e_0^2) c_0^j c_0^2 c_1^{-1} + e_0^1 \end{aligned}$$

Thus $x_3 = c_0^2 c_0^3 c_1^{-1} f(e_0^1, e_0^2)$.
In a similar way one calculates :

$$s_{(c_0,c_1)}s_{(d_0,d_1)}(e_0^2) = \sum_{j=1}^{n-2} e_0^j f(e_0^1, e_0^2) c_0^j c_0^1 c_1^{-1} + e_0^2.$$

and $y_3 = c_0^1 c_0^3 c_1^{-1} f(e_0^1, e_0^2)$.
Filling in these two expressions in the formula concerning commutativity gives :

$$c_0^2 c_0^3 c_1^{-1} f(e_0^1, e_0^2) z_1 + c_0^1 c_0^3 c_1^{-1} f(e_0^1, e_0^2) z_3 = f(e_0^1, e_0^2) c_0^2 c_0^3 c_1^{-1}.$$

Hence :

$$c_0^2 z_1 + c_0^1 z_3 = c_0^2$$

where we used the fact that by the choice of e_0^1 and e_0^2, $f(e_0^1, e_0^2) \neq 0$. As $\{e_0^j \,|\, 1 \leq j \leq m\}$ forms a base, we can choose c_0^1 and c_0^2 to be arbitrary elements of k. This means in particular that :

$$xz_1 + yz_3 = x, \; \forall x, y \in k$$

Thus $z_1 = 1$ and $z_3 = 0$.
In a completely analogous way one deduces from equation (3.5) that $z_2 = 0$ and $z_4 = 1$.
Thus $s_{(a_0,a_1)}s_{(b_0,b_1)}$ has matrix representation of the form :

$$\begin{pmatrix} \lambda & 0 & 0 \\ 0 & I_3 & 0 \\ 0 & 0 & \lambda^{-1} \end{pmatrix}$$

This means:

$$s_{(a_0,a_1)}s_{(b_0,b_1)}(v_0, v_1) = (v_0\lambda, v_1\lambda^2), \; \forall (v_0, v_1) \in R_{0,1}.$$

In concrete terms :

$$a_0 f(a_0, v_0)a_1^{-2}b_1 + b_0(f(a_0, b_0)f(v_0, a_0)a_1^{-2} + f(v_0, b_0)a_1^{-1}) + v_0 a_1^{-1}b_1 \\ = v_0\lambda, \; \forall v_0 \in V_0. \quad (3.6)$$

Let $v_0 \in V_0 \setminus \langle a_0, b_0 \rangle$.
Then equation (3.6) yields :

$$a_0 f(v_0, a_0)a_1^{-2}b_1 + b_0(f(a_0, b_0)f(a_0, v_0)a_1^{-2} + f(b_0, v_0)a_1^{-1}) = 0.$$

And as a_0 and b_0 are linearly independent :

$$\begin{aligned} f(a_0, v_0) &= 0 \\ f(a_0, b_0)f(a_0, v_0)a_1^{-2} + f(b_0, v_0)a_1^{-1} &= 0. \end{aligned}$$

But then we find that for every $v_0 \in V_0 \setminus \langle a_0, b_0 \rangle$, $f(a_0, v_0) = 0$, which is only possible if $a_0 \in Rad(f)$, contradicting the initial assumption on a_0.

<u>Second case</u> : $Rad(f) = \{0\}$.
Remark that in this case if $char(k) = 2$, $dim(V) = 2n$, $n \in \mathbb{N}$ as f is a symplectic form on V. As $dim(V) = 5$ this case can only occur if

$char(k) \neq 2$. Choose a coordinatization of the set associated to the decomposition $V = e_{-1}k \oplus V_0 \oplus e_1 k$ with labelling set $R_{0,1} = \{(v_0, v_1) \in V_0 \times k \mid q(v_0) + v_1 = 0\}$ and let B_0 be an ordered base of V_0. As $dim(V_0) = 3$ and $k \neq \mathbb{F}_3$ Lemma 1.29 in [24] implies that every element of $Z(Fix_{T\mathcal{MO}(V,q,k)}$ $(\{(\infty), (0,0)\}))$ has a matrix representation with respect to the ordered base $\{e_{-1}, B_0, e_1\}$ of the form :

$$\begin{pmatrix} \mu & 0 & 0 \\ 0 & I_3 & 0 \\ 0 & 0 & \mu^{-1} \end{pmatrix}$$

with $\mu \in k$. But then we find for (a_0, a_1), (b_0, b_1), $(v_0, v_1) \in R_{0,1}$:

$$s_{(a_0,a_1)} s_{(b_0,b_1)}(v_0, v_1) = (v_0 \nu, v_1 \nu^2)$$

for some $\nu \in k$. By similar arguments as in the case where $Rad(f) \neq \{0\}$ the implies $f(a_0, v_0) = 0$, $\forall v_0 \in V_0$, a contradiction.

This means that if $\mathcal{MO}(V, q, k, \sigma)$ is commutative, $dim(V) = 3$ or 4 or $codim(Rad(f)) = 2$.

That the converse holds if $dim(V) = 3$ or 4 follows from Lemmas 111 and 112.
Suppose $codim(Rad(f)) = 2$. Choose as usual a coordinatization associated to the decomposition $V = e_{-1}k \oplus V_0 \oplus e_1 k$ and with labelling set $R_{0,1} = \{(v_0, v_1) \in V_0 \times k \mid q(v_0) + v_1 = 0\}$. Let B_0 be an ordered base of V_0. Then we find in this case that $f|_{V_0} = 0$. Let $g \in Fix_{T\mathcal{MO}(V,q,k)}$ $(\{(\infty)\ (0,0)\})$. As g preserves the forms q and f this implies that g defines a linear transformation of V preserving V_0.
Moreover :

$$q(g(v_0)) = q(v_0),\ \forall v_0 \in V_0$$

yields :

$$q(g(v_0) + v_0) = 0,\ \forall v_0 \in V_0.$$

As $g(v_0) \in V_0$ and q is anisotropic on V_0 this shows :

$$g(v_0) = v_0,\ \forall v_0 \in V_0.$$

This means that g has a matrix representation of the form :

$$\begin{pmatrix} \mu & 0 & 0 \\ 0 & I_{|B_0|} & 0 \\ 0 & 0 & \mu^{-1} \end{pmatrix}$$

with respect to the ordered base $\{e_{-1}, B_0, e_1\}$. Hence we find $Z(Fix_{T\mathcal{MO}(V,q,k)}$ $(\{(\infty)\ (0,0)\})) = Fix_{T\mathcal{MO}(V,q,k)}\ (\{(\infty)\ (0,0)\})$. □

Lemma 121 *A polar line $Pol(k,\sigma)$ with $1 \in Tr(\sigma)$ is commutative if and only if k is a generalized quaternion algebra and σ its standard involution and $Pol(k,\sigma) \cong \mathcal{P}(Z(k))$.*

proof :

Fix a coordinatization for $Pol(k,\sigma)$. By Lemma 92 and section 3.12.2 we can assume that the point set is given by $\{(0,\theta)|\theta \in Tr(\sigma)\}$ with $1 \in Tr(\sigma)$. As $Pol(k,\sigma)$ is assumed to be commutative $Z(Fix_{TPol(k;\sigma)}\ \{(x), (y)\}) = Fix_{TPol(k;\sigma)}$ for any two points (x), $(y) \in Pol(k;\sigma)$. In particular $Z(Fix_{TPol(k;\sigma)} = Fix_{TPol(k;\sigma)}$ and the following equation should hold for any θ, $\theta' \in Tr(\sigma)$:

$$(s_\theta s_1)(s_{\theta'} s_1)(v) = (s_{\theta'} s_1)(s_\theta s_1)(v),\ \forall v \in Tr(\sigma).$$

Suppose that if k is a generalized quaternion algebra σ is not its standard involution. Lemma 47 implies then that $Tr(\sigma)$ generates k as a ring. But then the above equation yields that $[\theta,\theta'] \in Z(k)$, $\forall \theta,\theta' \in Tr(\sigma)$. By Lemma 49 we find that k is a generalized quaternion algebra and σ is its standard involution a contradiction. Hence the only possibility left is that k is a generalized quaternion algebra with standard involution σ. That the converse of the Lemma holds is a straightforward check. □

Corollary 122 *Let $\mathcal{M}U(V,q,k,\sigma)$ be a unitary Moufang set defined by a $(\sigma,-1)$ quadratic form q such that $1 \in Tr(\sigma)$. If $\mathcal{M}U(V,q,k,\sigma)$ is commutative, k is a generalized quaternion algebra with standard involution σ.*

proof :

Choose a coordinatization of $\mathcal{M}U(V,q,k,\sigma)$ associated to the decomposition $V = e_{-1}k \oplus V_0 \oplus e_1 k$ with labelling set $R_{0,1}$. Then the set $Y = \{(0,\theta)|\ \theta \in Tr(\sigma)\}$ defines a Moufang subset of $\mathcal{M}U(V,q,k,\sigma)$ isomorphic to $Pol(k,\sigma)$. But as $\mathcal{M}U(V,q,k,\sigma)$ is commutative, $Pol(k,\sigma)$ should be commutative.

Using Lemma 121 we find that k is a generalized quaternion algebra with standard involution σ. □

For an extended polar line (cfr. p96) defined over a generalized quaternion algebra we can be more precise.

Lemma 123 *Let $\mathcal{M}U(V,q,k,\sigma)$ be an extended polar line defined by a $(\sigma,-1)$-quadratic form q with $1 \in Tr(\sigma)$. Suppose that k is a generalized quaternion algebra k with standard involution σ. Then $\mathcal{M}U(V,q,k,\sigma)$ is commutative if and only if $dim(V)=2$, and $\mathcal{M}(V,q,k,\sigma) \cong \mathcal{P}(Z(k))$.*

proof :

Choose a coordinatization of the Moufang set associated to the decomposition $V = e_{-1}k \oplus V_0 \oplus e_1 k$ with labelling set $R_{0,1} = (v_0, v_1) \in V_0 \times k\ |q(v_0)+v_1 = 0\}$. Suppose $V_0 \neq 0$. The assumption on $\mathcal{M}U(V,q,k,\sigma)$ implies that $f|_{V_0} = 0$ and by Lemma 109 $R_1 \subset Fix(\sigma)$.
Let (v_0,v_1), $(w_0,w_1) \in R_1$. Due to the commutativity of $Fix_{T\mathcal{M}U(V,q,k,\sigma)}$ $\{(\infty),\ (0,0)\ \}$ we find :

$$\begin{aligned}&(s_{(v_0,v_1)}s_{(0,1)})(s_{(w_0,w_1)}s_{(0,1)})((u_0,u_1))\\ &= (s_{(w_0,w_1)}s_{(0,1)})(s_{(v_0,v_1)}s_{(0,1)})((u_0,u_1)),\ \forall (u_0,u_1) \in R_{0,1}\end{aligned}$$

Using the matrix representations of s_x as explained in section 3.12.2 one easily checks that this yields :

$$v_1 w_1 = w_1 v_1,\ \forall v_1, w_1 \in R_1 \tag{3.7}$$

Choose $(a_0,a_1) \in R_{0,1}$ such that $a_0 \neq 0$. Remark that as $Tr(\sigma) = Z(k)$, $a_1 \notin Z(k)$.
We find by (3.7) :

$$a_1 \lambda^\sigma a_1 \lambda = \lambda^\sigma a_1 \lambda a_1,\ \forall \lambda \in k.$$

Put $\{\lambda^\sigma a_1 \lambda\ |\lambda \in k\} = \Lambda_{a_1}$. Clearly $\Lambda_{a_1} \subset C_k\{a_1\}$. Suppose $\Lambda_{a_1} \subset Z(k)(a_1)$. This means that for all $\lambda \in k$:

$$\lambda^\sigma a_1 \lambda = a_1 z_1 + z_2,\ \text{for } z_1, z_2 \in Z(k).$$

But then Lemma 53 implies that $a_1 \in Z(k) = Tr(\sigma)$ a contradiction as q is anisotropic on V_0.

The only possibility left is that $V_0 = 0$. This means that the point set of $\mathcal{M}U(V,q,k,\sigma)$ is given by $\{(0,\theta)\,|\theta \in Tr(\sigma) = Z(k)\} \cup \{(\infty)\}$. Consider the projective Moufang set $\mathcal{P}(Z(k))$ Coordinatize in a canonical way. Then one easily checks that the map β give by :

$$\begin{aligned} \beta((\theta)) &= (0,\theta), \forall\theta \in Z(k) \\ \beta((\infty)) &= (\infty) \end{aligned}$$

defines a Moufang set isomorphism.
Conversely if $dim(V) = 2$ we find that $\mathcal{M}U(V,q,k,\sigma)$ is isomorphic to $\mathcal{P}(Z(k))$ by Lemma 97. Therefore it defines in this case a commutative Moufang set.

□

3.15.2 The isomorphism problem for projective Moufang sets.

In this section we investigate all possible Moufang sets under consideration isomorphic to a projective one.
The proof given below can be found on pp147-150 of [29]. One of the first to prove it with other notations was L. K. Hua in [12]. We restate it here for the sake of completeness and as the techniques used in it illustrate some basic strategies which will be used later on.

Proposition 124 *Consider two projective Moufang sets $\mathcal{P}(k)$ and $\mathcal{P}(k')$ defined over division rings k and k' then :*

$$\mathcal{P}(k) \cong \mathcal{P}(k') \Leftrightarrow k \cong k' \text{ or } k \overset{op}{\cong} k'.$$

<u>proof</u> :

Let $\mathcal{P}(k)$ and $\mathcal{P}(k')$ be isomorphic under β. Without loss of generality we can coordinatize $\mathcal{P}(k)$ and $\mathcal{P}(k')$ in such a way that $\beta((0)) = (0)$,$\beta((\infty)) = (\infty)$ and $\beta((1)) = (1)$. The map β induces a bijection between k and k' also denoted by β and defined by :

$$\beta((x)) = (\beta(x)),\ x \in k.$$

Translating the fact that β is a Moufang set isomorphism yields :
for v, $w \in k$:

$$\begin{aligned} \beta u((\infty);(0),(v))(w) &= \beta((v+w)) \\ &= (\beta(v+w)) \\ &= \beta u(\infty;0,v)\beta^{-1}\beta(w) \\ &= u((\infty);(0),(\beta(v))(\beta(w)) \\ &= (\beta(v)+\beta(w)) \end{aligned}$$

Hence β defines an isomorphism between the additive groups on k and k'.
To derive further properties of β we use the automorphisms $s_{(v)}$, for $v \neq 0$.
From section 3.12.2 we know that $s_{(v)}$ has matrix representation with respect to the base used for coordinatization :

$$\begin{pmatrix} 0 & v \\ -v^{-1} & 0 \end{pmatrix}$$

Hence $s_v(w) = (-vw^{-1}v), \quad \forall w \neq 0 \in k$.
Applying β we obtain :

$$\beta(-vw^{-1}v) = (-\beta(v)\beta(w^{-1})\beta(v)).$$

In particular if $v = 1$

$$\beta(w^{-1}) = (\beta(w))^{-1}, \, \forall w \neq\in k, \, \forall w \neq 0$$

and if $w = 1$

$$\beta(v^2) = (\beta(v))^2, \, \forall v \in k.$$

For v,$w \in k$ one deduces :

$$\begin{aligned} \beta(v^2)+\beta(vw)+\beta(wv)+\beta(w^2) &= \beta((v+w)^2) \\ &= (\beta(v)+\beta(w)^2 \\ &= \beta(v^2)+\beta(v)\beta(w)+\beta(w)\beta(v)+\beta(w^2) \end{aligned}$$

Or $\beta(vw + wv) = \beta(v)\beta(w) + \beta(w)\beta(v)$.
The properties of β deduced so far yield for $v,\ w \in k \setminus \{0\}$:

$$\begin{aligned}
&(\beta(vw) - \beta(v)\beta(w))(\beta(vw))^{-1}(\beta(vw) - \beta(w)\beta(v) \\
&= (1 - \beta(v)\beta(w)(\beta(vw))^{-1})(\beta(vw) - \beta(w)\beta(v)) \\
&= \beta(vw) - \beta(v)\beta(w) - \beta(w)\beta(v) + \beta(v)\beta(w)\beta((vw)^{-1})\beta(w)\beta(v) \\
&= \beta(vw) + \beta(v)\beta(v^{-1}w)\beta(v) - (\beta(v)\beta(w) + \beta(w)\beta(v)) \\
&= \beta(vw) + \beta(wv) - (\beta(v)\beta(w) + \beta(w)\beta(v)) \\
&= 0
\end{aligned}$$

For any fixed $v_0 \in k \setminus \{0\}$, the additive group on k is thus union of two subgroups $L_1 = \{w \in k | \beta(v_0 w) = \beta(v_0)\beta(w)\}$ and $L_2 = \{w \in k | \beta(v_0 w) = \beta(w)\beta(v_0)\}$. This is only possible if $L_1 = k$ or $L_2 = k$. Analogously k is union of the two additive subgroups $R_1 = \{v \in k | \beta(vw) = \beta(v)\beta(w),\ \forall w \in k\}$ and $R_2 = \{v \in k | \beta(vw) = \beta(w)\beta(v),\ \forall w \in k\}$. This implies that $R_1 = k$ and $\beta(vw) = \beta(v)\beta(w),\ \ \forall v, w \in k$ or $k = R_2$ and $\beta(vw) = \beta(w)\beta(v),\ \ \forall v, w \in k$. Hence β defines a field isomorphism or anti-isomorphism between k and k'.
Conversely suppose $k \cong k'$ or $k \cong k'^{op}$ under β. Choose coordinate systems for both $\mathcal{P}(k)$ and $\mathcal{P}(k')$ and define the bijection (also denoted by β) between both point sets by :

$$\begin{aligned}
\beta(v) &= (\beta(v)) \\
\beta(\infty) &= (\infty)
\end{aligned}$$

It is an easy exercise to check β defines a Moufang set isomorphism between $\mathcal{P}(k)$ and $\mathcal{P}(k')$. □

Proposition 125 *A non-commutative projective Moufang set $\mathcal{P}(k)$ is isomorphic to a Moufang set $\mathcal{M}(V', q', \sigma', k')$ if and only if k is a generalized quaternion algebra and $\mathcal{M}(V', q', \sigma', k')$ is an orthogonal Moufang set $\mathcal{MO}(V', q', k')$ with $dim(V') = 6$ and $k' \cong Z(k)$.*

proof :

Suppose the form associated to q' is given by f'. Remark that by Lemma 119 $Z(k) \neq k$.
Lemma 104 shows that $\mathcal{M}(V', q', k', \sigma')$ is of type 2, of type 3 with $dim(V') = 2$ or of type 4 with $codim(Rad(f')) = 2$

Let $\mathcal{P}(k)$ be defined in the 2 dimensional right k-vector space E. Choose a base $\{e_1, e_2\}$ of E inducing a coordinatization of $\mathcal{P}(k)$ such that $\langle e_1\rangle = (\infty)$ and $\langle e_2\rangle = (0)$.

First case : $\mathcal{M}(V', q', k', \sigma')$ is an orthogonal Moufang set $\mathcal{MO}(V', q', k')$.
As usual we suppose that the Moufang set isomorphism between $\mathcal{MO}(V', q', k')$ and $\mathcal{P}(k)$ is given by β. Suppose $\mathcal{P}(k)$ is defined in the 2-dimensional right k-vector space E. Let e_1, e_2 be a base of E inducing a coordinatization of $\mathcal{P}(k)$ such that $(\infty) = \langle e_1\rangle$, and $(0) = \langle e_2\rangle$. Choose a coordinatization of $\mathcal{MO}(V', q', k')$ associated to the decomposition $V' = e'_{-1}k' \oplus V'_0 \oplus e_1k'$ with labelling set $R'_{0,1} = \{(v'_0, v'_1) \in V'_0 \times k' \mid q(v'_0) + v'_1 = 0\}$. Let B'_0 be an ordered base of V'_0. Without loss of generality we can assume $\beta((\infty)) = (\infty)$ and $\beta((0)) = (0, 0)$.
Consider the Moufang subset $Y_{(v'_0,v'_1)} = \{(v'_0z', v'_1z'^2) \mid z' \in k'\} \cup \{(\infty)\}$. Clearly this determines a Moufang subset of $\mathcal{MO}(V', q', k')$. Let $\mathcal{P}(k')$ be the projective Moufang set with k' as ground field coordinatized in a canonical way. Define the bijection $\alpha_{(v'_0,v'_1)}$ from $\mathcal{P}(k')$ to $Y_{(v'_0,v'_1)}$ as follows :

$$\begin{aligned}\alpha_{(v'_0,v'_1)}(z') &= (v'_0z', v'_1z'^2),\ z' \in k'\\ \alpha_{(v'_0,v'_1)}(\infty) &= (\infty).\end{aligned}$$

Using Lemma 41 one checks that $\alpha_{(v'_0,v'_1)}$ defines a Moufang set isomorphism.
Let $(v) \in \mathcal{P}(k)$. Denote the set $\{(vz) \mid z \in Z(k)\} \cup \{(\infty)\}$ as $Y_{(v)}$. One easily shows that this set determines a Moufang subset of $\mathcal{P}(k)$. Let $\mathcal{P}(Z(k))$ be the projective Moufang set defined over $Z(k)$ coordinatized in a canonical way. Then one checks that the bijection $\alpha_{(v)}$ from $\mathcal{P}(Z(k))$ to $Y_{(v)}$ given by :

$$\begin{aligned}\alpha_{(v)}(z) &= (vz),\ z \in k\\ \alpha_{(v)}(\infty) &= (\infty)\end{aligned}$$

induces a Moufang set isomorphism.
Let $(v) \in \mathcal{P}(k)$ such that $\beta(v) = (v'_0, v'_1)$ with $v'_0 \notin Rad(f')$. We show that $\beta(Y_{(v)}) = Y_{(v'_0,v'_1)}$.
If $(vz) \in Y_{(v)}$, we find that $s_{(vz)}s_{(v)}$ has matrix representation with respect to the base $\{e_1, e_2\}$:

$$\begin{pmatrix} -z & 0 \\ 0 & -z^{-1} \end{pmatrix}.$$

Hence $s_{(vz)}s_{(v)} \in Z(Fix_{T\mathcal{P}(k)}\ (\{(\infty), (0)\}))$. Let $\beta(vz) = (w'_0, w'_1)$. We thus find $s_{(w'_0,w'_1)}\ s_{(v'_0,v'_1)} \in Z(Fix_{\mathcal{MO}(V',q',k')}\ \{(\infty), (0,0)\})$.

Remark that as $\mathcal{P}(k)$ is not commutative, $dim(V') \geq 5$ and $codim(Rad(f')) > 2$ and by Lemma 116, $k \neq \mathbb{F}_2$ or $\mathbb{F}_3$. The techniques used in the proof of Lemma 120 show that $s_{(w'_0,w'_1)}\ s_{(v'_0,v'_1)}$ has a matrix representation with respect to the ordered base $\{e_{-1},\ B'_0,\ e'_1\}$:

$$\begin{pmatrix} \mu' & 0 & 0 \\ 0 & I_{|B'_0|} & 0 \\ 0 & 0 & {\mu'}^{-1} \end{pmatrix}$$

where $\mu' \in k'$.
Thus we find in any case for a $\mu' \in k'$:

$$s_{(w'_0,w'_1)} s_{(v'_0,v'_1)}(u'_0, u'_1) = (u'_0\mu', {\mu'}^2 u'_1),\ \forall (u'_0, u'_1) \in R'_{0,1}. \tag{3.8}$$

By Lemma 106 we find for $(u'_0, u'_1) \in R'_{0,1}$:

$$\begin{aligned} &(s_{(w'_0,w'_1)} s_{(v'_0,v'_1)}((u'_0, u'_1)))_0 \\ &= w'_0 f'(w'_0, u'_0){w'_1}^{-2} v'_1 + v'_0(f'(w'_0, v'_0) f'(u'_0, w'_0){w'_1}^{-2} + f'(u'_0, v'_0){w'_1}^{-1}) \\ &+ u'_0 {w'_1}^{-1} v'_1 \\ &= u'_0\mu' \end{aligned}$$

Hence by (3.8) we find for $u'_0 \in V'_0 \setminus \langle v'_0,\ w'_0\rangle$ (remark that such u'_0 exist as $dim(V') \geq 5$) :

$$w'_0 f'(w'_0, u'_0){w'_1}^{-2} v'_1 + v'_0(f'(w'_0, v'_0) f'(u'_0, w'_0){w'_1}^{-2} + f'(u'_0, v'_0){w'_1}^{-1}) = 0. \tag{3.9}$$

Suppose that v'_0 and w'_0 are linearly independent. Then the above equation implies that :

$$f(w'_0, u'_0) = 0,\ \forall u'_0 \in V'_0$$

hence $w'_0 \in Rad(f')$. Equation (3.9) therefore becomes :

$$v'_0 f'(u'_0, v'_0){w'_1}^{-1} = 0,\ \forall u'_0 \in V'_0 \setminus \langle v'_0, w'_0\rangle$$

and thus :

$$f'(v'_0, u'_0) = 0,\ \forall u'_0 \in V'_0$$

contradicting the assumption on v'_0.
Therefore the only possibility left is that $w'_0 \in \langle v'_0\rangle$ and hence $\beta(vz) = (w'_0, w'_1) \in Y_{(v'_0,v'_1)}$.
Conversely let $(v'_0 z', v'_1 {z'}^2) \in Y_{(v'_0,v'_1)}$. By (1) we find $s_{(v'_0 z', v'_1 {z'}^2)}\ s_{(v'_0,v'_1)} \in$

$Z(Fix_{T\mathcal{MO}(V',q',k')}\ \{(\infty),\ (0,0)\})$. Hence if $\beta^{-1}\ (v_0'z', v_1'z'^2) = (w)$, also $s_{(w)}$ $s_{(v)} \in Z(Fix_{T\mathcal{P}(k)}\ \{(\infty),(0)\})$. In the proof of Lemma 119 we saw that every element of $Z(Fix_{T\mathcal{P}(k)}\ \{(\infty),\ (0)\})$ has as matrix representation with respect to the ordered base $\{e_1,\ e_2\}$:

$$\begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix}$$

with $t_1,\ t_2 \in Z(k)$ and $t_1t_2 \in [k,k]$. As $s_{(w)}\ s_{(v)}$ has matrix representation with respect to the base $\{e_1,\ e_2\}$ of the form :

$$\begin{pmatrix} 0 & w \\ -w^{-1} & 0 \end{pmatrix}\begin{pmatrix} 0 & v \\ -v^{-1} & 0 \end{pmatrix} = \begin{pmatrix} -wv^{-1} & 0 \\ 0 & -w^{-1}v \end{pmatrix}$$

this means that there exists a $z \in Z(k)$ such that $w = vz$ and thus w $\beta^{-1}(v_0'z', v_1'z'^2) \in Y_{(v)}$.
Thus we proved so far that $\beta(Y_{(v)}) = Y_{\beta((v))}$ if $(\beta((v)))_0 \notin Rad(f')$.
Remark that this property is equivalent to the statement that if $(\beta((v)))_0$ $\notin Rad(f')$ then for $z \in Z(k)$:

$$(\beta(vz))_0 = (\beta(v))_0z', \text{ for a } z' \in k'.$$

As a next step we show that there exists a field isomorphism α from $Z(k)$ to k' such that :

$$(\beta(vz))_0 = (\beta(v)_0z^{\alpha}),\ \forall z \in k. \tag{3.10}$$

Choose a fixed $\bar{v} \in k$ such that $(\beta(\bar{v}))_0 \notin Rad(f)$. Then we prove that β induces a Moufang set isomorphism from $Y_{(\bar{v})}$ to $Y_{\beta((\bar{v}))}$. Hence $\alpha_{(\bar{v})}\circ\ \beta\ \circ\alpha_{\beta((\bar{v})))}$ induces a Moufang set isomorphism from $\mathcal{P}(Z(k))$ to $\mathcal{P}(k')$. By Proposition 124 we know that this isomorphism is induced by a field isomorphism α from $Z(k)$ to k'. By the construction of $\alpha_{(\bar{v})}$ and $\alpha_{\beta((\bar{v}))}$ it follows that :

$$(\beta((\bar{v}z)))_0 = (\beta((\bar{v})))_0z^{\alpha}.$$

Let $u \in k$ such that $(\beta((u)))_0$ is linearly independent from $(\beta((\bar{v})))_0$. Two cases occur :
1. $(\beta((u))))_0 \in Rad(f')$.
Consider the equations :

$$\begin{aligned} (\beta(((u+\bar{v})z)))_0 &= (\beta((u+\bar{v})))_0z' \\ &= (\beta((u)))_0z' + (\beta((\bar{v})))_0z' \\ &= (\beta((uz)))_0 + (\beta((\bar{v})z)))_0 \\ &= (\beta((uz)))_0 + (\beta((\bar{v})))_0z^{\alpha}. \end{aligned} \tag{3.11}$$

for appropriate $z' \in k'$. If $(\beta((u)z)))_0$ would be not contained in $Rad(f')$, we find by what is already proved $(\beta((uzz^{-1})))_0 = (\beta((u)))_0 \notin Rad(f')$ a contradiction. Hence $(\beta((uz)))_0 \in Rad(f')$ and equation (3.11) shows that :

$$z' = z^\alpha$$

and :

$$(\beta((u)z)))_0 = \beta((u))_0 z^\alpha.$$

2. $(\beta((u))))_0 \notin Rad(f')$.
The equations :

$$\begin{aligned}(\beta(((u+\bar{v})z)))_0 &= (\beta((uz)))_0 + (\beta((\bar{v}z)))_0 \\ &= (\beta((u))))_0 z' + (\beta((\bar{v})))_0 z^\alpha \\ &= (\beta((u+\bar{v})))_0 z'' \\ &= (\beta((u)))_0 z'' + (\beta((\bar{v})))_0 z''\end{aligned}$$

for appropriate z', $z'' \in k'$ show that $z' = z'' = z^\alpha$ and $(\beta((uz)))_0 = (\beta((u)))_0 z^\alpha$.
If $(\beta((u)))_0$ is linearly dependent on $(\beta((\bar{v})))_0$ we choose a $w \in k$ such that $(\beta((w)))_0$ is linearly independent on $(\beta((\bar{v})))_0$.
The equations :

$$\begin{aligned}(\beta(((u+w)z)))_0 &= (\beta((u))))_0 z^\alpha + (\beta((w)))_0 z^\alpha \\ &= (\beta((uz)))_0 + (\beta((wz)))_0 \\ &= (\beta((uz)))_0 + (\beta((w)))_0 z^\alpha\end{aligned}$$

show $(\beta((uz))))_0 = (\beta((u)))_0 z^\alpha$.
Therefore we proved property (3.10).
But this means that $dim(V') = dim(k|_{Z(k)})$.
Let $\beta((1)) = (e'_0, e'_1)$. If $(v'_0, v'_1) \in R'_{0,1} \setminus\{(0,0)\}$ we consider the Moufang subset of $\mathcal{MO}(V', q', k')$ determined by the set $Y_{\langle e'_0, v'_0\rangle} = \{(e'_0 z'_1 + v'_0 z'_2, e'_1 {z'_1}^2$ $+ v'_1 {z'_2}^2 - f'(e'_0, v'_0 z'_1 z'_2) \,|z'_1, z'_2 \in k'\} \cup\{(\infty)\}$. Using property (**??**) we see that $\beta^{-1}\ (Y_{\langle e'_0, v'_0\rangle}) = \{(z_1 + v z_2)\ |z_1,\ z_2 \in k\} \cup\{(\infty)\}$. Call this set $Y_{1,v}$. Because β is a Moufang set isomorphism $Y_{1,v}$ is a Moufang subset of $\mathcal{P}(k)$. As (∞), (1), $(0) \in Y_{1,v}$, $s_{(1)}$ should stabilize $Y_{1,v}$ and $(-v^{-1}) = s_{(1)}((v)) \in Y_{1,v}$.
This means that there exist z_1^v and $z_2^v \in Z(k)$ with :

$$v^{-1} = z_1^v + v z_2^v$$

equivalently :

$$v^2 z_2^v + v z_1^v + 1 = 0.$$

As v was chosen arbitrarily this shows that every element of k is solution of a quadratic equation over $Z(k)$. Lemma 51 implies that k is a generalized quaternion algebra. That conversely every Moufang set of the form $\mathcal{P}(k)$ where k is a generalized quaternion algebra is isomorphic to some orthogonal Moufang set of the form $\mathcal{MO}(V', q', k')$ with $dim(V') = 6$ follows from Lemma 110.

Second case : $\mathcal{M}(V', q', k', \sigma')$ is a hermitian Moufang set.

In this case Lemma 104 implies $dim(V') = 2$ and $\mathcal{M}(V', q', k', \sigma') \cong \mathcal{P}(Fix(\sigma'))$. Using Proposition 124 we have $k \cong Fix(\sigma')$, contradicting the fact that k is non-commutative.

Third case: $\mathcal{M}(V', q', k', \sigma')$ is a unitary Moufang set.

By Lemma 104 $codim(Rad(f')) = 2$. Choose a decomposition of V' as $V' = e'_{-1}k' \oplus V'_0 \oplus e'_1 k'$ with associated coordinatization over the labelling set $R'_{0,1} = \{(v'_0, v'_1) \in V'_0 \times k | q'(v'_0) + v'_1 = 0\}$. Let B'_0 be an ordered base of V'_0. By Lemma 92 and section 3.12.2 we can assume that q' is a $(\sigma, -1)$-quadratic form, $R'_{0,1} \cap \{0\} \times k' = \{(0, x') | x' \in Tr(\sigma')\}$, $1 \in Tr(\sigma')$ and $\beta((1)) = (0, 1)$. Consider the Moufang subset of $\mathcal{M}(V', q', k', \sigma')$ determined by the subspace $e'_{-1}k' \oplus e'_1 k'$ of V'. Call this subset Y_0. Clearly $Y_0 \cong Pol(k', \sigma')$. Let θ'_1 and θ'_2 be two elements of $Tr(\sigma')$. Denote by v_1 and v_2 the elements of k such that $\beta(v_i) = (0, \theta'_i)$, $1 \leq i \leq 2$. By assumption on β, $\beta s_{v_i} s_1 \beta^{-1} = s_{(0,\theta'_i)} s_{(0,1)}$. Moreover as $s_{v_1} s_1 s_{v_2} s_1 = s_{v_1 v_2} s_1$ we find:

$$\begin{aligned} s_{\theta'_1} s_1 s_{\theta'_2} s_1 &= s_{\beta(v_1 v_2)} s_1 \\ &= s_{\tilde{\theta}'} s_1 \end{aligned}$$

with $\beta(v_1 v_2) = \tilde{\theta}'$.

This implies :

$$s_{\theta'_1} s_1 s_{\theta'_2} s_1(\theta') = s_{\tilde{\theta}'} s_1(\theta'),\ \forall \theta' \in Tr(\sigma')$$

or equivalently :

$$\theta'_1 \theta'_2 \theta' \theta'_2 \theta'_1 = \tilde{\theta}' \theta' \tilde{\theta}',\ \forall \theta' \in Tr(\sigma).$$

This means that there exists a $\lambda' \in C_{k'}(Tr(\sigma))$ with:

$$\theta_1 \theta_2 = \tilde{\theta} \lambda'.$$

From $\theta_1'{\theta_2'}^2\theta_1' = \tilde{\theta}'\tilde{\theta}'$ it follows $\lambda'\lambda'^{\sigma} = 1$ and :

$$\theta_1'\theta_2'\theta'\theta_2'\theta_1' = \theta_2'\theta_1'\theta'\theta_1'\theta_2',\ \forall\theta' \in Tr(\sigma).$$

Lemma 49 shows that k' is a generalized quaternion algebra with standard involution σ'.
If $char(k) \neq 2$, Corollary 105 shows that $\mathcal{M}U(V', q', k', \sigma') \cong \mathcal{P}(Z(k'))$. As $\mathcal{P}(k)$ was assumed to be non-commutative this leads to a contradiction.
Let $char(k) = 2$. Suppose $dim(V_0') \geq 1$. Remember that as $\mathcal{P}(k)$ has commutative root groups the same should hold for $\mathcal{M}U(V', q', k')$. Remark that by Lemmas 103 and 109 $R_1 \subset Fix(\sigma')$. Let (u), $(v) \in \mathcal{P}(k)$ such that $\beta(u) = (u_0', u_1')$ and $\beta(v) = (v_0', v_1')$.
As :

$$\beta s_{(u)} s_{(1)} s_{(v)} s_{(1)} \beta^{-1} = \beta s_{(uv)} s_{(1)} \beta^{-1}$$

we find :

$$s_{(u_0',u_1')} s_{(0,1)} s_{(v_0',v_1')} s_{(0,1)} = s_{(w_0',w_1')} s_{(0,1)}$$

with $(w_0', w_1') = \beta(uv)$.
Using the results of section 3.13 we know that with respect to the ordered base the element $\{e_{-1}', B_0', e_1'\}$ $s_{(u_0',u_1')}$ $s_{(0,1)}$ $s_{(v_0',v_1')}$ $s_{(0,1)}$ has as matrix representation :

$$\begin{pmatrix} u_1'v_1' & 0 & 0 \\ 0 & I_{|B_0'|} & 0 \\ 0 & 0 & {u_1'}^{-1}{v_1'}^{-1} \end{pmatrix}$$

and $s_{(w_0',w_1')}$ $s_{(0,1)}$ has matrix representation :

$$\begin{pmatrix} w_1' & 0 & 0 \\ 0 & I_{|B_0'|} & 0 \\ 0 & 0 & {w_1'}^{-1} \end{pmatrix}.$$

Because these two matrices should act in a same way on $\mathcal{M}U(V', q', k', \sigma')$ we find $u_1'v_1' = w_1'$. We already remarked that $R_1' \in Fix(\sigma')$, in particular u_1', v_1', $w_1' \in Fix(\sigma')$. But then

$$\begin{aligned} (u_1'v_1')^{\sigma'} &= u_1'v_1' \\ &= v_1'u_1' \\ &= (w_1')^{\sigma'} \\ &= w_1' \end{aligned}$$

shows $v_1'u_1' = u_1'v_1'$. The automorphism $s_{(v'0,v_1')}\ s_{(0,1)}\ s_{(u_0',u_1')}\ s_{(0,1)}$ has a matrix representation with respect to the ordered base $\{e_{-1}',\ B_0',\ e_1'\}$ of the form :

$$\begin{pmatrix} v_1'u_1' & 0 & 0 \\ 0 & I_{|B_0'|} & 0 \\ 0 & 0 & v_1'u_1' \end{pmatrix}.$$

Thus we see $s_{(v_0',v_1')}\ s_{(0,1)}\ s_{(u_0',u_1')}\ s_{(0,1)} = s_{(u_0',u_1')}\ s_{(0,1)}\ s_{(v_0',v_1')}\ s_{(0,1)}$. Sending this equation over to $\mathcal{P}(k)$ via β^{-1} yields :

$$s_{(u)}s_{(1)}s_{(v)}s_{(1)} = s_{(v)}s_{(1)}s_{(u)}s_{(1)}$$

equivalently :

$$s_{(uv)}s_{(1)} = s_{(vu)}s_{(1)}.$$

This is only possible if $uv = vuz$, for an element $z \in Z(k)$. As u and v where chosen arbitrarily we find

$$[u, v] \in Z(k),\ \forall u, v \in k.$$

Lemma 50 shows that this is only possible if $Z(k) = k$. But then Lemma 119 implies that $\mathcal{P}(k)$ is commutative, a contradiction. □

Proposition 126 *A commutative projective Moufang set $\mathcal{P}(k)$ is isomorphic to a Moufang set $(X', (U_{x'})_{x'\in X'})$ isomorphic to $\mathcal{M}(Q(V', q', \sigma', k'))$ or to $\mathcal{P}(\bar{k}', l'; k')$ if and only if :*

- *(i) $(X', (U_{x'})_{x'\in X'})$ is an orthogonal Moufang set $\mathcal{MO}(V', q', k')$ and one of the following possibilities occurs :*
  (i.a) $dim(V') = 3$ and $k \cong k'$,
  (i.b) $dim(V') = 4$, $codim(Rad(f)) \neq 2$ and $k \cong k''$, where k'' is a quadratic Galois extension of k',
  (i.c) $char(k) = 2$, $codim(Rad(f')) = 2$ and there exists a constant $c' \in k'$ such that the subset $\{c'q'(w')|w' \in Rad(f)\}$ of k' is a subfield of k' isomorphic to k,
- *(ii) $(X', (U_{x'})_{x'\in X'})$ is a hermitian Moufang set $\mathcal{MH}(V', q', k', \sigma')$, $dim(V') = 2$ and $k \cong Fix(\sigma)$,*

(iii) $(X', (U_{x'})_{x'\in X'})$ *is a unitary Moufang set* $\mathcal{MU}(V', q', k', \sigma')$ *defined over a generalized quaternion algebra* k' *with standard involution* σ'*,* $dim(V')$ $= 2$ *and* $k \cong Z(k')$*,*

(iv) $(X', (U_{x'})_{x'\in X'})$ *is an indifferent Moufang set of the form* $\mathcal{P}(\bar{k}', l'; k')$*, and* $k \cong l' = k'$*.*

Choose a coordinatization of $\mathcal{M}(V', q', k', \sigma)$ with associated decomposition $V' = e'_{-1}k' \oplus V'_0 \oplus e'_1 k'$ and labelling set $R'_{0,1} = \{(v'_0, v'_1) \in V'_0 \times k | q(v'_0) + v'_1 = 0\}$. If β denotes the isomorphism from $\mathcal{P}(k)$ to $\mathcal{M}(V', q', \sigma', k')$ we can assume without loss of generality that $\beta(0) = (0, 0)$ and $\beta(\infty) = (\infty)$.

<u>First case</u> : $(X', (U_{x'})_{x'\in X'})$ is an orthogonal Moufang set of the form $\mathcal{MO}(V', q', k'))$.
As $\mathcal{P}(k)$ is commutative Lemma 120 implies that $dim(V'_0) = 1$, $dim(V'_0) = 2$ or $f|_{V'_0} = 0$. We investigate these cases separately.

1. $dim(V'_0) = 1$
Lemma 111 shows that $\mathcal{MO}(V', q', k') \cong \mathcal{P}(k')$. Hence by Proposition 124 we see that $\mathcal{MO}(V', q', k')$ is isomorphic to $\mathcal{P}(k)$ if and only if $k \cong k'$.
2. $dim(V'_0) = 2$.
Lemma 112 shows that $\mathcal{MO}(V', q', k') \cong k''$, where k'' is a quadratic Galois extension of k'. Hence Proposition 124 implies that in this case $\mathcal{MO}(V', q', k')$ is isomorphic to $\mathcal{P}(k)$ if and only if $k \cong k''$.
3. $f'|_{V'_0} = 0$
Without loss of generality we can assume that after an possible multiplication of q' with a constant c', we can work in the proportional Moufang set $\mathcal{MO}(V', cq', k')$, with proportional coordinate system associated to the decomposition $e'_{-1}c'^{-1}k' \oplus V'_0 \oplus e'_1 k'$ (cfr. section 3.12.2) with labelling set $\bar{R}_{0,1}$. Denote $cq' = \bar{q}$ and $c'f' = \bar{f}$. As $\mathcal{MO}(V', q', k')$ is isomorphic to $\mathcal{MO}(V, \bar{q}, k')$ in a canonical way under ψ_c, β induced an isomorphism $\bar{\beta} = \psi_c \circ \beta$ between $\mathcal{P}(k)$ and $\mathcal{MO}(V', \bar{q}, k')$ satisfying :

$$\begin{aligned}\bar{\beta}(0) &= (0,0)\\ \bar{\beta}(\infty) &= (\infty)\\ \bar{\beta}(1) &= (\bar{e}_0, 1).\end{aligned}$$

We show $\bar{R}_1$ is a subfield of k'.
As $Rad(\bar{f}) = 0$, $\bar{R}_1$ is an additive subgroup of k.

Let $\bar{u}_1$, $\bar{v}_1 \in \bar{R}_1$. This means that there exists $\bar{u}_0, \bar{v}_0 \in V_0'$ such that $(\bar{u}_0, \bar{u}_1)$ and $(\bar{v}_0, \bar{v}_1) \in \bar{R}_{0,1}$. Suppose $u, v \in k$ with $\beta(u)(\bar{u}_0, \bar{u}_1)$, $\beta(v) = (\bar{v}_0, \bar{v}_1)$ and $\beta(uv) = (\bar{w}_0, \bar{w}_1)$ then :

$$\begin{aligned} s_{(\bar{v}_0,\bar{v}_1)} s_{(\bar{e}_0,1)} s_{(\bar{u}_0,\bar{u}_1)} s_{(\bar{e}_0,1)} (\bar{e}_0, 1) &= (\bar{e}_0 \bar{u}_1 \bar{v}_1, (\bar{u}_1 \bar{v}_1)^2) \\ &= \beta(s_{(v)} s_{(1)} s_{(u)} s_{(1)}(1)) \\ &= \beta(s_{(uv)} s_{(1)}(1) \\ &= s_{(\bar{w}_0,\bar{w}_1)}(\bar{e}_0, 1) \\ &= (\bar{e}_0 \bar{w}_1, \bar{w}_1^2). \end{aligned}$$

Thus we have :

$$\bar{u}_1 \bar{v}_1 = \bar{w}_1 \in \bar{R}_{0,1}.$$

If $\bar{u}_1 \in \bar{R}_1$, also $\bar{u}_1^{-1} \in \bar{R}_1$ This shows that $\bar{R}_1$ is a subfield of k'.
Hence we can consider the projective Moufang set $\mathcal{P}(\bar{R}_1)$ with a coordinatization.
Consider the bijection γ from $\mathcal{MO}(V', \bar{q}, k')$ to $\mathcal{P}(\bar{R}_1)$ determined by :

$$\begin{aligned} \gamma((\bar{v}_0, \bar{v}_1)) &= (\bar{v}_1) \\ \gamma((\infty)) &= (\infty). \end{aligned}$$

Then one easily checks that γ defines a Moufang set isomorphism. As $\mathcal{P}(k)$ is isomorphic to $\mathcal{MO}(V', \bar{q}, k')$ we deduce using Proposition 124 that $k \cong \bar{R}_1$.

Second case: $(X', (U_{x'})_{x' \in X'})$ is a hermitian Moufang set $\mathcal{M}\ H(V',q',k',\sigma')$.
Using Lemma 104 we find $dim(V') = 2$ and $\mathcal{M}(V', q', k', \sigma')$ is isomorphic to $\mathcal{P}(Fix(\sigma))$. Hence $\mathcal{P}(k) \cong \mathcal{P}(Fix(\sigma'))$ and $k \cong Fix(\sigma')$ by Proposition 124.

Third case: $(X', (U_{x'})_{x' \in X'})$ is a unitary Moufang set $\mathcal{MU}(V', q', k', \sigma')$.
As $\mathcal{P}(k)$ is commutative and has commutative root groups. Lemma 104 implies that $\mathcal{MU}(V', q', k', \sigma')$ is a polar line or extended polar line. Moreover by Lemma 123 we see that $dim(V') = 2$ and $\mathcal{MU}(V', q', k', \sigma') \cong \mathcal{P}(Z(k'))$. Proposition 124 implies that in this case thus $\mathcal{P}(k) \cong \mathcal{MU}(V', q', k', \sigma')$ if and only if $k \cong Z(k')$.

Fourth case : $(X', (U_{x'})_{x' \in X'})$ is of the form $\mathcal{P}(\bar{k}', l'; k')$
Suppose $\mathcal{P}(k) \cong \mathcal{P}(\bar{k}', l'; k')$ under β. After coordinatization of both Moufang sets, β induces a map from k to l', also denoted by β, and defined by

$\beta(v) = (\beta(v)),\ \forall v \in k$. Without loss of generality we can assume $\beta(0) = (0)$, $\beta(\infty) = (\infty)$ and $\beta(1) = (1)$.
Let $v_1, v_1 \in k$ and $w'_1, w'_2 \in l'$ with $\beta(v_1) = w'_1$ and $\beta(v_2) = w'_2$. Then :

$$\begin{aligned}
\beta(s_{(v_1 v_2)} s_{(1)}(1) &= s_{(\beta(v_1 v_2))} s_{(1)}(1) \\
&= (\beta(v_1 v_2)^2) \\
&= ((\beta(v_1)\beta(v_2))^2) \\
&= ((w'_1 w'_2)^2) \\
&= s_{(w'_1)} s_{(1)} s_{(w'_2)} s_{(1)}(1) \\
&= \beta(s_{(v_1)} s_{(1)} s_{(v_2)} s_{(1)}(1).
\end{aligned}$$

This implies that $w'_1 w'_2 \in l'$. As w'_1 and w'_2 where chosen arbitrarily and k' is generated by l' as a ring it follows that $l' = k'$. But in this case $\mathcal{P}(\bar{k}', l', k') \cong \mathcal{P}(k')$, and hence $k \cong l'$ by Proposition 124.

That the converse of the proposition holds in the cases $(i).a$, $(i).b$, (ii) and (iii) is an easy consequence of Proposition 124.
Remains to check that whenever $(X', (U_{x'})_{x' \in X'})$ is an orthogonal Moufang set of the form $\mathcal{MO}(V', q', k')$ with $codim(Rad(f')) = 2$ such that for some constant $c' \in k'$ the set$\{c'q'(w')\ |w' \in Rad(f')\}$ is a field isomorphic to k then $\mathcal{P}(k) \cong (X', (U_{x'})_{x' \in X'})$.
Under the conditions we coordinatize $\mathcal{MO}(V', q', k')$ associated to the decomposition $V' = e'_{-1}k \oplus V'_0 \oplus e'_1 k$ with labelling set $R'_{0,1} = \{(v'_0, v'_1)\ |q(v'_0) + v'_1 = 0\}$. We have $\{c'q'(v'_0)\ |v'_0 \in V'_0\} = \{c'q'(w')|\ w' \in Rad(f')\}$. Denote this set as k''. By the assumptions we have that k'' is a field such that $k \cong k''$, hence $\mathcal{P}(k) \cong \mathcal{P}(k'')$. Let $\mathcal{MO}(V', c'q', k')$ be the Moufang set proportional to $\mathcal{MO}(V, q, k)$ with factor c' coordinatized using the decomposition $V' = \bar{e'}_{-1}k' \oplus V'_0 \oplus \bar{e'}_1 k'$ with $\bar{e'}_{-1} = e'_{-1}c'^{-1}$ and $\bar{e'}_1 = e'_1$.
Consider a canonical coordinatization of $\mathcal{P}(k'')$. Remark that for any $c'q'(v'_0)$, v'_0 is the unique vector w'_0 of V'_0 such that $c'q'(w'_0) = c'q'(v'_0)$. Therefore we can define the bijection α from $\mathcal{P}(k'')$ to $\mathcal{MO}(V', c'q', k')$ by :

$$\begin{aligned}
\alpha(v') &= (v'_0, v') \\
\alpha(\infty) &= (\infty)
\end{aligned}$$

where for $v' \in k''$, v'_0 is the unique vector of V'_0 such that $c'q'(v'_0) = v'$. Using Lemma 41 one easily checks that α defines a Moufang set isomorphism. Hence $\mathcal{MO}(V', q', k') \cong \mathcal{MO}(V', c'q', k') \cong \mathcal{P}(k'')$. This shows that $\mathcal{P}(k) \cong$

$\mathcal{P}(k'') \cong \mathcal{MO}(V', q', k')$. □

3.15.3 The isomorphism problem for orthogonal Moufang sets.

In this section we investigate possible isomorphisms between orthogonal Moufang sets and other Moufang sets mentioned in the list of section 3.14

Proposition 127 *An orthogonal Moufang set $\mathcal{MO}(V, q, k)$ is isomorphic to a classical or indifferent Moufang set $(X', (U_{x'})_{x' \in X'})$ if and only if one of the following holds :*

(i) *$(X', (U_{x'})_{x' \in X'})$ is a projective Moufang set $\mathcal{P}(k')$ and one of the following subcases occurs :*
(i.a) $Z(k') \neq k'$, k' is a generalized quaternion algebra, $dim(V) = 6$ and $k \cong Z(k')$,
(i.b) $dim(V) = 3$, $Z(k') = k'$ and $k \cong k'$,
(i.c) $dim(V) = 4$, $codim(Rad(f)) \neq 2$, $Z(k') = k'$ and $\bar{k} \cong k'$, where $\bar{k}$ is the quadratic Galois extension of k determined by $\mathcal{MO}(V, q, k)$,
(i.d) $codim(Rad(f)) = 2$, there exists a constant c such that the set $\{cq(w) | w \in Rad(f)\}$ is a field isomorphic to k',

(ii) *$(X', (U_{x'})_{x' \in X'})$ is an orthogonal Moufang set $\mathcal{MO}(V', q', k', \sigma')$ and one of the following subcases occurs :*
(ii.a) $dim(V) = dim(V') = 3$, and $k \cong k'$,
(ii.b) $dim(V) = 3$, $dim(V') = 4$, $codim(Rad(f') \neq 2$ and $k \cong \bar{k}'$, where $\bar{k}'$ is the quadratic Galois extension of k' determined by $\mathcal{MO}(V', q', k')$,
(ii.c) $dim(V') = 3$, $dim(V) = 4$, $codim(Rad(f) \neq 2$ and $k' \cong \bar{k}$, where $\bar{k}$ is the quadratic Galois extension of k determined by $\mathcal{MO}(V, q, k)$,
(ii.d) $dim(V) = dim(V') = 4$, $codim(Rad(f)) \neq 2$, $codim(Rad(f')) \neq 2$ and $\bar{k} \cong \bar{k}'$, where $\bar{k}$ is the quadratic Galois extension of k determined by $\mathcal{MO}(V, q, k)$ and $\bar{k}'$ is the quadratic Galois extension of k' determined by $\mathcal{MO}(V', q', k')$,
(ii.d) $codim(Rad(f)) = 2$, $dim(V') = 3$, there exists a constant $c \in k$

such that the set $\{cq(w)\ |w \in Rad(f)\}$ *is a subfield of* k *isomorphic to* k',
$(ii.e)$ $codim(Rad(f)) = 2$, $dim(V') = 4$, $codim(Rad(f')) \neq 2$ *and there exists a constant* $c \in k$ *such that the set* $\{cq(w)\ |w \in Rad(f)\}$ *is a field isomorphic to a quadratic Galois extension* $\bar{k}'$ *determined by* $\mathcal{MO}(V', q', k')$,
$(ii.f)$ $codim(Rad(f')) = 2$, $dim(V) = 3$, *there exists a constant* $c' \in k$ *such that the set* $\{c'q'(w')\ |w' \in Rad(f')\}$ *is a subfield of* k' *isomorphic to* k,
$(ii.g)$ $codim(Rad(f')) = 2$, $dim(V) = 4$, $codim(Rad(f)) \neq 2$ *and there exists a constant* $c' \in k'$ *such that the set* $\{c'q'(w')\ |w' \in Rad(f')\}$ *is a field isomorphic to the quadratic Galois extension* $\bar{k}$ *determined by* $\mathcal{MO}(V, q, k)$,
$(ii.h)$ $codim(Rad(f)) = codim(Rad(f') = 2$, β *induces a bijection from* φ *from* $\{q(w)\ |w \in Rad(f)\}$ *to* $\{q'(w')|\ w' \in Rad(f')\}$, *there exist constants* $c \in k$, c', $d' \in k'$ *such that* $1 \in \{cq(w)\ |w \in Rad(f)\}$, $1 \in \{\ c'q'(w')\ |w' \in Rad(f')\}$ *and an isomorphism* α *from the field generated by* $\{cq(w)\ |w \in Rad(f)\}$ *to the field generated by* $\{c'q'(w')\ |w' \in Rad(f')\}$ *such that :*

$$d'\varphi(q(w)) = (c(q(w)))^{\alpha},\ \forall w \in Rad(f).$$

$(ii.i)$ $\mathcal{MO}(V, q, k)$ *is not commutative and there exists a bijective semi-linear transformation* φ *from* V *to* V' *and a constant* $c' \in k'$ *such that :*

$$\begin{array}{lcl} \beta(\langle x\rangle) & = & \langle\varphi(x)\rangle,\ \forall x \in V \text{ with } \langle x\rangle \in \mathcal{MO}(V, q, k), \\ c'(f(x, y)^{\alpha} & = & f'(\tilde{\beta}(x), \tilde{\beta}(y)),\ \forall x, y \in V, \\ c'(q(x))^{\alpha} & = & q'(\tilde{\beta}(x)),\ \forall x \in V, \end{array}$$

where f *and* f' *are the forms associated to* q *and* q',

(iii) $(X', (U_{x'})_{x' \in X'}$ *is a hermitian Moufang set of the form* $\mathcal{MH}(V', q', k', \sigma')$ *with* $dim(V') = 2$ *and* $\mathcal{MO}(V, q, k)) \cong \mathcal{P}(Fix(\sigma)), \cong \mathcal{M}(V', q', k', \sigma')$,

(iv) $(X'(U_{x'})_{x' \in X'})$ *is an extended polar line* $\mathcal{MU}(V', q', k', \sigma')$ *defined over a generalized quaternion algebra* k' *isomorphic to* $\mathcal{MO}(V, q, k)$. *If* $char(k) \neq 2$, $dim(V') = 2$ *and one of the following subcases occurs :*

(iv.a) $dim(V)=3$, σ' *is the standard involution and* $k \cong Z(k')$,
(iv.b) $dim(V)=4$, σ' *is the standard involution and* $\bar{k} \cong Z(k')$, *where* $\bar{k}$ *is the quadratic Galois extension determined by* $\mathcal{M}O(V,q,k)$,
(iv.c) $dim(V)=3$, σ' *is not the standard involution and* $\mathcal{M}O(V,q,k)$ *is isomorphic to the orthogonal Moufang set determined by* $\mathcal{M}(V',q',k',\sigma')$.

(v) $X',(U_{x'})_{x'\in X'})$ *is an indifferent Moufang set* $\mathcal{P}(\bar{k}',l';k')$ *and one of the following subcases occurs :*
(v.a) $dim(V)=3$ *and* $k \cong l' = k'$,
(v.b) $dim(V)=4$, $codim(Rad(f')) \neq 2$ *and* $\bar{k} \cong l' = k'$, *where* $\bar{k}$ *is the quadratic Galois extension of* k *determined by* $\mathcal{M}O(V,q,k)$,
(v.c) $codim(Rad(f))=2$, β *induces a bijection* φ *from* $\{q(w)\ |w \in Rad(f)\}$ *to* $\bar{l}'$, *there exist constants* $c \in k$, $c' \in k'$ *such that* $1 \in \{cq(w)\ |w \in Rad(f)\}$ *and an isomorphism* α *from the field generated by* $\{cq(w)\ |w \in Rad(f)\}$ *to* k' *such that :*

$$c'(\varphi(q(w)) = (c(q(w)))^{\alpha}.$$

<u>proof</u> :

<u>First case</u> : $(X',(U_{x'})_{x'\in X'})$ is a projective Moufang set of the form $\mathcal{P}(k')$. If $Z(k') \neq k'$ we refer to Proposition 125. If $Z(k')=k'$ we refer to Proposition 126.

<u>Second case</u>: $(X',(U_{x'})_{x'\in X'})$ is a non commutative orthogonal Moufang set of the form $\mathcal{M}O(V',q',k')$.
Remark that this implies that $dim(V) \geq 5$ and $dim(V') \geq 5$ and $k \neq \mathbb{F}_2$ or $\mathbb{F}_3$ by Lemma 116. Choose coordinatizations of both Moufang sets with associated decompositions $V = e_{-1}k \oplus V_0 e_1 k$ and $V' = e'_{-1}k' \oplus V'_0 e'_1 k'$. and labelling sets $R_{0,1} = \{(v_0,v_1) \in V_0 \times k | q(v_0)+v_1 = 0\}$ and $R'_{0,1} = \{(v'_0,v'_1) \in V'_0 \times k | q'(v'_0)+v'_1 = 0\}$. Let B_0 and B'_0 be ordered bases of V_0 and V'_0. Without loss of generality we can assume $\beta((0,0)) = (0,0)$ and $\beta((\infty)) = (\infty)$. Using the results of section 3.12.3 we can assume $(0,1) \in R_{0,1}$, $(0,1) \in R'_{0,1}$ and $\beta((0,1)) = (0,1)$. As for any coordinate $(v_0,v_1) \in R_{0,1}$, v_1 is completely determined by v_0, β induces a bijection from V_0 to V'_0 also denoted by β and defined by :

$$\beta(v_0,v_1) = (\beta(v_0),v'_1)$$

The element v'_1 will be denoted in the sequel by $\beta^{v_0}(v_1)$.
Expressing that β is a Moufang set isomorphism implies that

for (v_0, v_1), $(w_0, w_1) \in R_{0,1}$:

$$\begin{aligned} & \beta((v_0,v_1) \oplus (w_0,w_1) \\ = \; & (\beta(v_0+w_0), \beta^{v_0+w_0}(v_1+w_1-f(v_0,w_0)) \\ = \; & (\beta(v_0)+\beta(w_0), \beta^{v_0}(v_1)+\beta^{w_0}(w_1)-f'(\beta(v_0),\beta(w_0)) \\ = \; & \beta((v_0,v_1)) \oplus \beta((w_0,w_1)). \end{aligned}$$

This means that β defines an additive bijection from V_0 to V_0'.
Our next claim is that β induces a semi-linear transformation from V_0 to V_0'.
We give different proofs depending on $char(k)$.
1. First case : $char(k) \neq 2$.
Let $(v_0, v_1) \in R_{0,1}$ and $\lambda \in k$.
Using Lemma 107 we find that $s_{(v_0,v_1)}\ s_{(v_0\lambda, v_1\lambda^2)} \in Z(Fix_{T\mathcal{MO}(V,q,k)}\ \{(\infty), (0,0)\})$. Namely for $(w_0, w_1) \in R_{0,1}$ we have :

$$s_{(v_0,v_1)} s_{(v_0\lambda, v_1\lambda^2)}(w_0, w_1) = (w_0\lambda^{-2}, w_1\lambda^{-4}).$$

Thus it follows that $s_{(\beta(v_0),\beta^{v_0}(v_1))}\ s_{(\beta(v_0\lambda),\beta^{v_0\lambda}(v_1\lambda^2))} \in Z(Fix_{T\mathcal{MO}(V',q',k')}\ \{(\infty), (0,0)\}$. Suppose $\beta(v_0)$ and $\beta(v_0\lambda)$ are linearly independent. As $dim(V') \geq$ we can choose an $a_0' \in V_0'$, such that $a_0' \notin \langle \beta(v_0),\ \beta(v_0\lambda)\ \rangle$. Denote $W_0' = \langle\ a_0',\ \beta(v_0),\ \beta(v_0\lambda)\ \rangle$ and $W' = e_{-1}'k' \oplus W_0' \oplus e_1'k'$. As q' is a non degenerate quadratic form of Witt index 1 on W' we can consider the Moufang subset $\mathcal{MO}(W', q', k')$ of $\mathcal{MO}(V', q', k')$. We find that $s_{(\beta(v_0),\beta^{v_0}(v_1))}\ s_{(\beta(v_0\lambda),\beta^{v_0\lambda}(v_1\lambda^2))} \in Z(Fix_{T\mathcal{MO}(W',q',k')}\ \{(\infty), (0,0)\})$. But as $dim(W_0') \geq 3$ and $k \neq \mathbb{F}_3$, Theorem 1.29 in [24] implies that the restriction of $s_{(\beta(v_0),\beta^{v_0}(v_1))}\ s_{(\beta(v_0\lambda),\beta^{v_0\lambda}(v_1\lambda^2))}$ on W' has a matrix representation with respect to the ordered base $\{e_{-1}'$, $\beta(v_0)$, $\beta(v_0\lambda)$, a_0', e_1' $\}$ of the form :

$$\begin{pmatrix} \lambda' & 0 & 0 \\ 0 & I_3 & 0 \\ 0 & 0 & \lambda'^{-1} \end{pmatrix}.$$

This yields for $w_0' \in W_0'$ that :

$$\begin{aligned} & (s_{(\beta(v_0),\beta^{v_0}(v_1))} s_{(\beta(v_0\lambda),\beta^{v_0\lambda}(v_1\lambda^2))})((w_0', q'(w_0')))_0 \\ & = \beta(v_0) f'(w_0', \beta(v_0))(\beta^{v_0}(v_1))^{-2}\beta^{v_0\lambda}(v_1\lambda^2) \\ & +\beta(v_0\lambda)(f'(\beta(v_0),\beta(v_0\lambda))f'(\beta(v_0),w_0')(\beta^{v_0}(v_1))^{-2} \\ & +f'(\beta(v_0\lambda),w_0')(\beta^{v_0}(v_1))^{-1}) + w_0'(\beta^{v_0}(v_1))^{-1}\beta^{v_0\lambda}(v_1\lambda^2) \\ & = w_0'\lambda' \end{aligned}$$

This means that for every $w_0' \in W_0' \setminus \langle \beta(v_0),\, \beta(v_0\lambda)\rangle$:

$$\begin{aligned}&\beta(v_0)f'(w_0', \beta(v_0))(\beta^{v_0}(v_1))^{-2}\beta^{v_0\lambda}(v_1\lambda^2)\\ &+\beta(v_0\lambda)(f'(\beta(v_0), \beta(v_0\lambda))f'(\beta(v_0), w_0')(\beta^{v_0}(v_1))^{-2}\\ &+f'(\beta(v_0\lambda), w_0')(\beta^{v_0}(v_1))^{-1})\\ &= 0.\end{aligned}$$

And as $\beta(v_0)$, $\beta(v_0\lambda)$ are linearly independent this shows :

$$f'(w_0', \beta(v_0)) = 0,\ \forall w_0' \in W_0' \setminus \langle \beta(v_0), \beta(v_0\lambda)\rangle$$

yielding that $\beta(v_0) \in Rad(f'|_{W'})$, a contradiction as f' is not degenerate on W'.
This shows that $\beta(v_0)$ and $\beta(v_0\lambda)$ are linearly dependent.
And in this way we find :

$$\beta(v_0\lambda) = \beta(v_0)\lambda',\ \lambda' \in k', \forall \lambda \in k, v_0 \in V_0.$$

As $dim(V_0) \geq 3$ we can use Lemma 54 to see that β defines a bijective semi-linear transformation with an associated field isomorphism α such that :

$$\beta(v_0\lambda) = \beta(v_0)\lambda^{\alpha},\ \forall \lambda \in k, v_0 \in V_0.$$

2. Second case $char(k) = 2$.
Let $(v_0, v_1) \in R_{0,1}$ with $\beta(v_0) \notin Rad(f')$ and $\lambda \in k$. Then we show that $\beta(v_0\lambda) = \beta(v_0)\lambda',\ \lambda' \in k'$.
By Lemma 107 we have :

$$s_{(v_0,v_1)}s_{(v_0\lambda,v_1\lambda^2)}(w_0, w_1) = (w_0\lambda^2, w_1\lambda^4),\ \forall (w_0, w_1) \in R_{0,1},$$

and $s_{(v_0,v_1)}s_{(v_0\lambda,v_1\lambda^2)} \in Z(Fix_{T\mathcal{M}O(V,q,k)}\ \{(0,0),(\infty)\})$ and also $s_{(\beta(v_0),\beta^{v_0}(v_1))}$ $s_{(\beta(v_0\lambda),\beta^{v_0}(\lambda(v_1\lambda^2))} \in Z(Fix_{T\mathcal{M}O(V',q',k')}\{(0,0),(\infty)\}$.
To simplify notations and calculations we reduce the situation to the case where $dim(V') = 5$ to prove that if $\beta(v_0) \notin Rad(f')$, $\beta(v_0)$ and $\beta(v_0\lambda)$ are linearly dependent. Indeed, due to the conditions on $\mathcal{M}O(V,q,k)$ and $\beta(v_0)$ we can choose a subspace $W_0' \subset V_0'$ containing $\beta(v_0)$ and $\beta(v_0\lambda)$ sucht that $\beta(v_0)$ $\notin Rad(f'|_{W_0'})$ and such that the Moufang subset $\mathcal{M}O(W', q', k')$ is not commutative with $W' = e_{-1}'k' \oplus W_0' \oplus e_1'k'$ and $dim(W') = 5$. As $s_{(\beta(v_0\lambda),\beta^{v_0}(\lambda(v_1\lambda^2))}$ $\in Z(Fix_{T\mathcal{M}O(V',q',k')}\{(0,0),(\infty)\}$ we also find that $s_{(\beta(v_0\lambda),\beta^{v_0}(\lambda(v_1\lambda^2))} \in Z(Fix_{T\mathcal{M}O(W',q',k')}\{(0,0),(\infty)\}$.
Hence we are reduced to the case where $dim(V') = 5$.

Suppose $\beta(v_0)$ and $\beta(v_0\lambda)$ are linearly independent.
We give matrix representations of $s_{(\beta(v_0),\beta^{v_0}(v_1))}s_{(\beta(v_0\lambda),\beta^{v_0\lambda}(v_1\lambda^2))}$ with respect to certain bases.
Suppose firstly that $\beta(v_0\lambda) \in Rad(f')$.
Then we choose an ordered base $B'_0 = \{{e'_0}^1, {e'_0}^2, {e'_0}^3\}$ such that ${e'_0}^1 = \beta(v_0)$ and ${e'_0}^3 = \beta(v_0\lambda)$. Using Lemma 106 one we find that with respect to the ordered base $\{e'_{-1}, B'_0, e'_1\}$, $s_{(\beta(v_0\lambda),\beta^{v_0\lambda}(v_1\lambda^2))}$ has a matrix representation of the form :

$$\begin{pmatrix} \lambda' & 0 & 0 & 0 & 0 \\ 0 & 1 & x' & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & {\lambda'}^{-1} \end{pmatrix}.$$

Consider an arbitrary element t of $Fix_{TMO(V',q',k')}\{(0,0),(\infty)\}$. Then this has a general matrix representation with respect to the ordered base $\{e'_{-1}, B'_0, e'_1\}$ of the form :

$$\begin{pmatrix} \mu' & 0 & 0 & 0 & 0 \\ 0 & g'_1 & h'_1 & 0 & 0 \\ 0 & g'_2 & h'_2 & 0 & 0 \\ 0 & g'_3 & h'_3 & 1 & 0 \\ 0 & 0 & 0 & 0 & {\mu'}^{-1}. \end{pmatrix}.$$

Expressing that t and $s_{(\beta(v_0),\beta^{v_0}(v_1))}$ $s_{(\beta(v_0\lambda),\beta^{v_0}(\lambda(v_1\lambda^2))}$ commute translates in the following set of equations :

$$g'_i x' \quad = \quad 0, 2 \le i \le 3.$$

Using Lemma 106 one can choose a t such that for some i, $2 \le i \le n-2$, $g'_i \neq 0$. Then the above equations show $x' = 0$.
Hence $s_{(\beta(v_0),\beta^{v_0}(v_1))}$ $s_{(\beta(v_0\lambda),\beta^{v_0}(\lambda(v_1\lambda^2))}$ has matrix representation of the form :

$$\begin{pmatrix} \lambda' & 0 & 0 \\ 0 & I_3 & 0 \\ 0 & 0 & {\lambda'}^{-1} \end{pmatrix}.$$

Suppose $\beta(v_0\lambda) \notin Rad(f)$.
If $f'(\beta(v_0), \beta(v_0\lambda) \neq 0$ we let $B'_0 = \{{e'_0}^1, {e'_0}^2, {e'_0}^3\}$ be an ordered base of V'_0 with ${e'_0}^1 = \beta(v_0)$, ${e'_0}^2 = \beta(v_0\lambda)$ and ${e'_0}^3$ the vector which spans $Rad(f')$.

By Lemma 106 one checks that $s_{(\beta(v_0),\beta^{v_0}(v_1))}\ s_{(\beta(v_0\lambda),\beta^{v_0(\lambda}(v_1\lambda^2))}$ has a matrix representation with respect to the ordered base $\{e'_{-1},\ B'_0,\ e'_1\}$ of the form :

$$\begin{pmatrix} \lambda' & 0 & 0 & 0 & 0 \\ 0 & x'_1 & x'_2 & 0 & 0 \\ 0 & x'_3 & x'_4 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & (\lambda')^{-1} \end{pmatrix}$$

If $f'(\beta(v_0),\beta(v_0\lambda) = 0$ we choose the base $B'_0 = \{\ {e'_0}^1,\ {e'_0}^2,\ {e'_0}^3\}$ of V'_0 such that ${e'_0}^2 = \beta(v_0)$ and $\langle {e'_0}^3 = Rad(f')$. Remark that by choice of ${e'_0}^3$, ${e'_0}^3$ $\in \beta(v_0)^\perp \cap = (\beta(v_0\lambda))^\perp = \langle\ \beta(v_0),\beta(v_0\lambda)\rangle$. Using Lemma 106 we find as matrix representation of $s_{(\beta(v_0),\beta^{v_0}(v_1))}\ s_{(\beta(v_0\lambda),\beta^{v_0(\lambda}(v_1\lambda^2))}$ with respect to the ordered base $\{e'_{-1},\ B'_0,\ e'_1\}$:

$$\begin{pmatrix} \lambda' & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & x'_1 & 1 & 0 & 0 \\ 0 & x'_3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & (\lambda')^{-1} \end{pmatrix}$$

We set as general matrix representation of an element t of $Fix_{T\mathcal{M}O(V',q',k')}$ $\{(0,0),(\infty)\}$ with respect to the ordered bases $\{e'_{-1},B'_0,e'_1\}$ the following form :

$$\begin{pmatrix} \mu' & 0 & 0 & \dots & 0 \\ 0 & g'_1 & h'_1 & 0 & 0 \\ 0 & g'_2 & h'_2 & 0 & 0 \\ 0 & g'_3 & h'_3 & 1 & 0 \\ 0 & 0 & 0 & 0 & {\mu'}^{-1}. \end{pmatrix}$$

Let $f'(\beta(v_0),\beta(v_0\lambda)) = 0$.
Translating the fact that t and $s_{(\beta(v_0),\beta^{v_0}(v_1))}\ s_{(\beta(v_0\lambda),\beta^{v_0(\lambda}(v_1\lambda^2))}$ commute yields the following set of equations :

$$\begin{aligned} g'_1x'_1 &= x'_1 \\ h'_1x'_1 &= 0 \\ g'_1x'_3 &= x'_3 \\ h'_1x'_3 &= 0. \end{aligned}$$

As a next step we calculate g'_i and h'_i for a special case. Let b'_0 and c'_0 $\in V'_0$ with $f({e'_0}^1, c'_0) = 0$, and $f({e'_0}^1, b'_0) \neq 0$ (remark that this is possible as ${e'_0}^1 \notin Rad(f')$). Write $b'_0 = \sum_{j=1}^3 {e'_0}^j {b'_0}^j$. Using Lemma 106 we calculate :

$$\begin{aligned} s_{(b'_0,b'_1)} s_{(c'_0,c'_1)}({e'_0}^1) &= b'_0 f({e'_0}^1, b'_0) {b'_1}^{-1} + {e'_0}^1 \\ &= \sum_{j=1}^3 {e'_0}^j {b'_0}^j f({e'_0}^1, b'_0) {b'_1}^{-1} + {e'_0}^1. \end{aligned}$$

Thus in particular for $r_{(b'_0,b'_1)} r_{(c'_0,c'_1)}$, $g'_1 = (1 + {b'_0}^1 f({e'_0}^1, b'_0){b'_1}^{-1})$ and $g'_2 = {b'_0}^2 f({e'_0}^1, b'_0){b'_1}^{-1}$ and the equation from above becomes :

$$x'_1 + x'_1({b'_0}^1 f({e'_0}^1, b'_0){b'_1}^{-1}) = x'_1.$$

Equivalently :

$$x'_1 {b'_0}^1 = 0.$$

As $f'({e'_0}^1, b'_0) = f'({e'_0}^1, \sum_{j=2}^3 {e'_0}^j {b'_0}^j)$ we can choose ${b'_0}^1$ arbitrarily in the above formula. Hence $x'_1 = 0$. In a completely similar way one shows $x'_3 = 0$. Thus in this case $s_{(\beta(v_0),\beta^{v_0}(v_1))}\ s_{(\beta(v_0\lambda),\beta^{v_0}(\lambda(v_1\lambda^2))}$ has a matrix representation of the form (with respect to the base $B = \{\ e'_{-1},\ B'_0,\ e'_1\}$) :

$$\begin{pmatrix} \lambda' & 0 & 0 \\ 0 & I_3 & 0 \\ 0 & 0 & {\lambda'}^{-1} \end{pmatrix}.$$

Suppose finally that $f'(\beta(v_0), \beta(v_0\lambda)) \neq 0$. Expressing in this case that $s_{(\beta(v_0),\beta^{v_0}(v_1))}\ s_{(\beta(v_0\lambda),\beta^{v_0}(\lambda(v_1\lambda^2))}$ commutes with an arbitrary element t of $Fix_{\mathcal{MO}(V,q,k)}$ $\{(0,0),\ (\infty)\}$ with matrix representation with respect to the ordered base $\{e'_{-1}, B'_0, e'_1\}$:

$$\begin{pmatrix} \mu' & 0 & 0 & 0 & 0 \\ 0 & g'_1 & h'_1 & 0 & 0 \\ 0 & g'_2 & h'_2 & 0 & 0 \\ 0 & g'_3 & h'_3 & 1 & 0 \\ 0 & 0 & 0 & 0 & {\mu'}^{-1}. \end{pmatrix}$$

yields the following set of equations :

$$g'_3 x'_1 + h'_3 x'_3 = g'_3 \qquad (3.12)$$

$$g'_3 x'_2 + h'_3 x'_4 = h'_3. \qquad (3.13)$$

We calculate in some special cases g_3' and h_3'. Let $c_0' \in V_0'$ with $f'({e_0'}^1, c_0') = 0$ and $f'({e_0'}^2, c_0') = 0$ and $b_0' = \sum_{j=1}^3 {e_0'}^j {b_0'}^j$. Choose c_1', $b_1' \in k$ with (b_0', b_1'), $(c_0', c_1') \in R_{0,1}'$.
Then we find :

$$\begin{aligned} s_{(b_0',b_1')} s_{(c_0',c_1')}({e_0'}^1) &= \sum_{j=1}^3 {e_0'}^j {b_0'}^j f({e_0'}^1, {e_0'}^2) {b_0'}^2 {b_1'}^{-1} + {e_0'}^1 \\ s_{(b_0',b_1')} s_{(c_0',c_1')}({e_0'}^2) &= \sum_{j=1}^3 {e_0'}^j {b_0'}^j f({e_0'}^1, {e_0'}^2) {b_0'}^1 {b_1'}^{-1} + {e_0'}^2. \end{aligned}$$

Consequently we have in this case :

$$\begin{aligned} f_3' &= {b_0'}^2 {b_0'}^3 f'({e_0'}^1, {e_0'}^2) {b_1'}^{-1} \\ g_3' &= {b_0'}^1 {b_0'}^2 f'({e_0'}^1, {e_0'}^2) {b_1'}^{-1}. \end{aligned}$$

But then equations (3.12) and (3.13) become :

$$\begin{aligned} {b_0'}^2 {b_0'}^3 f'({e_0'}^1, {e_0'}^2) {b_1'}^{-1} x_1' + {b_0'}^1 {b_0'}^2 f'({e_0'}^1, {e_0'}^2) {b_1'}^{-1} x_3' &= {b_0'}^1 {b_0'}^2 f'({e_0'}^1, {e_0'}^2) {b_1'}^{-1} \\ {b_0'}^2 {b_0'}^3 f'({e_0'}^1, {e_0'}^2) {b_1'}^{-1} x_2' + {b_0'}^1 {b_0'}^2 f'({e_0'}^1, {e_0'}^2) {b_1'}^{-1} x_4' &= {b_0'}^1 {b_0'}^2 f'({e_0'}^1, {e_0'}^2) {b_1'}^{-1}. \end{aligned}$$

After simplification :

$$\begin{aligned} {b_0'}^2 x_1' + {b_0'}^1 x_3' &= {b_0'}^2 \\ {b_0'}^2 x_2' + {b_0'}^1 x_4' &= {b_0'}^1. \end{aligned}$$

As b_0' was chosen arbitrarily, these equations should hold for all ${b_0'}^1$ and ${b_0'}^2$ in k. This is only possible if $x_2' = x_3' = 0$ and $x_1' = x_4' = 1$.
This means that also in this case $s_{(\beta(v_0),\beta^{v_0}(v_1))}$ $s_{(\beta(v_0\lambda),\beta^{v_0}(\lambda(v_1\lambda^2))}$ has a matrix representation of the form :

$$\begin{pmatrix} \lambda' & 0 & 0 \\ 0 & I_3 & 0 \\ 0 & 0 & \lambda'^{-1} \end{pmatrix}.$$

In any case we thus find that for every $(v_0', v_1') \in R_{0,1}'$:

$$s_{(\beta(v_0),\beta^{v_0}(v_1))} s_{(\beta(v_0\lambda),\beta^{v_0\lambda}(v_1\lambda^2))}(v_0', v_1') = (v_0'\lambda', v_1'\lambda'^2).$$

Using Lemma 106 and projecting on the first coordinate gives that this is equivalent to :

$$\begin{aligned}&\beta(v_0)f'(v_0',\beta(v_0))(\beta^{v_0}(v_1))^{-2}\beta^{v_0\lambda}(v_1\lambda^2)\\&+\beta(v_0\lambda)f'(\beta(v_0),\beta(v_0\lambda))f'(\beta(v_0),v_0')(\beta^{v_0}(v_1))^{-2}\\&+f'(\beta(v_0\lambda),v_0')(\beta^{v_0}(v_1))^{-1})+v_0'(\beta^{v_0}(v_1))^{-1}\beta^{v_0\lambda}(v_1\lambda^2)\\&=v_0'\lambda'\end{aligned}$$

As $dim(V_0') \geq 3$, $\langle\beta(v_0),\beta(v_0\lambda)\rangle \neq V_0'$ and this implies that for every $v_0' \notin \langle\beta(v_0)\ \beta(v_0\lambda)\rangle$:

$$f'(\beta(v_0),v_0') = 0.$$

or equivalently $\beta(v_0) \in Rad(f')$, contradicting the assumption on $\beta(v_0)$. This proves $\beta(v_0)$ and $\beta(v_0\lambda)$ are linearly dependent.
So far we thus showed the following property : if for $w_0 \in V_0$, $\beta(w_0) \notin Rad(f')$ then :

$$\beta(w_0\lambda) = \beta(w_0')\lambda',\ \lambda \in k', \forall\lambda' \in k'.$$

Suppose $w_0 \in V_0$ with $\beta(w_0) \in Rad(f')$ and $\lambda \in k$. Then we choose a $u_0 \in V_0$ such that $\beta(u_0) \notin Rad(f')$.
Let $\beta(u_0\lambda) = \beta(u_0)\lambda'$. We have $\beta(w_0+u_0) \notin Rad(f')$.
Hence :

$$\begin{aligned}\beta((w_0+u_0)\lambda) &= \beta(w_0+u_0)\lambda'',\lambda'' \in k'\\&= \beta(w_0\lambda)+\beta(u_0\lambda)\\&= \beta(w_0\lambda)+\beta(u_0)\lambda'.\end{aligned}$$

If $\beta(w_0\lambda) \notin Rad(f')$, $\beta(w_0) = \beta((w_0\lambda)\lambda^{-1})$ implies $\beta(w_0) \notin Rad(f')$ a contradiction. Hence $\beta(w_0\lambda) \in Rad(f')$.
Then the equation from above implies :

$$\beta(w_0\lambda) = \beta(w_0)\lambda'.$$

We find in all cases for $\lambda \in k$ and $v_0 \in V_0$:

$$\beta(v_0\lambda) = \beta(v_0)\lambda',\ \lambda' \in k'.$$

Thus β defines in all cases an additive map from V_0 to V_0' preserving vector lines. As $dim(V_0) \geq 3$ Lemma 54 implies that β is a semi-linear transformation from V_0 to V_0' with an associated isomorphism α.

This shows that in all cases β defines a semi-linear transformation from V_0 to V_0' with an associated field isomorphism α.
Let $(v_0, v_1) \in R_{0,1}$. Then the equation :

$$\begin{aligned}\beta s_{(0,1)}(v_0, v_1) &= \beta(v_0 v_1^{-1}, v_1^{-1}) \\ &= (\beta(v_0)(v_1^{-1})^{\alpha}, \beta^{v_0}(v_1)) \\ &= (\beta(v_0)(\beta^{v_0}(v_1))^{-1}, (\beta^{v_0}(v_1))^{-1})\end{aligned}$$

shows that :

$$\beta^{v_0}(v_1) = v_1^{\alpha}.$$

This implies that $\beta^{v_0}(v_1)$ is independent of v_0.
Define the semi-linear transformation φ from V to V' in the following way. If $x = e_{-1}x_{-1} + x_0 + e_1 k \in V$ we set

$$\varphi(x) = e'_{-1}x_{-1}^{\alpha} + \beta(x_0) + e'_1 x_1^{\alpha}.$$

By the definition of φ we have :

$$\beta(\langle x\rangle) = \langle\varphi(x)\rangle,\ \forall\langle x\rangle \in \mathcal{M}O(V, q, k).$$

We check that φ preserves the forms f and q.
Let (v_0, v_1), $(w_0, w_1) \in R_{0,1}$.
Then $\beta(v_0, v_1) = (\beta(v_0), v_1^{\alpha})$ implies :

$$q'(\beta(v_0)) = (q(v_0))^{\alpha},\ \forall v_0 \in V_0.$$

From :

$$\begin{aligned}\beta((v_0, v_1) \oplus (w_0, w_1)) &= (\beta(v_0 + w_0), (v_1 + w_1 - f(v_0, w_0))^{\alpha}) \\ &= (\beta(v_0) + \beta(w_0), v_1^{\alpha} + w_1^{\alpha} - f'(\beta(v_0), \beta(w_0)) \\ &= \beta((v_0, v_1)) \oplus \beta((w_0, w_1))\end{aligned}$$

we deduce :

$$(f(v_0, w_0))^{\alpha} = f'(\beta(v_0), \beta(w_0)),\ \forall v_0, w_0 \in V_0.$$

Let $x = e_{-1}x_{-1} + x_0 + e_1 x_1$, $y = e_{-1}y_{-1} + y_0 + e_1 y_1 \in V$.
We have :

$$\begin{aligned}q(x) &= x_{-1}x_1 + q(x_0) \\ f(x, y) &= x_{-1}y_1 + x_1 y_{-1} + f(x_0, y_0).\end{aligned}$$

Applying α to this formulas yields :

$$\begin{aligned}(q(x))^\alpha &= x_{-1}^\alpha x_1^\alpha + (q(x_0))^\alpha \\ &= x_{-1}^\alpha x_1^\alpha + q'(\beta(x_0)) \\ &= q'(\varphi(x))\end{aligned}$$

and :

$$\begin{aligned}(f(x,y))^\alpha &= x_{-1}^\alpha y_1^\alpha + x_1^\alpha y_{-1}^\alpha + (f(x_0,y_0))^\alpha \\ &= x_{-1}^\alpha y_1^\alpha + x_1^\alpha y_{-1}^\alpha + f'(\beta(x_0),\beta(y_0)) \\ &= f'(\varphi(x),\varphi(y)).\end{aligned}$$

This proves φ meets the conditions of the theorem.
Throughout the proof we assumed $(0,1) \in R_{0,1}$, $(0,1) \in R'_{0,1}$ and $\beta((0,1)) = (0,1)$. In general this is not always the case and we have to use a possible multiplication of the forms. Namely suppose $\beta(v_0,v_1) = (v'_0,v'_1)$ for $(v_0,v_1) \in R_{0,1}$ and $(v'_0,v'_1) \in R'_{0,1}$. Then we can consider the forms $v_1^{-1}q$ and ${v'_1}^{-1}q'$. Using proportional coordinate systems for the sets $\mathcal{MO}(V, v_1^{-1}q, k)$ and $\mathcal{MO}(V', {v'_1}^{-1}q', k')$ as explained in section 3.12.3 we see that the isomorphism $\varphi^{-1}_{v_1^{-1}} \circ \beta \circ \varphi_{{v'_1}^{-1}}$ satisfies $\beta((0,0)) = (0,0)$, $\beta((\infty)) = (\infty)$ and $\beta((0,1)) = (0,1)$.
Therefore we find a bijective semi-linear transformation φ such that :

$$\begin{aligned}\varphi_{v_1^{-1}} \circ \beta \circ \varphi_{{v'_1}^{-1}}(\langle x\rangle) &= \langle \varphi(x)\rangle,\ \forall \langle x\rangle \in \mathcal{MO}(V,q,k) \\ (q(x))^\alpha &= v_1^\alpha {v'_1}^{-1} q'(\varphi(x)),\ \forall x \in V \\ (f(x,y))^\alpha &= v_1^\alpha {v'_1}^{-1} f'(\varphi(x),\varphi(y)),\ \forall x,y \in V.\end{aligned}$$

As one easily checks that $\varphi_{v_1^{-1}} \circ \beta \circ \varphi_{{v'_1}^{-1}}(\langle x\rangle) = \beta(\langle x\rangle)$ and φ meets the conditions of the theorem.

Conversely let φ be a bijective semi-linear transformation meeting the requirements of the Proposition. Then Lemma 102 shows that β defined by :

$$\beta(\langle x\rangle)\langle \varphi(x)\rangle,\ \langle x\rangle \in \mathcal{MO}(V,q,k)$$

determines a Moufang set isomorphism form $\mathcal{MO}(V,q,k)$ to $\mathcal{MO}(V',q',k')$.

<u>Third case</u> : $dim(V) \leq 4$, $codim(Rad(f)) \neq 2$ and $(X', (U_{x'})_{x' \in X'})$ is an

orthogonal Moufang set.
If $dim(V') \geq 5$ the proof of the second case implies that β^{-1} induces a semi-linear transformation from V' to V. It follows that then $dim(V) = dim(V')$, a contradiction as $dim(V) \leq 4$ by assumption. Hence we have $dim(V') \leq 4$. Four cases occur :
1. $dim(V) = dim(V') = 3$.
Lemma 111 implies $\mathcal{MO}(V, q, k) \cong \mathcal{P}(k)$ and $\mathcal{MO}(V', q', k') \cong \mathcal{P}(k')$. Hence by Proposition 124 we see that $\mathcal{MO}(V, q, k) \cong \mathcal{MO}(V', q', k')$ if and only if $k \cong k'$.
2. $dim(V) = 3$, $dim(V') = 4$ and $codim(Rad(f')) \neq 2$.
Using Lemmas 111 and 112 we see that $\mathcal{MO}(V, q, k) \cong \mathcal{P}(k)$ and $\mathcal{MO}(V', q', k')$ $\cong \mathcal{P}(k'')$, where k'' is the quadratic Galois extension of k' determined by k'. Hence by Proposition 124 we have that $\mathcal{MO}(V, q, k) \cong \mathcal{MO}(V', q', k')$ if and only if $k \cong k''$.
3. $dim(V) = 3$, $dim(V') = 4$ and $codim(Rad(f')) = 2$.
We refer to the proof of the sixth case.
4. $dim(V) = 4$ and $dim(V') = 3$.
The situation is similar as when $dim(V) = 3$ and $dim(V') = 4$.
5. $dim(V) = dim(V') = 4$ and $codim(Rad(f)) = 2$.
By Lemma 120 we then know that $\mathcal{MO}(V, q, k)$ is commutative. Hence $\mathcal{MO}(V', q', k')$ is commutative and $codim(Rad(f')) = 2$ by the same Lemma. We refer to the proof of the sixth case.
6. $dim(V) = dim(V') = 4$ and $codim(Rad(f)) \neq 2$. By Lemma 120 we have $codim(Rad(f')) \neq 2$. Lemma 112 implies that $\mathcal{MO}(V, q, k) \cong \mathcal{P}(\bar{k})$, where $\bar{k}$ is the quadratic Galois extension of k determined by $\mathcal{MO}(V, q, k)$ and similarly $\mathcal{MO}(V', q', k') \cong \mathcal{P}\ (\bar{k}')$ where $\bar{k}'$ is the quadratic Galois extension of k' determined by $\mathcal{MO}(V', q', k')$. Proposition 124 shows that in this case $\mathcal{MO}(V, q, k) \cong \mathcal{MO}(V', q', k')$ if and only if $\bar{k} \cong \bar{k}'$.

<u>Fourth case</u>: $\mathcal{M}(V', q', k', \sigma')$ is of type 4.
Denote the isomorphism from $\mathcal{MO}(V, q, k)$ to $\mathcal{M}(V', q', k', \sigma')$ by β. Without loss of generality we can assume that q' is a $(\sigma, -1)$-quadratic form and $1 \in Tr(\sigma')$ (cfr. see Lemma 92 and section 3.12.2). Choose decompositions $V = e_{-1}k \oplus V_0 \oplus e_1 k$ and $V' = e'_{-1}k' \oplus V'_0 \oplus e'_1 k'$ with associated coordinatizations over the labelling sets $R_{0,1} = \{(v_0, v_1) \in V_0 \times k | q(v_0) + v_1 = 0\}$ and $R'_{0,1} = \{(v'_0, v'_1) \in V'_0 \times k'\ | q'(v'_0) + v'_1 = 0\}$. Remark that the assumptions on q' yield $R'_{0,1} \cap \ \{0\} \times k = \{(0, x') | x' \in Tr(\sigma')\}$ and $(0, 1) \in R'_{0,1}$.
As $\mathcal{MO}(V, q, k)$ has commutative root groups the same should hold for

$\mathcal{M}(V',q',k',\sigma')$. Lemma 109 shows that $f'_{|V'_0} = 0$ and $R'_1 \subset Fix(\sigma')$.
Suppose that if k' is a generalized quaternion algebra, σ' is not the standard involution. Then Lemma 47 implies that k' is generated as a ring by $Tr(\sigma')$. Let $v' \in Tr(\sigma')$. By assumption there exists a $(v_0,v_1) \in R_{0,1}$ with $\beta((v_0,v_1)) = (0,v')$. Let $\lambda \in k$. Set $\beta(v_0\lambda, v_1\lambda^2) = (w'_0,w'_1)$. By Lemma 107 we have $s_{(v_0,v_1)}\ s_{(v_0\lambda,v_1\lambda^2)} \in Z(Fix_{T\mathcal{MO}(V,q,k)}\{(0,0),(\infty)\})$. Hence after applying β we get :

$$s_{(0,v')}s_{(w'_0,w'_1)} \in Z(Fix_{T\mathcal{M}(V',q',k',\sigma')}\{(0,0),(\infty)\}).$$

Using the matrix representations of section 3.13 this means :

$$\begin{pmatrix} 0 & 0 & v' \\ 0 & I_{|B_0|} & 0 \\ {v'}^{-1} & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & w'_1 \\ 0 & I_{|B_0|} & 0 \\ {w'_1}^{-1} & 0 & 0 \end{pmatrix} = \begin{pmatrix} \lambda'_1 & 0 & 0 \\ 0 & I_{|B_0|} & 0 \\ 0 & 0 & \lambda'_2 \end{pmatrix}$$

Hence $v'{w'_1}^{-1} = \lambda'_1$ and ${v'}^{-1}w'_1 = \lambda'_2$.
By assumption we have :

$$\begin{pmatrix} \lambda'_1 & 0 & 0 \\ 0 & I_{|B'_0|} & 0 \\ 0 & 0 & \lambda'_2 \end{pmatrix} \in Z(Fix_{T\mathcal{M}U(V',q',k',\sigma'}\{(\infty),(0,0)\}).$$

This means that in particular for every $\theta' \in Tr(\sigma')$ the automorphism with matrix representation with respect to the ordered base $\{e'_{-1},\ B'_0,\ e'_1\}$:

$$\begin{pmatrix} [\theta',\lambda'_1] & 0 & 0 \\ 0 & I_{|B'_0|} & 0 \\ 0 & 0 & [{\theta'}^{-1},\lambda'_2] \end{pmatrix}$$

should act as the identity on $\mathcal{M}U(V',q',k',\sigma')$. As $Tr(\sigma')$ generates k' as a ring we find that $[\theta',\lambda'_1] = [\theta',\lambda'_2] = z'_{\theta'}$ with $z'_{\theta'} \in Z(k')$ with ${z'_{\theta'}}^{\sigma'} z'_{\theta} = 1$. This means that there exists for every $\theta' \in Tr(\sigma')$ a $z'_{\theta'} \in Z(k')$ such that :

$$\theta'\lambda'_1 = \lambda'_1\theta' z'_{\theta'}.$$

Let $\theta' \notin Z(k')$. Then :

$$\begin{aligned} (1+\theta')\lambda'_1 &= (1+\theta')\lambda'_1 z'_{(1+\theta')} \\ &= \lambda'_1 + \lambda'_1\theta' z'_{\theta'} \end{aligned}$$

shows :

$$z'_{(1+\theta')} + \theta' z'_{(1+\theta')} = 1 + \theta' z_{\theta'}$$

and thus :

$$z'_{\theta'} = z'_{(1+\theta')} = 1.$$

This means $\lambda'_1 \in Z(k')$. In a similar one shows that $\lambda'_2 \in Z(k')$.
Set $z' = \lambda'_1$ then we thus find :

$$\beta(v_0\lambda, v_1\lambda^2) = (w'_0, v'z'),\ z' \in Z(k') \tag{3.14}$$

By assumption there exists a $(e_0, e_1) \in R_{0,1}$ such that $\beta((e_0, e_1)) = (0, 1)$. Call the Moufang subset of $\mathcal{MO}(V, q, k)$ determined by the set $\{(e_0\mu_1 + v_0\mu_2, e_1\mu_1^2 + v_1\mu_2^2 + \mu_1\mu_2 f(e_0, y_0)) | \mu_i \in k, 1 \leq i \leq 2\}$ as $Y_{\langle e_0, v_0\rangle}$. Formula (3.14) yields that for every $(u'_0, u'_1) \in \beta(Y)$, u'_1 can be written as $\mu'_1 + v'\mu'_2$, with $\mu'_i \in Z(k')$, $1 \leq i \leq 2$. In particular $\beta(s_{(e_0,e_1)}(v_0, v_1)) \in \beta(Y)$ and thus $s_{(0,1)}(0, v') = (0, {v'}^{-1}) \in \beta(Y)$. But then ${v'}^{-1}$ can be written as $\nu'_1 + v'\nu'_2$, for $\nu'_i \in Z(k')$, $1 \leq i \leq 2$.
Equivalently :

$${v'}^2\nu'_2 + v'\nu'_1 + 1 = 0.$$

As v' was chosen arbitrarily we conclude that every element of $Tr(\sigma')$ is solution of a quadratic equation with coëfficiënts in $Z(k')$.
Lemma 52 shows that k' is a generalized quaternion algebra.
But then we find by Lemma 47 that in k' is in any case a generalized quaternion algebra.
Let $char(k) \neq 2$. As $\mathcal{MO}(V, q, k)$ has commutative root groups Lemma 104 implies that $dim(V') = 2$. Two cases occur according if σ' is the standard involution in k' or not.
If σ' is the standard involution in k' Lemma 105 implies that $\mathcal{MU}(V', q', k', \sigma') \cong \mathcal{P}(Z(k'))$. Therefore Proposition 125 implies that or $dim(V) = 3$ and $k \cong Z(k')$ or $dim(V) = 4$ and $\bar{k} \cong Z(k')$ where $\bar{k}$ is the quadratic Galois extension of k determined by $\mathcal{MO}(V, q, k)$.
If σ' is not the standard involution Lemma 115 shows that $\mathcal{MU}(V', q', k', \sigma')$ is isomorphic to a non commutative orthogonal Moufang set $\mathcal{MO}(V'', q'', Z(k'))$ with $dim(V'') = 5$. In this case we thus find that $\mathcal{MO}(V, q, k)$ should be isomorphic to $\mathcal{MO}(V'', q'', Z(k'))$.

<u>Fifth case</u> : $(X', (U_{x'})_{x' \in X'})$ is an indifferent Moufang set of the form $\mathcal{P}(\bar{k}, l; k)$.
Choose a coordinate system for $\mathcal{MO}(V, q, k)$ with associated decomposition

$V = e_{-1}k \oplus V_0 \oplus e_1 k$ and labelling set $R_{0,1} = \{(v_0, v_1) \in V_0 \times k | q(v_0) + v_1 = 0\}$ and a coordinate system for $\mathcal{P}(\bar{k}, l; k)$. Let B_0 be an ordered base of V_0. Without loss of generality we can assume $\beta((0,0)) = (0)$ and $\beta((\infty)) = (\infty)$. As $\mathcal{P}(\bar{k}', l'; k')$ is a commutative Moufang set the same should hold for $\mathcal{MO}(V, q, k)$.
Hence Lemma 120 implies that $dim(V) = 3$, $dim(V) = 4$ or $codim(Rad(f)) = 2$.
1. $dim(V) = 3$.
Lemma 111 shows that $\mathcal{MO}(V, q, k) \cong \mathcal{P}(k)$. By Proposition 126 we see that $\mathcal{MO}(V, q, k) \cong \mathcal{P}(\bar{k}', l'; k')$ if and only if $k \cong l' = k'$.
2.$dim(V) = 4$ and $codim(Rad(f)) \neq 2$.
Lemma 112 shows $\mathcal{MO}(V, q, k) \cong \mathcal{P}(\bar{k})$, where $\bar{k}$ is a quadratic Galois extension of k determined by $\mathcal{MO}(V, q, k)$. Proposition 126 yields $\mathcal{MO}(V, q, k) \cong \mathcal{P}(\bar{k}', l'; k')$ if and only if $\bar{k} \cong l' = k'$.
3. $codim(Rad(f)) = 2$.
Remark that in this case $f|_{V_0} = 0$ and $Rad(f) = V_0$. Let $e_0 \in V_0$ with $q(e_0) \neq 0$. Set $c = q(e_0)^{-1}$. Denote $l = \{cq(w)\ |w \in Rad(f)\} = \{cq(v_0)\ |v_0 \in V_0\}$. We check that l meets the requirements of Proposition 84.
(a) If $cq(v_0) \in l$ we have :

$$c^{-1}(q(v_0))^{-1} = cq(v_0(q(v_0)^{-1}c^{-1})) \in l$$

and hence $l = l^{-1}$.
(b) By construction we have $1 \in l$.
(c) If $cq(v_0) \in l$ and $\lambda \in k$ we find :

$$(cq(v_0))\lambda^2 = cq(v_0\lambda) \in l$$

and thus l is a vector space over k^2.
Denote the field generated by l in k as h. But then Proposition 84 shows that l determines a Moufang subset of h namely the indifferent Moufang set $\mathcal{P}(l; h)$.
We show that $\mathcal{P}(l; h)$ is isomorphic to $\mathcal{MO}(V, q, k)$.
Choose a coordinatization of $\mathcal{P}(l; h)$ and let γ be the bijection from $\mathcal{P}(l; h)$ to $\mathcal{MO}(V, q, k)$ defined by :

$$\begin{aligned} \gamma(cq(v_0)) &= (v_0, v_1) \\ \gamma(\infty) &= (\infty). \end{aligned}$$

Remark that as $f|_{V_0} = 0$, for $(v_0, v_1) \in R_{0,1}$, v_0 is completely determined by v_1. Therefore we see that γ is well defined.
We use Lemma 41 to show that γ defines a Moufang set isomorphism. It will be enough if we prove that the mappings $\gamma_{(\infty)}$ and $\gamma_{(0)}$ defined by :

$$\begin{aligned} \gamma_{(\infty)}(u((\infty);(0),(x))) &= \gamma \circ u((\infty);(0),(x)) \circ \gamma^{-1},\ x \in l \\ \gamma_{(0)}(u((0);(\infty),(x))) &= \gamma \circ u((0);(\infty),(x)) \circ \gamma^{-1},\ x \in l \end{aligned}$$

define bijections from $U_{(0)}$ to $U_{(0,0)}$ and from $U_{(\infty)}$ to $U_{(\infty)}$.
Let $cq(v_0) \in l$ with $q(v_0) = v_1$ then we find for $(w_0, w_1) \in R_{0,1}$:

$$\begin{aligned} \gamma u((\infty),(0),(cq(v_0))\gamma^{-1}((w_0,w_1)) &= \gamma u((\infty);(0),(cq(v_0))((cq(w_0))) \\ &= \gamma((cq(v_0+w_0))) \\ &= (v_0+w_0, v_1+w_1) \\ &= u((\infty);(0,0),(v_0,v_1))((w_0,w_1)). \end{aligned}$$

Moreover as (w_0, w_1) was chosen arbitrarily and $\gamma u((\infty);(0),(cq(v_0)))\ ((\infty)) = (\infty)$ we see that :

$$\gamma_{(\infty)}(u((\infty);(0),(cq(v_0)))) = u((\infty);(0,0),\gamma((cq(v_0)))).$$

Thus $\gamma_{(\infty)}$ bijection from $U_{(\infty)}$ to $U_{(\infty)}$. Remark that for $\mathcal{P}(l;k)$, $s_{(1)}\ U_{(\infty)}\ s_{(1)}^{-1} = U_{(0)}$ and similarly for $\mathcal{MO}(V,q,k)$, $s_{(e_0,e_1)}\ U_{(\infty)}\ e_{(e_0,e_1)}^{-1} = U_{(0,0)}$. As $\gamma((1)) = (e_0, q(v_0))$ it suffices to show that $\gamma \circ s_{(1)}\gamma^{-1} = s_{(e_0,e_1)}$ in order to prove that $\gamma_{(0)}$ defines a bijection from $U_{(0)}$ to $U_{(0,0)}$.
We find for $(v_0, v_1) \in R_{0,1}$:

$$\begin{aligned} \gamma s_{(1)}\gamma^{-1}((v_0,v_1)) &= \gamma s_{(1)}((cq(v_0))) \\ &= \gamma((c^{-1}q(v_0)^{-1})) \\ &= (v_0 v_1^{-1} c^{-1}, c^{-2} v_1^{-1}) \\ &= (v_0 v_1^{-1} q(e_0), q(e_0)^2 v_1^{-1}) \\ &= s_{(e_0,e_1)}((v_0,v_1)). \end{aligned}$$

and :

$$\begin{aligned} \gamma s_{(1)}\gamma^{-1}((\infty)) &= (0,0) \\ &= s_{(e_0,e_1)}((\infty)) \\ \gamma s_{(1)}\gamma^{-1}((0,0)) &= (\infty) \\ &= s_{(e_0,e_1)}((0,0)). \end{aligned}$$

This proves $\gamma\circ\ s_{(1)}\circ\gamma^{-1} = s_{(e_0,e_1)}$ and γ defines a Moufang set isomorphism. As $V_0 = Rad(f)$ we find $\{q(v_0)\ |v_0 \in V_0\} = \{q(w)\ |w \in Rad(f)\}$. Moreover as for any $v_0 \in V_0$, v_0 is completely determined by $q(v_0)$, β induces a bijection φ from $\{q(w)\ |w \in Rad(f)\}$ to $\bar{l}'$ if we set :

$$\varphi(q(v_0)) = \beta(v_0, v_1),\ \forall v_0 \in Rad(f) = V_0.$$

Proposition 131 implies that $\mathcal{MO}(V, q, k) \cong \mathcal{P}(l; h)$ is isomorphic to $\mathcal{P}(\bar{k}', l'; k')$ if and only if there exists a constant $c' \in k'$ and a field isomorphism α h to k' such that :

$$c'\beta\gamma((cq(v_0))) = (cq(v_0))^{\alpha}$$

or equivalently :

$$c'(\varphi(q(v_0))) = (cq(v_0)))^{\alpha}.$$

This closes the fifth case.

Sixth case : $codim(Rad(f)) = 2$ and $(X', (U_{x'})_{x'\in X'})$ is an orthogonal Moufang set $\mathcal{MO}(V', q', k')$.
Choose a coordinatization of $\mathcal{MO}(V, q, k)$ associated to the decomposition $V = e_{-1}k \oplus V_0 \oplus e_1 k$ with labelling set $R_{0,1} = \{(v_0, v_1) \in V_0 \times k | q(v_0) + v_1 = 0\}$. Choose as in proof of the fifth case a constant $c \in k$, such that $\mathcal{P}(l, h)$ is isomorphic under γ to $\mathcal{MO}(V, q, k)$ where $l = \{cq(v_0)\ |v_0 \in V_0\}$ and h is the field generated by l. As in this case $\mathcal{MO}(V, q, k)$ is commutative Lemma 120 shows that $dim(V') = 2$, $dim(V') = 3$ or $codim(Def(f')) = 2$.
We distinguish three cases :
1. $dim(V') = 3$.
Using Lemmas 111 and 131 we see that in this case $\mathcal{MO}(V, q, k) \cong \mathcal{MO}(V', q', k')$ if and only if $h = l \cong k'$.
2. $dim(V') = 4$ and $codim(Rad(f')) \neq 2$.
Using Lemmas 112 and 131 we see that $\mathcal{MO}(V, q, k) \cong \mathcal{MO}(V', q', k')$ if and only if $h = l \cong \bar{k}'$ where $\bar{k}'$ is the quadratic Galois extension of k' determined by $\mathcal{MO}(V', q', k')$.
3. $codim(Rad(f')) = 2$.
Choose a coordinatization of $\mathcal{MO}(V', q', k')$ associated to the decomposition $V' = e'_{-1}k' \oplus V'_0 \oplus e'_1 k'$ with labelling set $R'_{0,1} = \{(v'_0, v'_1) \in V'_0 \times k'\ |q'(v'_0) + v'_1 = 0\}$.
In view of the conditions on f and f' we have $Rad(f) = V_0$ and $Rad(f') = V'_0$. This means that for $v_0 \in V_0$, v_0 is completely determined by $q(v_0)$. Similarly

for $v_0' \in V_0'$, v_0' is completely determined by $q'(v_0'))$. Therefore β induces a bijection from $\{q(w)\,|w \in Rad(f)\} = \{q(v_0)\,|v_0 \in V_0\}$ to $\{q'(w')\,|w' \in Rad(f')\}$ $= \{q'(v_0'))\ |v_0' \in V_0'\}$ is we set :

$$\varphi(q(v_0)) = (\beta((v_0, q(v_0))))_1,\ \forall v_0 \in Rad(f).$$

Similarly as for $\mathcal{M}O(V,q,k)$ we know that there exists a indifferent Moufang set $\mathcal{P}(l';h')$ and an isomorphism γ' from $\mathcal{P}(l';h')$ to $\mathcal{M}O(V',q',k')$ such that $l' = \{c'q'(w')\ |w' \in Rad(f')\}$ and h' is the field generated by l'. Remember that γ' is given by :

$$\begin{aligned} \gamma'((c'q'(v_0'))) &= (v_0', q'(v_0')) \\ \gamma'((\infty)) &= ((\infty)). \end{aligned}$$

But then $\gamma'^{-1}\,\beta\,\gamma$ defines a Moufang set isomorphism from $\mathcal{P}(l;h)$ to $\mathcal{P}(l';h')$. Proposition 131 shows that $\mathcal{M}O(V,q,k) \cong \mathcal{P}(l;h)$ is thus isomorphic to $\mathcal{M}O(V',q',k') \cong \mathcal{P}(l';h')$ if and only if there exists an isomorphism from h to h' and a constant $a' \in h'$ such that

$$\gamma'^{-1}\beta\gamma(cq(v_0)) = (a'(cq(v_0)^{\alpha})),\ \forall v_0 \in Rad(f) = V_0$$

The is clearly equivalent to :

$$c'(\varphi(q(v_0))) = a'(cq(v_0))^{\alpha}, \forall v_0 \in Rad(f) = V_0,$$

or if we set $c'a'^{-1} = d'$:

$$d'(\varphi(q(v_0))) = (cq(v_0))^{\alpha}, \forall v_0 \in Rad(f).$$

This closes the sixth case.

□

3.15.4 The isomorphism problem for hermitian Moufang sets.

In this section we will investigate the possible isomorphisms between a hermitian Moufang set and the other Moufang sets in the list of section 3.14.

Proposition 128 *A hermitian Moufang set $\mathcal{M}H(V,q,k,\sigma)$ with associated form f is isomorphic under β to a classical or indifferent Moufang set $(X',(U_{x'})_{x'\in X'})$ if and only if one of the following holds :*

(i) $(X',(U_{x'})_{x'\in X'})$ *is a projective Moufang set of the form* $\mathcal{P}(k')$, $dim(V) = 2$ *and* $Fix(\sigma)\cong k'$,

(ii) $(X',(U_{x'})_{x'\in X'})$ *is an orthogonal Moufang set* $\mathcal{M}O(V',q',k')$, $dim(V) = 2$ *and* $\mathcal{P}(Fix(\sigma'))\cong \mathcal{M}O(V',q',k')$,

(iii) $(X',(U_{x'})_{x'\in X'})$ *is a hermitian Moufang set* $\mathcal{M}(H(V',q',k',\sigma'))$ *and one of the following subcases occurs:*
$(ii.a)$ $dim(V) = dim(V') = 2$ *and* $Fix(\sigma)\cong Fix(\sigma')$,
$(ii.b)$ $dim(V) = dim(V') = 3$, $Fix(\sigma)\cong Fix(\sigma')$ *and* $\mathcal{M}H(V,q,k,\sigma)$ $\cong \mathcal{M}H'(V',q',k',\sigma')$,
$(ii.c)$ $dim(V) > 3$ *and* β *induces a bijective semi-linear transformation from* V *to* V' *preserving the forms i.e. there exists a collineation* φ *(with associated field isomorphism* α*) from* V *to* V' *and a constant* $c'\in Fix(\sigma')$ *such that :*

$$\begin{aligned}
\beta(\langle x\rangle) &= \langle\varphi(x)\rangle,\ \forall\langle x\rangle\in\mathcal{M}(H(V,q,k,\sigma))\\
c(f(x,y))^{\alpha} &= f'(\varphi(x),\varphi(y)),\ \forall x,y\in V\\
c(q(x))^{\alpha} &= q'(\varphi(x)),\ \forall x\in V,
\end{aligned}$$

(iv) $(X',(U_{x'})_{x'\in X'})$ *is a unitary Moufang set* $\mathcal{M}U(V',q',k',\sigma')$ *defined over a quaternion algebra* k' *with standard involution* σ', $dim(V') = 2$ *and* $\mathcal{M}H(V,q,k,\sigma)\cong Fix(\sigma)\cong Z(k')\cong \mathcal{M}U(V',q',k',\sigma')$.

(v) $(X',(U_{x'})_{x'\in X'})$ *is a unitary Moufang set* $\mathcal{M}U(V',q',k',\sigma')$ *with* $dim(V') = 3$, k' *is a generalized quaternion algebra with standard involution* σ' *which determines a hermitian Moufang set* $\mathcal{M}H(\bar V,\bar q,\bar k,\bar\sigma)$ *with* $dim(\bar V) = 4$ *and isomorphic to* $\mathcal{M}H(V,q,k,\sigma)$.

(vi) $(X',(U_{x'})_{x'\in X'})$ *is an indifferent Moufang set* $\mathcal{P}(\bar k',l';k')$, $dim(V) = 2$ *and* $Fix(\sigma)\cong l' = k'$,

proof :

First case :$(X', (U_{x'})_{x' \in X'})$ is a projective Moufang set $\mathcal{P}(k')$.
We refer to Proposition 125.

Second case : $(X', (U_{x'})_{x' \in X'})$ is an orthogonal Moufang set $\mathcal{MO}(V', q', k')$.
We refer to Proposition 127.

Third case : $(X', (U_{x'})_{x' \in X'})$ is a hermitian Moufang set $\mathcal{MH}(V', q', k', \sigma')$.
Using Lemma 92 and section 3.12.2 we can assume q is a $(\sigma, -1)$-quadratic and q' is a $(\sigma', -1)$- quadratic form. We have $Tr(\sigma) = Fix(\sigma)$ and $Tr(\sigma')$ $= Fix(\sigma')$. Choose coordinatizations of both Moufang sets associated to the decompositions $V = e_{-1}k \oplus V_0 \oplus e_1 k$ and $V' = e'_{-1}k' \oplus V'_0 \oplus e'_1 k'$ with labelling sets $R_{0,1} = \{(v_0, v_1) \in V_0 \times k | q(v_0) + v_1 = 0\}$ and $R'_{0,1} = \{(v'_0, v'_1) \in V'_0 \times k | q'(v'_0) + v'_1 = 0\}$. Without loss of generality we can assume that $\beta((0,0)) = (0,0)$ and $\beta((\infty)) = (\infty)$. Let g and f be the forms on V such that $q(x) = g(x,x) + k_{\sigma,-1}$, $\forall x \in V$ and $q(x+y) = q(x) + q(y) + f(x,y)$, $\forall x, y \in k$. Similarly g' and f' denote the forms on V' such that $q'(x')$ $= g'(x', x')$, $\forall x' \in V'$ and $q'(x' + y') = q'(x') + q'(y') + f'(x', y')$, $\forall$ x', y' $\in V'$. By the proof of Lemma 104 we have for $\mathcal{MH}(V, q, k, \sigma)$ that $Z(U_{(\infty)}) =$ $\{ u((\infty); (0,0), (0,t)) |\ t \in Fix(\sigma)\}$ and similarly for $\mathcal{MH}(V', q', k', \sigma')$ that $Z(U_{(\infty)}) = \{ u((\infty); (0,0), (0,t')\ | t' \in Fix(\sigma')\}$. As β induces an isomorphism between root groups and $\beta u((\infty); (0,0), (0,t))\beta^{-1} = u((\infty); (0,0), \beta((0,t)))$ we find that $\beta\{(0,t) | t \in Fix(\sigma)\} = \{(0,t') | t' \in Fix(\sigma')\}$. This means that β induces a bijection (also denoted by β) from $Fix(\sigma)$ to $Fix(\sigma')$ defined by :

$$\beta((0,t)) = (0, \beta(t)),\ \forall t \in Fix(\sigma) \tag{3.15}$$

Upon a possible multiplication of q' with a certain constant we can thus also assume that $\beta((0,1)) = (0,1)$. Denote $Y = \{\ (0,t) | t \in Fix(\sigma)\}$ and $Y' = \{(0,t')\ | t' \in Fix(\sigma')\}$. Lemma 118 implies that Y $\cup\{(\infty)\}$ and Y' $\cup\{(\infty)\}$ define Moufang subsets of k and k' which are isomorphic to $Fix(\sigma)$ and $Fix(\sigma')$. In view of the isomorphism from $\mathcal{P}(Fix(\sigma))$ to Y and from $\mathcal{P}(Fix(\sigma'))$ to Y' Proposition 124 implies that the map β from Y to Y' defines a field isomorphism from $Fix(\sigma)$ to $Fix(\sigma')$.
Using property (3.15) one shows that β is independent of the first coordinate i.e. if $(v_0, v_1) \in R_{0,1}$ and $(v_0, \bar{v}_1) \in R_{0,1}$ with $\beta(v_0, v_1) = (v'_0, v'_1)$ then $\beta(v_0, \bar{v}_1) = (v'_0, \bar{v}'_1)$.
Namely if (v_0, v_1) and $(v_0, \bar{v}_1) \in R_{0,1}$, then $(v_0, v_1) \ominus (v_0, \bar{v}_1) = (0, t)$ for some $t \in Fix(\sigma)$. Hence $\beta((v_0, v_1) \ominus (v_0, \bar{v}_1)) = (0, t')$,$t' \in Fix(\sigma')$, and

$\beta(v_0, \bar{v}_1) = (v_0', v_1') \oplus (0, t') = (v_0', v_1' + t')$. This implies that β induces a bijection from V_0 to V_0' (which we also denote by β) defined by :

$$(\beta(v_0, v_1))_0 = \beta(v_0),\ \forall (v_0, v_1) \in R_{0,1}.$$

To simplify the calculations we introduce following notation.
If $(v_0, v_1) \in R_{0,1}$ we set :

$$\beta(v_0, v_1) = (\beta(v_0), \beta^{v_0}(v_1)),$$

where the superscript stresses a possible dependence on v_0.
Remark that $\beta(t) = \beta^0(t)\ \forall t \in Fix(\sigma)$.
If $t \in Fix(\sigma))$ we define the transformation m_t by :

$$m_t = s_{(0,t)} s_{(0,1)}.$$

Note that the action of m_t is given by :

$$\begin{array}{rcl} m_t(\infty) & = & (\infty) \\ m_t(v_0, v_1) & = & (v_0 t, v_1 t^2),\ \forall (v_0, v_1) \in R_{0,1}. \end{array}$$

We show the following property :
For every $v_0 \neq 0$ there exists at least one $\xi_{v_0} \notin Fix(\sigma)$ with

$$\beta(v_0 \xi_{v_0}) = \beta(v_0)\xi'_{v_0},\ \xi'_{v_0} \notin Fix(\sigma'). \tag{3.16}$$

For $v_0 \neq 0 \in V_0$ we choose a $v_1 \in k$ with $(v_0, v_1) \in R_{0,1}$ and consider $s_{(0,1)}(v_0, v_1) = (v_0 v_1^{-1}, v_1^{-1})$. Sending this equation over to $\mathcal{M}H(v', q', k', \sigma')$ via β implies :

$$\beta(v_0 v_1^{-1}) = \beta(v_0)(\beta^{v_0}(v_1))^{-1}.$$

As the form q is anisotropic on V_0, v_1 is not contained in $Fix(\sigma)$, and as $\beta\{(0,t)|t \in Fix(\sigma)\} = \{(0,t')|t' \in Fix(\sigma')\}$ it follows that $\beta^{v_0}(v_1) \notin Fix(\sigma')$. This means we can set $\xi_{v_0} = v_1^{-1}$ and (3.16) holds.
As every $s \in k$ can be written as $a + \xi_{v_0} b,\ a, b \in Fix(\sigma)$ we calculate :

$$\begin{array}{rcl} (\beta((v_0 s, s^\sigma s v)))_0 & = & (\beta(m_a((v_0, v_1))) \oplus m_b((v_0 \xi_{v_0}, \xi_{v_0}^\sigma \xi_{v_0} v_1)))_0 \\ & = & (m_{\beta(a)}\beta((v_0, v_1)))_0 + (m_{\beta(b)}\beta((v_0)\xi'_{v_0}, (\xi'_{v_0})^{\sigma'}\xi'_{v_0})))_0 \\ & = & \beta(v_0)(\beta(a) + \xi'_{v_0}\beta(b)) \end{array}$$

where we used the fact that $\beta(v_0\xi_{v_0}) = \beta(v_0)\ \xi'_{v_0}$ and $\beta(m_t((w_0,w_1))) = (\beta(w_0)\beta(t),\ \beta^{w_0}(w_1)(\beta(t))^2),\ \forall\ (w_0,w_1) \in R_{0,1},\ \forall t \in Fix(\sigma)$. Thus we find for $v_0 \in V_0$ and $\lambda \in k$:

$$\beta(v_0\lambda) = \beta(v_0)\lambda'_{v_0}, \tag{3.17}$$

where the subscript v_0 denotes a possible dependence on v_0. By symmetrical arguments we find for $\lambda' \in k'$ and $v'_0 \in V'_0$ that :

$$\beta^{-1}(v'_0\lambda') = \beta^{-1}(v'_0)\lambda_{v'_0}, \tag{3.18}$$

where the subscript v'_0 denotes a possible dependence on v'_0. We distinguish two cases :

1. First case : $dim(V_0) = 1$.

If in this case $dim(V'_0) \geq 2$, formula (3.17) and Lemma 54 imply that β defines a semi-linear transformation from V_0 to V'_0 a contradiction. Hence $dim(V'_0) = 1$. And thus we find that for $\lambda \in k$, $v_0 \in V_0$:

$$\beta(v_0\lambda) = \beta(v_0)\lambda'_{v_0}$$

where λ'_{v_0} might depend on v_0.

2. Second case : $dim(V_0) = 2$.

In this case formula (4) and Lemma 54 imply that β^{-1} induces a semi-linear transformation from V'_0 to V_0 with an associated field isomorphism α^{-1}. Hence β induces a semi-linear transformation of V_0 to V'_0 with associated field isomorphism α.

The relation :

$$s_{(0,t)}(v_0,v_1) = (v_0v_1^{-1}t, tv_1^{-1}t)$$

gives after applying β :

$$\begin{aligned} \beta(v_0)\beta^{v_0}(v_1)^{-1}\beta(t) &= \beta(v_0)(v_1^{-1})^{\alpha}t^{\alpha} \\ \beta^{v_0v_1^{-1}t}(t^2v_1^{-1}) &= \beta(t)(\beta^{v_0}(v_1)^{-1}\beta(t) \end{aligned}$$

As $t^{\alpha} = \beta(t)$, $\forall t \in Fix(\sigma)$ the first of these equations implies that :

$$\beta^{v_0}(v_1) = v_1^{\alpha},$$

proving that β is independent from its second coordinate. Therefore we can drop the superscript v_0 in $\beta^{v_0}(v_1)$ and simply write $\beta(v_1)$ instead of $\beta^{v_0}(v_1)$.

Let $(v_0, v_1), (w_0, w_1) \in R_{0,1}$ then the equation :

$$\begin{aligned}\beta(v_0, v_1) \oplus \beta(w_0, w_1) &= (\beta(v_0 + w_0), \beta(v_1 + w_1 - f(v_0, w_0)) \\ &= (\beta(v_0) + \beta(w_0), (v_1 + w_1 - f(v_0, w_0))^\alpha) \\ &= (\beta(v_0) + \beta(w_0), \beta(v_1) + \beta(w_1) - f'(\beta(v_0), \beta(w_0)))\end{aligned}$$

yields :

$$f(v_0, w_0)^\alpha = f'(\beta(v_0), \beta(w_0)),\ \forall v_0, w_0 \in V_0.$$

As $(v_0, -g(x_0, x_0))) \in R_{0,1}$ and $(\beta(v_0), -g'(\beta(v_0, \beta(v_0)))) \in R'_{0,1}$ we have :

$$g'(\beta(v_0), \beta(v_0)) = (g(v_0, g(v_0)))^\alpha + k'_{\sigma',-1},\ \forall v_0 \in V_0.$$

Consider v_0, $w_0 \in V_0$ such that $f(v_0, w_0) \neq 0$ (such a pair of vectors always exists as $Rad(f) = 0$). Then we find for every $\lambda \in k$:

$$(f(v_0\lambda, w_0))^\alpha = f'(\beta(v_0\lambda), \beta(w_0)).$$

Yielding :

$$\lambda^\sigma (f(v_0, w_0))^\alpha = (\lambda)^{\alpha\sigma'} f'(\beta(v_0), \beta(w_0)).$$

This shows :

$$\lambda^{\sigma\alpha} = \lambda^{\alpha\sigma'},\ \forall \lambda \in k.$$

Let $\langle e_{-1}x_{-1} + x_0 + e_1 x_1\rangle \in \mathcal{M}H(V, q, \sigma, k)$ i.e.

$$-x_{-1}^\sigma x_1 + q(x_0) = 0.$$

Applying α to this equation gives :

$$-x_{-1}^{\alpha\sigma'} x_1^\alpha + q'(\beta(x_0)) = 0. \tag{3.19}$$

Define the semi-linear transformation φ from V_0 to V'_0 by :

$$\varphi(e_{-1}x_{-1} + x_0 + e_1 x_1) = e'_{-1}x_{-1}^\alpha + \beta(x_0) + e'_1,\ x_{-1}, x_1 \in k, x_0 \in V_0.$$

Then equation (3.19) implies that φ induces a bijection from the points of $\mathcal{M}H(V, q, k, \sigma)$ to the points of $\mathcal{M}H(V', q', k', \sigma')$ such that

$$\beta(\langle x\rangle) = \langle \varphi(x)\rangle,\ \forall \langle x\rangle \in \mathcal{M}H(V, q, k, \sigma).$$

Remains to check that φ preserves the forms q and f. Let $x = e_{-1}x_{-1} + x_0 + e_1 x_1 \in V$ then :

$$q(x) = -x_{-1}^\sigma x_1 + q(x_0).$$

Applying α to this expression gives :

$$\begin{aligned}(-x_{-1}^{\sigma}x_1 + q(x_0))^{\alpha} &= -x_{-1}^{\sigma\alpha}x_1^{\alpha} + q'(\beta(x_0)) \\ &= -x_{-1}^{\alpha\sigma'}x_1^{\alpha} + q'(\beta(x_0) \\ &= q'(\varphi(x)).\end{aligned}$$

Let $e_{-1}x_{-1} + x_0 + e_1x_1$ and $e_{-1}y_{-1} + y_0 + e_1y_1 \in V$ then :

$$f(e_{-1}x_{-1} + x_0 + e_1x_1, e_{-1}y_{-1} + y_0 + e_1y_1) = -x_{-1}^{\sigma}y_1 + f(x_0, y_0) + x_1^{\sigma}y_{-1}$$

Let $x = e_{-1}x_{-1} + x_0 + e_1x_1$ and $y = e_{-1}y_{-1} + y_0 + e_1y_1$. Then :

$$\begin{aligned}(f(x,y))^{\alpha} &= (-x_{-1}^{\sigma}y_1 + f(x_0, y_0) + x_1^{\sigma}y_{-1})^{\alpha} \\ &= -x_{-1}^{\alpha\sigma'}y_1^{\alpha} + f'(\beta(x_0), \beta(y_0)) + x_1^{\alpha\sigma'}y_{-1}^{\alpha} \\ &= f'(e'_{-1}x_{-1}^{\alpha} + \beta(x_0) + e'_1x_1^{\alpha}, e_{-1}y_1^{\alpha} + \beta(y_0) + e_1y_1^{\alpha}) \\ &= f'(\varphi\beta(x), \varphi\beta(y)).\end{aligned}$$

Throughout the proof we assumed that $\beta((0,1)) = (0,1)$. This might involve a possible multiplication of q' with a certain element of $Fix(\sigma')$.
Namely suppose $\mathcal{M}(V', q', k', \sigma'))$ is coordinatized using a decomposition $e'_{-1}k' \oplus V'_0 \oplus e'_1k'$. Let $\beta((0,1)) = (0, c')$, $c' \in Fix(\sigma')$. Then we consider the proportional Moufang set $\mathcal{M}(V', c'^{-1}q, k', \sigma')$ (cfr. section 3.12.3) coordinatized using the decomposition $V' = \bar{e'}_{-1}k' \oplus V'_0 \oplus \bar{e'}_1k'$, with $\bar{e'} = e'_{-1}c$ and $\bar{e'}_1 = e'_1$. Using this coordinate system the isomorphism $\psi_{c'^{-1}} \circ \beta$ from $\mathcal{M}H(V, q, k, \sigma)$ to $\mathcal{M}H(V', c^{-1}q', k', \sigma')$ clearly satisfies $\psi_{c'^{-1}} \circ\beta((0,0)) = (0,0)$, $\psi_{c'^{-1}} \circ\beta((\infty)) = (\infty)$ and $\psi_{c'^{-1}} \circ\beta((0,1)) = (0,1)$ and we can apply the proof so far developed. This means that there exists a semi-linear transformation φ with associated field isomorphism α from V to V' such that :

$$\begin{aligned}\psi_{c'^{-1}} \circ \beta(\langle x\rangle) &= \langle\varphi(x)\rangle, \forall\langle x\rangle \in \mathcal{M}H(V, q, k, \sigma) \\ (q(x))^{\alpha} &= c'^{-1}q'(\varphi(x)), \forall x \in V \\ (f(x,y))^{\alpha} &= c'^{-1}f'(\varphi(x), \varphi(y)), \forall x, y \in V.\end{aligned}$$

And thus we find that φ is a bijective semi-linear transformation from V to

V' with associated field isomorphism α such that :

$$\begin{aligned} \beta(\langle x\rangle) &= \langle\varphi(x)\rangle,\ \forall\langle x\rangle\in\mathcal{M}H(V,q,k,\sigma) \\ c'(q(x))^{\alpha} &= q'(\varphi(x)),\ \forall x\in V \\ c'(f(x,y))^{\alpha} &= f'(\varphi(x),\varphi(y)),\ \forall x,y\in V. \end{aligned}$$

Conversely let φ be a bijective semi-linear transformation from V to V' satisfying :

$$\begin{aligned} c'(q(x))^{\alpha} &= q'(\varphi(x)),\ \forall x\in V \\ c'(f(x,y))^{\alpha} &= f'(\varphi(x),\varphi(y)),\ \forall x,y\in V \end{aligned}$$

with $c'\in Fix(\sigma')$. Then Lemma 102 implies that the map from $\mathcal{M}H(V,q,k,\sigma)$ to $\mathcal{M}H(V',q',k',\sigma')$ defined by :

$$\beta(\langle x\rangle)=\langle\varphi(x)\rangle$$

determines a Moufang set isomorphism.

<u>Fourth case</u> : $(X',(U_{x'})_{x'\in X'})$ is a unitary Moufang set $\mathcal{M}U(V',q',k',\sigma')$.

Using Lemma 92 and the results from section 3.12.2 we can assume without loss of generality that q' is a $(\sigma',-1)$-quadratic form such that $1\in Tr(\sigma')$. Choose coordinatizations of both Moufang sets associated to the decompositions $V=e_{-1}k\oplus V_0\oplus e_1k$ and $V'=e'_{-1}k'\oplus V'_0\oplus e'_1k'$ with labelling sets $R_{0,1}$ $=\{(v_0,v_1)\in V_0\times k|q(v_0)+v_1=0$ and $R'_{0,1}=\{(v'_0,v'_1)\in V'_0\times k'|q'(v'_0)+v'_1=0\}$. Without loss of generality we can assume that $\beta((0,0))=(0,0)$ and $\beta((\infty))=(\infty)$ and $\beta((0,1))=(0,1)$.
By Lemma 103 we know that $\beta\ \{(0,t)\ |t\in Fix(\sigma)\}=\{(v'_0,v'_1)\ v'_0\in Rad(f')\}$. Remark that $\{(v'_0,v'_1)|\ v'_0\in Rad(f')\}\cup\{(\infty)\}$ determines a Moufang subset of $\mathcal{M}U(V',q',k',\sigma')$ namely $\mathcal{M}(Rad(f'),q',k',\sigma')$. One checks that $\{(0,t)|t\in Fix(\sigma)\}\cup\{(\infty)\}$ determines a Moufang subset of $\mathcal{M}H(V,q,k,\sigma)$ isomorphic to $\mathcal{P}(Fix(\sigma))$. Moreover in a similar way one checks that the set $\{(0,\theta')|\theta'\in Tr(\sigma)\}\cup\{(\infty)\}$ determines a Moufang subset of $\mathcal{M}U(Rad(f'),q',k',\sigma')$. As $\mathcal{P}(Fix(\sigma)$ is a commutative Moufang set Lemma 121 implies that k' is a generalized quaternion algebra with standard involution σ'.
For every (v'_0,v'_1), the element v'_0 is completely determined by v'_1. Therefore

we define a bijection (also denoted by β) from $Fix(\sigma)$ to $L' = \{t' | t' = q'(v'_0)$ for a $v'_0 \in Rad(f')$ if we set :

$$(\beta((0,t)))_1 = \beta(t), \forall t \in Fix(\sigma).$$

Remark that Lemma 109 shows that $L' \subset Fix(\sigma')$. Therefore we find that $L' \nsubseteq k'$. In the sequel we will denote for $t \in Fix(\sigma)$ the automorphism $s_{(0,t)}$ $s_{(0,1)}$ as m_t and for $t' \in L'$ with $q(t'_0) = t'$, $s_{(t'_0,t')}$ $s_{(0,1)}$ as $m_{t'}$. We show that L' is a field isomorphic to $Fix(\sigma)$.
By definition $1 \in L'$. Let a'_1, $b'_1 \in L'$ with $q'(a'_0) = a'$, $q'(b'_0) = b'_1$, $\beta(r) = a'_1$ and $\beta(t) = b'_1$.
Then the equations :

$$\begin{aligned} q'(a'_0 + b'_0) &= a'_1 + b'_1 \\ q'(a'_0 {a'_1}^{-1}) &= {a'_1}^{-1} \end{aligned}$$

show that $a'_1 + b'_1 \in L'$ and ${a'_1}^{-1} \in L'$.
Using the matrix representations of s_x as explained in section 3.13 we find :

$$\begin{aligned} \beta m_r m_t \beta^{-1} &= \beta m_{rt} \beta^{-1} \\ &= m_{\beta(rt)} \\ &= m_{a'_1} m_{b'_1} \\ &= m_{a'_1 b'_1}. \end{aligned}$$

Using the matrix representations this is only possible if $\beta(rt) = a'_1 b'_1\ z'$, with $z' \in Z(k')$ such that $z' z'^{\sigma} = 1$. But as k' is a generalized quaternion algebra with standard involution this implies $a'_1 b'_1 = \pm\beta(rt)$ and $a'_1 b'_1 \in L'$.
This proves that L' is a field and L' is a Moufang subset of $\mathcal{MU}(V', q', k', \sigma')$.
As β induces a bijction from $Fix(\sigma)$ to L' it determines an isomorphism from $\mathcal{P}(Fix(\sigma))$ to $\mathcal{P}(L')$. Proposition 124 shows that β induces a field isomorphism from $Fix(\sigma)$ to L'. Suppose $Rad(f') \neq 0$.
Then there exists at least one (a'_0, a'_1) with $a'_0 \neq 0$ and $a'_1 \in L'$. As q' is anisotropic on V'_0 we find $a'_1 \notin (k')$ and $dim(L')|_{Z(k')} = 2$ as k' is a generalized quaternion algebra and $L' \neq k'$. Because L' is 2 dimensional over $Z(k')$ we find $Z(k')(a'_1) = L'$.
Let $\lambda' \in k'$ arbitrarily then $q'(a'_0\lambda') = \lambda'^{\sigma'} a'_1 \lambda'$ shows :

$$\lambda'^{\sigma'} a'_1 \lambda' \in L' = (Z(k')(a'_1)), \forall \lambda' \in k'.$$

Lemma 53 implies that $a'_1 \in Z(k')$ a contradiction as q' is anisotropic on V'_0. This shows that for $\mathcal{M}U(V', q', k', \sigma')$ clearly $Rad(f') = 0$. But then we find $Z((R'_{0,1}, \oplus)) = \{(0, \theta') \, |\theta' \in Tr(\sigma')$ and $\beta\{\ (0,t) \ |t \in Fix(\sigma)\} = \{(0, \theta') \ |\theta' \in Tr(\sigma')$. This implies that β is independent of its second coordinate. Indeed if (v_0, v_1), $(v_0, \bar{v}_1) \in R_{0,1}$ we have :

$$\begin{aligned} \beta((v_0, v_1) \ominus (v_0, \bar{v}_1) &= \beta((0, v_1 - \bar{v}_1)) \\ &= (0, \beta(v_1 - \bar{v}_1) \end{aligned}$$

and hence $(\beta((v_0, v_1)))_0 = (\beta((v_0, \bar{v}_1)))_0$. This implies that we can define a bijection from V_0 to V'_0 (also denoted by β) in the following way. If $v_0 \in V_0$ we set :

$$(\beta((v_0, q(v_0))))_0 = \beta(v_0).$$

Thus we can introduce the following notation. If $(v_0, v_1) \in R_{0,1}$ we set :

$$\beta((v_0, v_1)) = (\beta(v_0), \beta^{v_0}(v_1))$$

where the superscript denotes a possible dependence on v_0. As for (v_0, v_1), $(w_0, w_1) \in R_{0,1}$

$$\begin{aligned} (\beta((v_0, v_1) \oplus (w_0, w_1)))_1 &= (\beta((v_0 + w_0, v_1 + w_1 - f(v_0, w_0)))_1 \\ &= (\beta((v_0, v_1)))_1 + (\beta((w_0, w_1)))_1 \end{aligned}$$

β is an additive map from V_0 to V'_0. We find for $(v_0, v_1) \in R_{0,1}$, $v_0 \neq 0$ and $\theta \in Fix(\sigma)$:

$$\begin{aligned} \beta(m_\theta(v_0, v_1)) &= \beta((v_0\theta, \theta^2 v_1)) \\ &= \beta((v_0\theta), \beta^{v_0\theta}(\theta^2 v_1))) \\ &= \beta((v_0)\beta(\theta), \beta(\theta)^2 \beta^{v_0}(v_1)) \\ &= m_{\beta(\theta)}(\beta(v_0), \beta^{v_0}(v_1)) \end{aligned}$$

and thus :

$$\beta(v_0\theta) = \beta(v_0)\beta(\theta) \, \forall \theta \in Fix(\sigma'). \tag{3.20}$$

Moreover :

$$\begin{aligned} \beta((v_0 v_1^{-1}, v_1^{-1})) &= \beta(s_{(0,1)}(v_0, v_1)) \\ &= s_{(0,1)}(\beta((v_0), \beta^{v_0}(v_1))) \\ &= (\beta(v_0)(\beta^{v_0}(v_1))^{-1}, (\beta^{v_0}(v_1))^{-1}) \end{aligned}$$

shows :

$$\beta(v_0 v_1^{-1}) = \beta(v_0)(\beta^{v_0}(v_1))^{-1}. \tag{3.21}$$

As for every $(v_0, v_1) \in R_{0,1}$ with $v_0 \neq 0$, $v_1 \notin Fix(\sigma)$ we have $Fix(\sigma)(v_1)$ $= k$, equations (3.20) and (3.21) show :

$$\beta(v_0\lambda) = \beta(v_0)\lambda'_{v_0}, \forall \lambda \in k, v_0 \in V_0, \tag{3.22}$$

where the superscript denotes a possible dependence on v_0.
We distinguish two cases :
1. $dim(V_0') \geq 2$.
Equation (3.22) and Lemma 54 imply that β induces a semi-linear transformation from V_0 to V_0' with α a field isomorphism from k to k'. But then we would have that $Z(k') = k'$ a contradiction.
2. $dim(V_0') = 1$.
In this case Lemma 114 implies that $\mathcal{M}(V', q', k', \sigma')$ is isomorphic to $\mathcal{M}(H, V, q, k)$ with $Fix(\sigma) \cong k'$.

Fifth case : $(X', (U_{x'})_{x' \in X'})$ is an indifferent Moufang set of the form $\mathcal{P}(\bar{k}', l'; k')$. If $\mathcal{M}H(V, q, k, \sigma) \cong \mathcal{P}(\bar{k}', l'; k')$ Lemma 104 implies that $dim(V) = 2$, and $\mathcal{M}H(V, q, k, \sigma) \cong \mathcal{P}(Fix(\sigma))$. The result now follows from Proposition 131.
□

3.15.5 The isomorphism problem for unitary Moufang sets.

In this section we will assume that the all quadratic forms q are $(\sigma, -1)$-quadratic forms. Lemma 92 and section 3.12.3 show that this does not put any restrictions on the forms.

Proposition 129 *A unitary Moufang set $\mathcal{M}U(V, q, k, \sigma))$ with non commutative root groups such that $Rad(f) = 0$ if $char(k) = 2$ and k is a generalized quaternion algebra with standard involution σ is isomorphic under β to a classical Moufang set $(X', (U_{x'})_{x' \in X'})$ if and only if one of the following holds :*

(i) $(X', (U_{x'})_{x'\in X'}$ *is a hermitian Moufang set* $\mathcal{M}H(V', q', k', \sigma')$, *with* $dim(V') = 4$, $dim(V) = 3$, k *is a generalized quaternion algebra and* $Fix(\sigma') \cong Z(k)$.

(ii) $(X', (U_{x'})_{x'\in X'})$ *is a unitary Moufang set* $\mathcal{M}U(V', q', k', \sigma')$ *and* β *induces a bijective semi-linear transformation* φ *satisfying :*

$$\begin{aligned} \beta(\langle x\rangle) &= \langle\varphi(x)\rangle,\ \forall\langle x\rangle \in \mathcal{M}U(V, q, k, \sigma) \\ (f(x, y))^{\alpha} &= c' f'(\varphi(x), \varphi(y)),\ \forall x, y \in V \\ q(x)^{\alpha} &= c' q'(\varphi(x)),\ \forall x \in V \end{aligned}$$

for some constant $c' \in k'$ *with :*

$$c'\lambda^{\sigma\alpha}c'^{-1} = \lambda^{\alpha\sigma'},\ \forall\lambda \in k.$$

proof :

As $\mathcal{M}U(V, q, k, \sigma)$ has by assumption non-commutative root groups, the set $(X', (U_{x'})_{x'\in X'})$ can only be a hermitian or unitary Moufang set.

First case : $(X', (U_{x'})_{x'\in X'})$ is a hermitian Moufang set $\mathcal{M}H(V', q', k', \sigma')$. Proposition 128 implies that $dim(V) = 3$, k is a generalized quaternion algebra with standard involution σ, $dim(V') = 4$. By Lemma 114 $\mathcal{M}U(V, q, k, \sigma)$ is isomorphic to a hermitian Moufang set $\mathcal{M}H(V_1, q_1, k_1, \sigma_1)$ with k_1 a quadratic Galois extension of $Z(k) = Fix(\sigma_1)$. Therefore Proposition 128 implies thus that $Fix(\sigma_1) = Z(k) \cong Fix(\sigma')$.
Conversely suppose $dim(V) = 3$, k is a generalized quaternion algebra with standard involution σ and $\mathcal{M}H(V', q', k', \sigma')$ is a hermitian Moufang set such that $dim(V') = 4$ and $Z(k) \cong Fix(\sigma')$. Lemma 114 implies that $\mathcal{M}H(V', q', k', \sigma')$ is isomorphic to a unitary Moufang set $\mathcal{M}U(V_1', q_1', k_1', \sigma_1')$ defines over a generalized quaternion algebra k_1' with center $Fix(\sigma')$ and with $dim(V') = 3$. As $Z(k_1') \cong Z(k)$ we find $k \cong k_1'$. As there is up to isomorphism only one unitary Moufang set $\mathcal{M}U(V, q, k, \sigma)$ with $dim(V) = 3$ we find that $\mathcal{M}H(V', q', k', \sigma') \cong \mathcal{M}U(V_1', q_1', k_1', \sigma_1') \cong \mathcal{M}U(V, q, k, \sigma)$.

Second case : $(X', (U_{x'})_{x'\in X'})$ is a unitary Moufang set $\mathcal{M}U(V', q', k', \sigma')$. Remark as mentioned in the beginning of this section q is assumed to be a $(\sigma, -1)$-quadratic form similarly q' is a $(\sigma', -1)$-quadratic. Let f be the

$(\sigma,-1)$-hermitian form associated to q, f' the $(\sigma',-1)$-hermitian form associated to q', $q(v) = g(v,v) + k_{\sigma,-1}$, $\forall v \in V$ and $q'(v') = g'(v',v') + k'_{\sigma,-1}$, $\forall v' \in V'$ where g is a σ-sesquilinear form and g' is a σ'-sesquilinear form. Using section 3.12.3 we can moreover assume $1 \in Tr(\sigma)$, $1 \in Tr(\sigma')$. Choose a coordinatization $\mathcal{M}(V,q,k,\sigma)$ associated to the decomposition $V = e_{-1}k \oplus V_0 \oplus e_1 k$ and a coordinatization of $\mathcal{M}(V',q',k',\sigma')$ with decomposition $V' = e'_{-1}k' \oplus V'_0 \oplus e'_1 k'$. Then the labelling sets $R_{0,1} = \{(v_0, v_1 \in V_0 \times k | q(v_0) + v_1 = 0\}$ and $R'_{0,1} = \{(v'_0, v'_1) | q'(v'_0) + v'_1 = 0\}$ satisfy $R_{0,1} \cap \{0\} \times k = \{(0,\theta) | \theta \in Tr(\sigma)\}$ and $R'_{0,1} \cap \{0\} \times k' = \{(0,\theta') | \theta' \in Tr(\sigma')\}$. Without loss of generality we can assume $\beta((0,0)) = (0,0)$, $\beta((0,1)) = (0,1)$ and $\beta((\infty)) = (\infty)$.
Let (v_0,v_1), $(w_0,w_1) \in R_{0,1}$.
The equation :

$$\begin{aligned}
\beta(u((\infty);(0,0),(v_0,v_1))(w_0,w_1)) &= \beta((v_0,v_1) \oplus (w_0,w_1)) \\
&= \beta((v_0,v_1) \oplus \beta((w_0,w_1)) \\
&= u((\infty);(0,0),\beta((v_0,v_1)))\beta((w_0,w_1)).
\end{aligned}$$

shows that β induces an isomorphism from $(R_{0,1},\oplus)$ to $(R'_{0,1},\oplus)$. Using Lemma 103 we have :

$$\begin{aligned}
Z((R_{0,1},\oplus)) &= \{(v_0,v_1) \in R_{0,1} | v_0 \in Rad(f)\} \\
Z((R'_{0,1},\oplus)) &= \{(v'_0,v'_1) \in R'_{0,1} | v'_0 \in Rad(f')\}.
\end{aligned}$$

Thus if $(v_0,v_1) \in R_{0,1}$ with $v_0 \in Rad(f)$ we have $(\beta((v_0,v_1)))_0 \in Rad(f')$. Note that for any vector $w_0 \in Rad(f)$ the only vector $u_0 \in V_0$ for which $q(u_0) = q(w_0)$ is w_0 and similarly for any $w'_0 \in V'_0$, $q'(w'_0)$ is completely determined by w'_0. Therefore we can define a bijection (also denoted by β) between the set $L = \{t \in k \,| q(v_0) = t,\, v_0 \in Rad(f)\}$ and the set $L' = \{t' \in k' \,| q'(v'_0) = t',\, v'_0 \in Rad(f')\}$ by :

$$(\beta(v_0,v_1))_1 = \beta(v_1),\ \forall v_0 \in Rad(f).$$

Remark that the equation $s_{(0,1)}((a_0,a_1)) = (a_0 a_1^{-1}, a_1^{-1})$, $\forall (a_0,a_1) \in Z(R_{0,1},\oplus)$ implies that $\beta(a_1^{-1}) = (\beta(a_1))^{-1}$, $\forall a_1 \in L$.

Using this map we show that k is a quaternion algebra with standard involution σ and if and only if k' is a generalized quaternion algebra with standard involution σ'. We show one direction. The other way follows by symmetric arguments.
One easily checks $Y = Z((R_{0,1},\oplus)) \cup \{(\infty)\}$ is a Moufang subset of

$\mathcal{M}U(V,q,k,\sigma)$ and similarly that $Y' = Z((R'_{0,1},\oplus)) \cup\{(\infty)\}$ is a Moufang subset of $\mathcal{M}\ U\ (V',q',k',\sigma')$. As $\beta(Y) = Y'$, Y is isomorphic as Moufang set to Y'.
The assumptions on $Rad(f)$ in the characteristic 2 case implies that in any case $Z((R_{0,1},\oplus)) = \{(0,\theta)\ |\theta \in Tr(\sigma)\}$. Lemma 123 implies that $(Y,(U_y)_{y\in Y})$ $\cong \mathcal{P}(Z(k))$. As $(Y',(U_{y'})_{y'\in Y'}) \cong (Y,(U_y)_{y\in Y})$ Lemma 121 shows that k' is a generalized quaternion algebra with standard involution.

As a next step we show
<u>**Claim 1**</u> :

$$\beta\{(0,\theta)|\theta \in Tr(\sigma)\} = \{(0,\theta')|\theta' \in Tr(\sigma')\}. \tag{3.23}$$

If $char(k) \neq 2$ then this follows from the fact that $Z(R_{0,1},\oplus) = \{(0,t)|t \in Tr(\sigma)\}$ and $Z(R'_{0,1},\oplus) = \{(0,t')|t' \in Tr(\sigma')\}$.
Hence we can assume $char(k) = 2$.
Suppose that for a $(0,\theta) \in R_{0,1}$ we have $\beta(0,\theta) = (a'_0,a'_1)$. As $(0,\theta) \in Z((R_{0,1},\oplus))$, clearly $a'_0 \in Rad(f')$.
We distinguish two subcases :

1. First subcase : k is a generalized quaternion algebra with standard involution σ.
If $char(k) = 2$, we have by assumption that $Rad(f) = \{0\}$. We already saw that the Moufang subset determined by $Y = Z((R_{0,1},\oplus)) = \{(0,t)\ |t \in Tr(\sigma)\}$ is isomorphic to the Moufang subset determined by $Y' = Z((R'_{0,1},\oplus))$. As $Rad(f) = 0$ we have that $(Y,(Stab_{U_y}(Y))_{y\in Y})$ is isomorphic to $\mathcal{P}(Z(k))$. But $(Y',(Stab_{U_{y'}}(Y')))$ is an extended polar line defined over a generalized quaternion algebra with standard involution and Lemma 123 yields that $Rad(f') = 0$.
And thus we find that (3.23) is also satisfied in this case.

2. Second subcase : if k is a generalized quaternion algebra σ is not its standard involution.
Remark that in this case by Lemma 47, k is generated as a ring by $Tr(\sigma)$. We write $\theta = z + z^\sigma$, $z \in k$.
Let $\mu \in Tr(\sigma)$ and $(v_0,v_1) \in R_{0,1}$. In the sequel we will denote the automorphism $s_{(0,\mu)}s_{(0,1)}$ as m_μ and the group $\langle u((\infty);(0,0),(v_0,v_1),m_\mu|\ \mu \in Tr(\sigma)\rangle$

by $S_{(v_0,v_1)}$.
To proceed we first prove a general property :

Consider λ in k and $(v_0, v_1) \in R_{0,1}$ with $\beta(v_0, v_1) = (v_0', v_1')$. Write λ as an expression of elements of $Tr(\sigma)$ i.e. :

$$\lambda = \sum_{j=1}^{n} \theta_{1,j}^{j} \dots \theta_{i(j),j},\ \theta_{k,j} \in Tr(\sigma)$$

We show by induction on the number of terms in the expression that there exists a $\psi_\lambda \in S_{(v_0,v_1)}$ such that :

$$\begin{aligned}(\psi_\lambda(v_0, v_1))_0 &= v_0\lambda \\ (\beta(\psi_\lambda(v_0, v_0))_0 &= v_0'\lambda' \end{aligned} \tag{3.24}$$

with $\lambda'\ \ sum_{j=1}^{n}\beta(\theta_{1,j})\ \dots\ \beta(\theta_{i(j),j})$.

If there is only one term in the expression i.e. we have for example $\lambda = \theta_1\theta_2 \dots \theta_n$ one checks that we can set $\psi_\lambda = m_{\theta_1} m_{\theta_2} \dots m_{\theta_n}$.
Suppose the claim is true for any expression of elements of $Tr(\sigma)$ with fewer terms that in $\lambda = \sum_{j=1}^{n} \theta_{1,j} \dots \theta_{i(j),j}$. To simplify notation we can assume that without loss of generality $i(n) = 2$. Put $\theta_{1,n}\ \theta_{n-1}$ and $\theta_{2,n} = \theta_n$. Consider $\xi = \sum_{j=1}^{n} \theta_{1,j} \dots \theta_{i(j),j}\theta_n^{-1}\theta_{n-1}^{-1}$. Remark that then $(\xi + 1)\ \theta_{n-1}\ \theta_n = \lambda$.
By induction we know that there exists a $\psi_\xi \in S_{(v_0,v_1)}$ such that :

$$\begin{aligned}(\psi_\xi(v_0, v_1))_1 &= v_0\xi \\ (\beta(\psi_\xi(v_0, v_1))_1 &= v_0'\xi'\end{aligned}$$

with $\xi'\ \sum_{j=1}^{n} \beta(\theta_{1,j}) \dots \beta(\theta_{i(j),j})\beta(\theta_n)^{-1}\beta(\theta_{n-1})^{-1}$.

Consider $m_{\theta_n} m_{\theta_{n-1}} u((\infty); (0,0), (v_0, v_1))\ \psi_\xi$. We find :

$$\begin{aligned} & m_{\theta_n} m_{\theta_{n-1}} u((\infty); (0,0), (v_0, v_1))\psi_\xi(v_0, v_1) \\ = \ & m_{\theta_n} m_{\theta_{n-1}} u((\infty); (0,0), (v_0, v_1))(v_0\xi, x_1) \\ = \ & m_{\theta_n} m_{\theta_{n-1}} (v_0(\xi + 1), x_1 + v_1 - f(v_0, v_0)) \\ = \ & (v_0((\xi + 1)\theta_{n-1}\theta_n), \theta_n\theta_{n-1}(x_1 + v_1 - f(v_0, v_0))\theta_{n-1}\theta_n) \\ = \ & (v_0\lambda, \theta_n\theta_{n-1}(x_1 + v_1 - f(v_0, v_0))\theta_{n-1}\theta_n) \end{aligned}$$

and :

$$\begin{aligned} & \beta(m_{\theta_n} m_{\theta_{n-1}} u((\infty); (0,0), (v_0, v_1))\psi_\xi(v_0, v_1)) \\ = \ & m_{\beta(\theta_n)} m_{\beta(\theta_{n-1})} u((\infty); (0,0), (v_0', v_1')(v_0'\xi', x_1') \\ = \ & m_{\beta(\theta_n)} m_{\beta(\theta_{n-1})} (v_0'(\xi' + 1), v_1' + x_1' - f'(v_0', v_0')) \\ = \ & (v_0'((\xi' + 1)\beta(\theta_{n-1})\beta(\theta_n), \beta(\theta_n)\beta(\theta_{n-1})(v_1' + x_1' - f'(v_0', v_0'))\beta(\theta_{n-1})\beta(\theta_n)) \\ = \ & (v_0'\lambda', \beta(\theta_n)\beta(\theta_{n-1})(v_1' + x_1' - f'(v_0', v_0'))\beta(\theta_{n-1})\beta(\theta_n'))). \end{aligned}$$

This shows that we can set $\psi_\lambda = m_{\theta_n}\, m_{\theta_{n-1}}\, u((\infty);(0,0),(v_0,v_1))\, \psi_\xi$ and the formula (3.24) is proved.
We proceed with the proof of formula (3.23). As the root groups of both Moufang sets are non commutative we can choose a $(w_0,w_1) \in R_{0,1}$ such that $w_0 \notin Rad(f)$ and $\beta(w_0,w_1) = (w_0',w_1')$ where $w_0' \notin Rad(f')$.
Using Lemma 108 we calculate :

$$\begin{aligned}\beta(w_0\theta,\theta w_1\theta) &= \beta s_{(0,\theta)} s_{(0,1)}(w_0,w_1)\\ &= s_{(a_0',a_1')} s_{(0,1)}(w_0',w_1')\\ &= (w_0'a_1', a_1'w_1'a_1'),\end{aligned}$$

where $\beta((0,\theta)) = (a_0',a_1')$, $a_0' \in Rad(f')$. Write z as an expression of elements of $Tr(\sigma)$ i.e. for example $z = \sum_{j=1}^m \mu_{1,j}\ldots\mu_{i(j),j}$. This implies :

$$\begin{aligned}\theta &= \sum_{j=1}^m \mu_{1,j}\ldots\mu_{i(j),j} + \sum_{j=1}^m \mu_{i(j),j}\ldots u_{1,j}\\ &= \sum_{j=1}^m (\mu_{1,j}\ldots\mu_{i(j),j} + \mu_{i(j),j}\ldots\mu_{1,j})\end{aligned}$$

Using (3.24) we find a $\psi_\theta \in S_{(w_0,w_1)}$ such that :

$$\begin{aligned}\psi_\theta(w_0,w_1) &= (w_0\theta, \theta\bar{w}_1\theta)\\ \beta(\psi_\theta(w_0,w_1)) &= (w_0'\theta', \theta'\bar{w}_1'\theta')\end{aligned}$$

with $(w_0,\bar{w}_1) \in R_{0,1}$ and $(w_0',\bar{w}_1') \in R_{0,1}'$ and :

$$\theta' \;= \textstyle\sum_{j=1}^m (\beta(\mu_{1,j})\ldots\beta(\mu_{i(j),j}) + \beta(\mu_{i(j),j})\ldots\beta(\mu_{1,j}))$$

The above equation clearly shows that $\theta' \in Tr(\sigma')$.
Using equation (3.25) we find :

$$\beta((w_0\theta,\theta w_1\theta) \oplus (0,\theta(\bar{w}_1 - w_1)\theta) = (w_0'\theta', \theta'\bar{w}_1'\theta')$$

But as $\beta(w_0\theta,\theta w_1\theta) = (w_0'a_1',\ a_1'\ w_1'a_1')$ this yields :

$$w_0'a_1' + (\beta(0,\theta(\bar{w}_1 - w_1)\theta))_0 = w_0'\theta'.$$

Moreover $\beta(0,\ \theta(\bar{w}_1 - w_1)\theta) \in Z(R_{0,1}',\oplus)$ and we have $(\beta(0,\theta(\bar{w}_1 - w_1)\theta))_0$ $\in Rad(f')$. As $w_0' \notin Rad(f')$ the above equation is only possible if $a_1' = \theta'$.

But then we find as the form q' is anisotropic on V_0' that $a_0' = 0$. By this the proof of (3.23) is complete.

As a result of equation (3.23), we have that β is independent of the second coordinate i.e. if (v_0, v_1), $(v_0, \bar{v}_1) \in R_{0,1}$ with $\beta(v_0, v_1) = (v_0', v_1')$ then $\beta(v_0, \bar{v}_1) = (v_0', \bar{v}_1')$.
Indeed this follows from :

$$\begin{aligned}\beta(v_0, \bar{v}_1) &= \beta((v_0, v_1) \oplus (0, \bar{v}_1 - v_1)) \\ &= (v_0', v_1') \oplus (0, \beta(\bar{v}_1 - v_1)) \\ &= (v_0', v_1' + \beta(\bar{v}_1 - v_1)).\end{aligned}$$

This means β induces a bijection between V_0 and V_0' also denoted by β and defined in the following way. If $v_0 \in V_0$ we choose a $v_1 \in k$ such that $(v_0, v_1) \in R_{0,1}$ and set :

$$(\beta(v_0, v_1))_0 = \beta(v_0).$$

It thus makes sense to introduce the following notation. For $(v_0, v_1) \in R_{0,1}$ we write :

$$\beta(v_0, v_1) = (\beta(v_0), \beta^{v_0}(v_1))$$

where the superscript denotes a possible dependence on v_0.
Remark that the equation $s_{(0,1)}$ $((v_0, v_1)) = (v_0 v_1^{-1}, v_1^{-1})$, $\forall (v_0, v_1) \in R_{0,1}$ implies that :

$$s_{(0,1)}(\beta(v_0), \beta^{v_0}(v_1)) = (\beta(v_0 v_1^{-1}, \beta^{v_0 v_1^{-1}}(v_1^{-1})).$$

Hence :

$$\begin{aligned}\beta(v_0)\beta^{v_0}(v_1)^{-1} &= \beta(v_0 v_1^{-1}) \\ (\beta^{v_0}(v_1))^{-1} &= \beta^{v_0 v_1^{-1}}(v_1^{-1}),\ \forall (v_0, v_1) \in R_{0,1}\end{aligned}$$

Let (v_0, v_1), $(w_0, w_1) \in R_{0,1}$. As β induces an isomorphism between root groups we find :

$$\begin{aligned}\beta(v_0 + w_0) &= (\beta((v_0, v_0) \oplus (w_0, w_1)))_0 \\ &= (\beta((v_0, v_1)))_0 + (\beta(w_0)))_0 \\ &= \beta(v_0) + \beta(w_0).\end{aligned}$$

This means that β defines an additive bijection from V_0 to V_0'.

Our next goal is to show :
Claim 2 : β induces a semi-linear transformation from V_0 to V_0' with certain associated field isomorphism α and such that for $(v_0, v_1) \in R_{0,1}$, $\beta((v_0, v_1)) = (\beta(v_0), v_1^\alpha)$.

Remark that the assumptions on $\mathcal{M}U(V, q, k, \sigma)$ implies $dim(V) \geq 3$ and $dim(V') \geq 3$.
We consider two subcases.

1. First subcase : k is a generalized quaternion algebra with standard involution σ.
If $dim(V) = 3$, Lemma 114 and Proposition 128 show that $dim(V') = 3$. This means in particular that we can choose $v_0, \in V_0$, $v_0' \in V_0'$ with $\langle v_0 \rangle = V_0$ and $\langle v_0' \rangle = V_0'$ with $\beta(v_0) = v_0'$. Let i, j and $\in k$ such that $k = Z(k) \oplus iZ(k)$ $jZ(k)$ $+jiZ(k)$ and :

if $char(k) \neq 2$:
$i^2 = \alpha_0$, $j^2 = \beta_0$, $ij = -ji$

if $char(k) = 2$:
$i^2 = i + \alpha_0$, $j^2 = \beta_0$, $ij = ji + j$,

with α_0, $\beta_0 \in Z(k) \setminus Z(k)^2$. Without loss of generality we can choose i such that $g(v_0, v_0) = -i$. Denote the norm function in k by N. Choose similar i', j' for k' and denote the norm function in k' by N'. We use the notations of the proof of Lemma 114. Let $\mathcal{M}H(V_1, q_1, k_1, \sigma_1)$ be the hermitian Moufang set isomorphic to $\mathcal{M}U(V, q, k, \sigma)$ and $\mathcal{M}H(V_1', q_1', k_1', \sigma_1')$ the hermitian Moufang set isomorphic to $\mathcal{M}U(V', q', k', \sigma')$ as constructed in the proof of Lemma 114. Suppose that g_1 is a σ_1-sesquilinear on V_1 such that $q_1\ (v_1) = g(v_1)\ +k_{(\sigma,-1)}$, $\forall v_1 \in V_1$ and similarly that g_1' is a σ_1'-sesquilinear form on V_1' with $q_1'(v_1') = g'(v_1', v_1')\ +(k_1')_{(\sigma_1',-1)}$, $\forall v_1' \in V_1'$. By construction we have $k_1 = Z(k)(i)$. Without loss of generality we can moreover assume that $k_1' = Z(k)(i')$. Suppose β_1 is the isomorphism from $\mathcal{M}H(V_1, q_1, k_1, \sigma_1)$ to $\mathcal{M}U(V, q, k, \sigma)$ and β_1' is the isomorphism from $\mathcal{M}H(E_1', q_1', k_1', \sigma_1')$ to $\mathcal{M}U(V', q', k', \sigma')$ as described in the proof of Lemma 114. Remark that β_1 is given by :

$$\begin{aligned} \beta_1((\infty)) &= (\infty) \\ \beta_1((v_0(z_1 + iz_2) + v_0 j(z_3 + iz_4), N(\lambda) + u) &= (v_0\lambda^\sigma, -\lambda v_1^\sigma \lambda^\sigma - u) \end{aligned}$$

where $\lambda = z_1 + iz_2 + jz_3 + jiz_4$ and similarly :

$$\begin{aligned}
\beta_1'((\infty)) &= (\infty) \\
\beta_1'((v_0'(z_1' + i'z_2') + v_0'j'(z_3' + i'z_4), N'(\lambda') + u') &= (v_0'\lambda'^{\sigma'}, -\lambda' v_1'^{\sigma'} \lambda'^{\sigma'} - u')
\end{aligned}$$

where $\lambda' = z_1' + i'z_2' + j'z_3' + j'i'z_4'$. Consider the isomorphism $\beta_1'^{-1}\beta\beta_1$ from $\mathcal{M}H(V_1, q_1, k_1, \sigma_1)$ to $\mathcal{M}H(V_1', q_1', k_1', \sigma_1')$ and denote it by $\tilde{\beta}$. By Proposition 128 we know that there exist a constant c' k_1' and a semi-linear transformation φ from V_1 to V_1' with associated field isomorphism α such that :

$$\begin{aligned}
\tilde{\beta}(\langle x_1 \rangle &= \langle \varphi(x_1) \rangle, \forall \langle x_1 \rangle \in \mathcal{M}H(V_1, q_1, k_1, \sigma_1) \\
c'(f_1(x_1, y_1))^\alpha &= f_1'(\varphi(x_1), \varphi(y_1)), \forall x_1, y_1 \in V_1 \\
c'(g_1(x_1, x_1))^\alpha &= g_1'(\varphi(x_1), \varphi(x_1)), \forall x_1 \in V_1.
\end{aligned}$$

As α defines an isomorphism from $Z(k)(i)$ to $Z(k')(i')$ we can assume without loss of generality that $i^\alpha = i'$ and $\alpha_0^\alpha = \alpha_0'$. As by assumption $\beta((0,1)) = (0,1)$, we have $\tilde{\beta}$ $((0,1)) = (0,1)$ and hence the proof of Proposition 128 implies that $c' = 1$. Remark that $\beta_1'^{-1}$ and β_1 induce semi-linear transformations (also denoted by $\beta_1'^{-1}$ and β_1) satisfying :

$$\begin{aligned}
\beta_1(v_0((z_1 + iz_2) + v_0 j(z_3 + iz_4)) &= (v_0((z_1 + iz_2 + jz_3 + jiz_4)) \\
\beta_1'(v_0'((z_1' + i'z_2') + v_0'j'(z_3' + i'z_4') &= v_0'(z_1' + iz_2' + j'z_3' + j'i'z_4')
\end{aligned}$$

Let $\tilde{\beta}(v_0 j) = \varphi(v_0 j) = w_0'$, with $w_0' = v_0'\lambda_0'$, and $\lambda_0' = \lambda_1' + i'\lambda_2' + j'\lambda_3' + j'i'\lambda_4'$.
Then we have $V_0' = v_0'(Z(k')i') \oplus w_0'(Z(k')i')$.
Suppose $char(k) \neq 2$.
We find $f(v_0, v_0 j) = 0$ and $g(v_0 j, v_0 j) = \beta_0 i$.
Hence :

$$\begin{aligned}
g_1'(w_0', w_0') &= (g_1(v_0 j, v_0 j))^\alpha \\
&= \beta_0^\alpha i^\alpha \\
&= -N'(\lambda_0')i' \\
f_1'(v_0', w_0') &= (f_1(v_0, v_0 j))^\alpha \\
&= 0 \\
&= -\lambda_0' i' + i'^\sigma \lambda_0'
\end{aligned}$$

The last equation implies that $\lambda_0'\ i' = -i'\ \lambda_0'$ and hence $\lambda_1' = 0$. This yields $(\lambda_0')^\sigma = -\lambda_0'$, and thus $N'(\lambda_0') = -\beta_0^\alpha$ gives $(\lambda_0')^2 = \beta_0^\alpha$. This means that we can extend α to an isomorphism from k to k' if we set $(z_1 + iz_2 + jz_3 + jiz_4)^\alpha$ $= z_1^\alpha + i'z_2^\alpha + \lambda_0' z_3^\alpha + \lambda_0' i' z_4^\alpha$.
Suppose $char(k) = 2$.
In this case we have $f_1(v_0, v_0 j) = 0$ and $g_1(v_0 j, v_0 j) = \beta_0 i$.
Hence :

$$\begin{aligned} g_1'(w_0', w_0') &= \beta_0^\alpha i' \\ &= N'(\lambda_0')i' \\ f_1'(v_0', w_0') &= 0 \\ &= \lambda_0' i' + i'\lambda_0' + \lambda_0' \end{aligned}$$

The first of these equations shows that $\beta_0^\alpha = N'(\lambda_0')$ while the last equation implies that $\lambda_4' = 0$ and $\lambda_3' = 1$.
From :

$$\begin{aligned} (1 + N'(\lambda_0'))i' &= (1 + \beta_0^\alpha)i' \\ &= (N((1+j))i)^\alpha \\ &= (g_1(v_0 + v_0 j, v_0 + v_0 j))^\alpha \\ &= g_1'(v_0' + v_0'\lambda_0', v_0' + v_0'\lambda_0') \\ &= N'((1+\lambda_0'))i' \\ &= (1 + (\lambda_0' + {\lambda_0'}^{\sigma'}) + N(\lambda_0'))i' \end{aligned}$$

if follows that $\lambda_0' = {\lambda_0'}^{\sigma'}$.
This implies that as in the characteristic non 2 case we can extend the isomorphism α to an isomorphism from k to k' if we set :

$$(z_1 + iz_2 + jz_3 + jiz_4)^\alpha = z_1^\alpha + i'z_2^\alpha + \lambda_0' z_3^\alpha + \lambda_0' i' z_4^\alpha.$$

We show that for $w_0 \in \langle v_0 \rangle$ and $\mu \in k$

$$\beta(w_0\mu) = \beta(w_0)\mu^\alpha.$$

We have :

$${\beta_1'}^{-1}\beta\beta_1(v_0(u_1 + iu_2) = v_0'(u_1 + iu_2)^\alpha.$$

Using the explicit expressions of β_1 and β_1' this leads to :

$$\beta(v_0(u_1+iu_2)^\sigma)=\beta(v_0)(u_1+iu_2)^{\alpha\sigma'}.$$

Moreover :

$$\beta_1'^{-1}\beta\beta_1(v_0j(u_3+iu_4))=v_0'\lambda_0'(u_3+iu_4)^\alpha$$

shows that :

$$\beta(v_0(ju_3+jiu_4)^\sigma)=\beta(v_0)(ju_3+jiu_4)^{\alpha\sigma'}.$$

As :

$$\beta(v_0j^2)=\beta(v_0)\beta_0=\beta(v_0)(\lambda_0')^2$$

one deduces that for any $w_0\in\langle v_0\ \rangle$ and μ as above $\beta(w_0\mu)=\beta(w_0)\ \mu^\alpha$. Hence β defines a semi-linear transformation from V_0 to V_0' with associated field isomorphism α.

This means that we can assume for the rest of this subcase that $dim(V)\geq 4$ and $dim(V')\geq 4$.
Suppose that $char(k)\neq 2$. We present a proof which holds whenever $Tr(\sigma)=Fix(\sigma)$, $Tr(\sigma')=Fix(\sigma')$ and $dim(V_0)\geq 2$.
If k is a generalized quaternion algebra with standard involution σ such that $char(k)\neq 2$ we have $Tr(\sigma)=Fix(\sigma)=Z(k)$. As we already saw this implies that k' is a generalized quaternion algebra with standard involution σ'. As $char(k')\neq 2$ it follows that $Tr(\sigma')=Fix(\sigma')=Z(k')$. This shows that if $dim(V_0)\geq 2$ and k is a generalized quaternion algebra with standard involution σ such that $char(k)\neq 2$ the proof which we will presents holds.

The first step consists in showing that for any two v_0, $w_0\in V_0$:

$$f(v_0,w_0)=0\Leftrightarrow f'(\beta(v_0),\beta(w_0))=0.$$

We show one direction (the other direction then follows by symmetric arguments).
Suppose for v_0, $w_0\in V_0$, $f(v_0,w_0)=0$. As we know that $\beta(0)=0$ we can assume $v_0\neq 0$ and $w_0\neq 0$. Choose v_1, $w_1\in k$ with (v_0,v_1), $(w_0,w_1)\in R_{0,1}$. Consider $(v_0v_1^{-1},v_1^{-1})\in R_{0,1}$.
We find :

$$\beta(v_0v_1^{-1},v_1^{-1})\ =\ \beta(s_{(0,1)}(v_0,v_1))$$

$$\begin{aligned} &= s_{(0,1)}(\beta(v_0), \beta^{v_0}(v_1)) \\ &= (\beta(v_0)(\beta^{v_0}(v_1))^{-1}, (\beta^{v_0}(v_1))^{-1}). \end{aligned}$$

Moreover $f(v_0 v_1^{-1}, w_0) = 0$ yields :

$$[(v_0 v_1^{-1}, v_1^{-1}), (w_0, w_1)] = 0.$$

Sending this equation over to $R'_{0,1}$ via β gives :

$$[(\beta(v_0)(\beta^{v_0}(v_1))^{-1}, (\beta^{v_0}(v_1))^{-1}), (\beta(w_0), \beta^{w_0}(w_1))] = 0$$

and hence :

$$f'(\beta(v_0)(\beta^{v_0}(v_1))^{-1}, \beta(w_0)) = f'(\beta(w_0), \beta(v_0)(\beta^{v_0}(v_1))^{-1}).$$

If $f'(\beta(v_0), \beta(w_0)) \neq 0$ this equation implies that :

$$\beta^{v_0}(v_1) = (\beta^{v_0}(v_1))^{\sigma}.$$

As by assumption $Tr(\sigma') = Fix(\sigma')$ and the form q' is anisotropic on V'_0 we see that $\beta(v_0) = 0$ and hence $v_0 = 0$, a contradiction.
Remark that Lemma 109 implies $Rad(f) = 0$ and $Rad(f') = 0$ as q and q' are forms of Witt index 2 with $Tr(\sigma) = Fix(\sigma)$ and $Tr(\sigma') = Fix(\sigma')$.
Let $\lambda \in k$ and $v_0 \in V_0$. As $v_0^{\perp}$ is sent via β to $(\beta(v_0))^{\perp}$ it follows that

$$\beta(v_0 \lambda) = \lambda'_{v_0},$$

with $\lambda'_{v_0} \in k'$ and where the subscript denotes a possible dependence on v_0.
As $dim(V_0) \geq 2$ Lemma 54 implies that β defines a semi-linear transformation from V_0 to V'_0 with an associated field isomorphism α.
Suppose that $char(k) = 2$.
In this case we know that by assumption $Rad(f) = \{0\}$ and $Rad(f') = \{0\}$. Denote in the sequel of this proof for $x \in k$, $L_x = Z(k)(x)$ i.e. L_x is the subfield of k generated over $Z(k)$ by x and similarly for $x' \in k'$, $L_{x'} = Z(k')(x')$. We show that for any $(v_0, v_1) \in R_{0,1}$

$$\beta(v_0 L_{v_1}) = \beta(v_0) L_{\beta^{v_0}(v_1)}. \tag{3.25}$$

From $s_{(0,1)}(v_0, v_1) = (v_0 v_1^{-1}, v_1^{-1})$ we get after applying β that $\beta(v_0 v_1^{-1}) = \beta(v_0)(\beta^{v_0}(v_1))^{-1}$. Let $z \in Z(k) = Tr(\sigma)$. Then the equation $s_{(0,z)}$ $s_{(0,1)}$

$((v_0, v_1)) = (v_0 z, z v_1 z)$ gives after applying β that $\beta(v_0 z) = \beta(v_0)\beta(z)$, $\forall z \in Z(k)$. As $L_{v_1} = Z(k)(v_1) = Z(k)(v_1^{-1})$ and $L_{\beta^{v_0}(v_1)} = L_{(\beta^{v_0}(v_1))^{-1}}$ we see that $\beta(v_0 L_{v_1}) \subset \beta(v_0) L_{\beta^{v_0}(v_1)}$. The other inclusion $\beta(v_0 L_{\beta^{v_0}(v_1)}) \subset \beta(v_0 L_{v_1})$ follows by similar arguments.
Let v_0, $w_0 \in V_0$. We prove :

$$f(v_0, w_0) = 0 \Leftrightarrow f'(\beta(v_0), \beta(w_0)) = 0.$$

We show one direction. The other direction follows by symmetric arguments. Let $f(v_0, w_0) = 0$ for some v_0, $w_0 \in V_0$. Firstly we show that :

$$f'(\beta(v_0), \beta(w_0)) = f'(\beta(w_0), \beta(v_0)). \tag{3.26}$$

Choose v_1, $w_1 \in k$ such that (v_0, v_1), $(w_0, w_1) \in R_{0,1}$. We have $[(v_0, v_1), (w_0, w_1)] = 0$. As β defines an isomorphism from $(R_{0,1}, \oplus)$ to $(R'_{0,1}, \oplus)$ we have :

$$[(\beta(v_0), \beta^{v_0}(v_1)), (\beta(w_0), \beta^{w_0}(w_1))] = 0$$

or equivalently

$$f'(\beta(v_0), \beta(w_0)) = f'(\beta(w_0), \beta(v_0)).$$

Suppose firstly that $f(v_0, w_0) = 0$ with $w_0 = v_0$. As also $f(v_0 v_1^{-1}, v_0) = 0$ in this case and $\beta((v_0 v_1^{-1}, v_1^{-1})) = \beta(s_{(0,1)}((v_0, v_1))) = s_{(0,1)}\ (\beta(v_0), \beta^{v_0}(v_1)) = (\beta(v_0)(\beta^{v_0}(v_1))^{-1}, (\beta^{v_0}(v_1))^{-1})$ we see that :

$$f'(\beta(v_0)(\beta^{v_0}(v_1))^{-1}, \beta(v_0)) = f'(\beta(v_0), \beta(v_0)(\beta^{v_0}(v_1))^{-1}).$$

Equivalently :

$$(\beta^{v_0}(v_1)^{-1})^{\sigma} f'(\beta(v_0), \beta(v_0)) = f'(\beta(v_0), \beta(v_0))(\beta^{v_0}(v_1))^{-1}).$$

Suppose $f'(\beta(v_0), \beta(v_0)) \neq 0$ then we find :

$$(\beta^{v_0}(v_1)^{-1})^{\sigma} = f'(\beta(v_0), \beta(v_0))\beta^{v_0}(v_1^{-1})(f'(\beta(v_0), \beta(v_0)))^{-1}. \tag{3.27}$$

As $(\beta^{v_0}(v_1)^{-1})^{\sigma} \in L_{\beta^{v_0}(v_1)}$ it follows that $f'(\beta(v_0), \beta(v_0))$ stabilizes $L_{\beta^{v_0}(v_1)}$ under conjugation. If $f'(\beta(v_0), \beta(v_0)) \notin L_{\beta^{v_0}(v_1)}$ it follows that as $Z(k)$, $\beta^{v_0}(v_1)$ also stabilizes $L_{\beta^{v_0}(v_1)}$ and k is a generalized quaternion algebra that k' stabilizes $L_{\beta^{v_0}(v_1)}$. A similar reasoning as the one used in the proof of Lemma 123 leads to a contradiction. Therefore we find that $f'(\beta(v_0), \beta(v_0)) \in$

$L_{\beta^{v_0}(v_1)}$. As $L_{\beta^{v_0}(v_1)}$ is a commutative field equation (3.27) becomes $(\beta^{v_0}(v_1))^\sigma$ $= \beta^{v_0}(v_1)$. From $q'(\beta(v_0)) = g'(\beta(v_0), \beta(v_0)) + Z(k) = \beta^{v_0}(v_1) + Z(k)$ we find that $g'(\beta(v_0), \beta(v_0)) \in Fix(\sigma')$. Hence $f'(\beta(v_0), \beta(v_0)) = g'(\beta(v_0), \beta(v_0))$ $+(g'(\beta(v_0), \beta(v_0)))^\sigma) = 0$, a contradiction. Therefore we find that
$f'(\beta(v_0), \beta(v_0)) = 0$.

Remains to consider the case where $f(v_0, w_0) = 0$ and $v_0 \neq w_0$. Suppose $f'(\beta(v_0), \beta(w_0)) \neq 0$. We find that $f(v_0 v_1^{-1}, w_0) = f(v_0, w_0 w_1^{-1}) =$ $f(v_0 v_1^{-1}, w_0 w_1^{-1}) = 0$. Moreover we have $\beta((v_0 v_1^{-1}, v_1^{-1})) = \beta(s_{(0,1)}((v_0, v_1)))$ $= (\beta(v_0)\beta^{v_0}(v_1)^{-1},\ \beta(v_0)\beta^{v_0}(w_1)^{-1}$ and similarly $\beta((w_0 v_1^{-1},\ w_1^{-1}))\ (\beta(w_0)$ $(\beta^{w_0}(w_1))^{-1},\ (\beta^{w_0}(w_1))^{-1})$. In view of (3.26) this means that the following equations hold :

$$
\begin{aligned}
f'(\beta(v_0), \beta(w_0)) &= f'(\beta(w_0), \beta(v_0)) \\
f'(\beta(v_0)(\beta^{v_0}(v_1))^{-1}, \beta(w_0)) &= f'(\beta(w_0), \beta(v_0)(\beta^{v_0}(v_1))^{-1}) \\
f'(\beta(v_0), \beta(w_0)(\beta^{w_0}(w_1))^{-1}) &= f'(\beta(w_0)(\beta^{w_0}(w_1))^{-1}, \beta(v_0)) \\
f'(\beta(v_0)(\beta^{v_0}(v_1))^{-1}, \beta(w_0)(\beta^{w_0}(w_1))^{-1}) &= f'(\beta(w_0)(\beta^{w_0}(w_1))^{-1}, \beta(v_0) \\
&\quad (\beta^{v_0}(v_1))^{-1}).
\end{aligned}
$$

Equivalently :

$$
\begin{aligned}
f'(\beta(v_0), \beta(w_0) &= f'(\beta(w_0), \beta(v_0)) &\quad (3.28)\\
(\beta^{v_0}(v_1)^{-1})^\sigma f'(\beta(v_0), \beta(w_0)) &= f'(\beta(w_0), \beta(v_0))(\beta^{v_0}(v_1))^{-1} &\quad (3.29)\\
f'(\beta(v_0), \beta(w_0))(\beta^{w_0}(w_1))^{-1} &= (\beta^{w_0}(w_1)^{-1})^\sigma f'(\beta(w_0), \beta(v_0)) &\quad (3.30)\\
(\beta^{v_0}(v_1)^{-1})^\sigma f'(\beta(v_0), \beta(w_0))(\beta^{w_0}(w_1))^{-1} &= (\beta^{w_0}(w_1)^{-1})^\sigma f'(\beta(w_0), \beta(v_0))\beta^{v_0}(v_1) \\
&\quad . &\quad (3.31)
\end{aligned}
$$

Using formulas (3.28), (3.29) and (3.30), formula (3.31) yields :

$$f'(\beta(v_0), \beta(w_0))(\beta^{v_0}(v_1))^{-1}(\beta^{w_0}(w_1))^{-1} = f'(\beta(v_0), \beta(w_0))(\beta^{w_0}(w_1))^{-1}(\beta^{v_0}(v_1))^{-1}.$$

If $f'(\beta(v_0), \beta(w_0)) \neq 0$ this implies :

$$\beta^{v_0}(v_1)\beta^{w_0}(w_1) = \beta^{w_0}(w_1)\beta^{v_0}(v_1)$$

If $\beta^{w_0}(w_1) \notin L_{\beta_0^v(v_1)}$ the above equation implies $\beta^{v_0}(v_1) \in Z(k)$ a contradiction as q is anisotropic on V_0. Thus $\beta^{w_0}(w_1) \in L_{\beta^{v_0}(v_1)}$ and we find that $L_{\beta^{w_0}(w_1)}$ $= L_{\beta^{v_0}(v_1)}$.

Equation (3.29) implies :

$$(\beta^{v_0}(v_1)^{-1})^\sigma = (f'(\beta(v_0), \beta(w_0))(\beta^{v_0}(v_1))^{-1}(f'(\beta(v_0), \beta(v_0))^{-1}.$$

As $(\beta^{v_0}(v_1)^{-1})^\sigma \in L_{\beta^{v_0}(v_1)} \setminus Z(k)$ this shows that $f'(\beta(v_0),\beta(v_0))$ stabilizes $L_{\beta^{v_0}(v_1)}$ via conjugation. If $f'(\beta(v_0),\beta(w_0)) \notin L_{\beta^{v_0}(v_1)}$ we find that k' stabilizes $L_{\beta^{v_0}(v_1)}$. A similar reasoning as the one used to prove Lemma 123 leads to a contradiction. Therefore we find that $f'(\beta(v_0),\beta(w_0)) \in L_{\beta^{v_0}(v_1)}$, a commutative field and hence $(\beta^{v_0}(v_1))^{\sigma'} = \beta^{v_0}(v_1)$. As $q'(\beta(v_0)) = \beta^{v_0}(v_1) + Tr(\sigma') = g'(\beta(v_0),\beta(v_0)) + Tr(\sigma')$ we see that $g'(\beta(v_0),\beta(v_0)) \in Fix(\sigma')$ and $f'(\beta(v_0),\beta(v_0)) = g'(\beta(v_0),\beta(v_0)) + (g'(\beta(v_0),\beta(v_0))^{\sigma'} = 0$. Similar arguments show that $f'(\beta(w_0),\beta(w_0)) = 0$, $\beta^{w_0}(w_1) \in Fix(\sigma')$ and $f'(\beta(v_0),\, \beta(w_0)) \in L_{\beta^{w_0}(w_1)}$. In particular we find by what is already proved that $f(w_0,w_0) = 0$. Hence $w_1 \in Fix(\sigma)$.
Suppose firstly that $w_0 \notin \langle v_0 \rangle$. Using Lemma 108 we find :

$$s_{(w_0,w_1)}s_{(0,1)}((v_0,v_1)) = (v_0w_1, w_1v_1w_1).$$

Applying β to this equation yields :

$$\begin{aligned}
\beta(v_0w_1) &= (\beta((v_0w_1, w_1v_1w_1)))_0 \\
&= (\beta(s_{(w_0,w_1)}((v_0,v_1))))_0 \\
&= (s_{(\beta(w_0),\beta^{w_0}(w_1))}s_{(0,1)}((\beta(v_0),\beta^{v_0}(v_1))_0 \\
&= \beta(w_0)\beta^{w_0}(w_1)^{-1}f'(\beta(v_0),\beta(w_0))\beta^{w_0}(w_1) + \beta(v_0)\beta^{w_0}(w_1).
\end{aligned}$$

As $\beta^{w_0}(w_1)f'(\beta(v_0),\beta(w_0))\beta^{w_0}(w_1) \in L_{\beta^{w_0}(w_1)}$ and $\beta^{w_0}(w_1) \in L_{\beta^{v_0}(v_1)}$ formula (3.25) implies that there exist $\theta_1 \in L_{\beta^{w_0}(w_1)}$ and $\theta_2 \in L_{\beta^{v_0}(v_1)}$ such that :

$$\begin{aligned}
\beta(w_0\theta_1) &= \beta(w_0)(\beta^{w_0}(w_1))^{-1}f'(\beta(v_0),\beta(w_0))\beta^{w_0}(w_1) \\
\beta(v_0\theta_2) &= \beta(v_0)\beta^{w_0}(w_1).
\end{aligned}$$

We find :

$$\beta(v_0w_1) = \beta(w_0\theta_1 + v_0\theta_2)$$

and thus :

$$v_0w_1 = w_0\theta_1 + v_0\theta_2.$$

As $w_0 \notin \langle v_0 \rangle$ this is only possible if $\theta_1 = 0$ and $\beta(w_0\theta_1) = \beta(w_0)\ (\beta^{w_0}(w_1))^{-1}$ $f'(\beta(v_0),\beta(w_0))\ \beta^{w_0}(w_1) = 0$. But then it follows that $f'(\beta(v_0),\beta(w_0)) = 0$, a contradiction against the assumption on $f(\beta(v_0),\beta(w_0))$. Therefore we find that if $w_0 \notin \langle v_0 \rangle$ and $f(v_0,w_0) = 0$ necessarily also $f'(\beta(v_0),\beta(w_0)) = 0$.
Remains the case where $w_0 \in \langle v_0 \rangle$. This means we can set $w_0 = v_0\mu$, $\mu \in k$. Without loss of generality we can thus assume that $w_1 = \mu^\sigma\ v_1\mu$.

Remark that if $\mu \in L_{v_1}$ we find $\beta(v_0\mu) = \beta(v_0)\ \mu'$ for some $\mu' \in L_{\beta^{v_0}(v_1)}$ and we find $f(\beta(v_0), \beta(v_0\mu)) = f(\beta(v_0), \beta(v_0))\mu' = 0$. Hence we can assume that $\mu \notin L_{v_1}$. Suppose $f'(\beta(v_0), \beta(v_0\mu)) \neq 0$. Then we already deduced that $f'(\beta(v_0), \beta(v_0)) = 0$, $f'(\beta(v_0\mu), \beta(v_0\mu)) = 0$, $L_{\beta^{v_0}(v_1)} = L_{\beta^{v_0\mu}(\mu^\sigma v_1 \mu)}$ and $f'(\beta(v_0),\ \beta(v_0\mu)) \in L_{\beta^{v_0}(v_1)}$. Using Lemma 108 we find :

$$(s_{(v_0,v_1)} s_{(0,1)}(v_0\mu, \mu^\sigma v_1 \mu))_0 = v_0 \mu v_1.$$

Applying β to this equation gives :

$$\begin{aligned}
&\beta(v_0\theta v_1)\\
&= (\beta(s_{(v_0,v_1)} s_{(0,1)}(v_0\mu, \mu^\sigma v_1\mu))_0\\
&= (s_{(\beta(v_0),\beta^{v_0}(v_1))} s_{(0,1)}(\beta(v_0\mu), \beta^{v_0\mu}(\mu^\sigma v_1 \mu)))_0\\
&= \beta(v_0)(\beta^{v_0}(v_1))^{-1} f'(\beta(v_0), \beta(v_0\mu))\beta^{v_0}(v_1) + \beta(v_0\mu)\beta^{v_0}(v_1).
\end{aligned}$$

Now as $\beta^{v_0}(v_1) \in L_{\beta^{v_0\mu}(\mu^\sigma v_1\mu)}$ and $(\beta^{v_0}(v_1))^{-1} f'(\beta(v_0), \beta(w_0))\ \beta^{v_0}(v_1) \in L_{\beta^{v_0}(v_1)}$ formula (3.25) implies that there exist $\theta_1 \in L_{v_1}$ and $\theta_2 \in L_{\mu^\sigma v_1 \mu}$ with :

$$\begin{aligned}
\beta(v_0\theta_1) &= \beta(v_0)(\beta^{v_0}(v_1))^{-1} f'(\beta(v_0), \beta(v_0\mu))\beta^{v_0}(v_1)\\
\beta(v_0\mu\theta_2) &= \beta(v_0\mu)(\beta^{v_0}(v_1))^{-1}.
\end{aligned}$$

But this means that we have :

$$\beta(v_0\mu v_1) = \beta(v_0\theta_1) + \beta(v_0\mu\theta_2), \tag{3.32}$$

and :

$$\mu v_1 = \theta_1 + \mu\theta_2.$$

As $\theta_2 \in L_{\mu^\sigma v_1 \mu}$ there exist z_1, $z_2 \in Z(k)$ with $\theta_2 = z_1 + \mu^\sigma v_1 \mu z_2$. This means that equation (3.32) becomes :

$$\mu v_1 = \theta_1 + \mu z_1 + \mu\mu^\sigma v_1 \mu z_2$$

or equivalently :

$$\mu(z_1 + \mu^\sigma v_1 \mu z_2 + v_1) = \theta_1.$$

As $\mu \notin L_{v_1}$ this is only possible if $\theta_1 = 0$ and $\beta(v_0\theta_1) = 0 = \beta(v_0)(\beta^{v_0}(v_1))^{-1}$ $f'(\beta(v_0), \beta(v_0\mu))\beta^{v_0}(v_1)$. We find $f'(\beta(v_0), \beta(v_0\mu) = 0$ contradicting the assumption on $f'(\beta(v_0), \beta(v_0\mu))$. Therefore we also find in this case that $f'(\beta(v_0), \beta(w_0)) = 0$.

This completes the proof that :

$$f(v_0, w_0) = 0 \Leftrightarrow f'(\beta(v_0), \beta(w_0)) = 0.$$

Now we can proceed as in the case where $char(k) \neq 2$ and k is a generalized quaternion algebra with standard involution to see that β induces a semi-linear transformation with an associated field isomorphism α such that $\beta(v_0\lambda) = \beta(v_0)\lambda^{\alpha}$, $\forall \lambda \in k$, $\forall v_0 \in V_0$.

2. Second subcase : the general case, if k is a generalized quaternion algebra σ is a non standard involution.
By Lemma 47 we have that k is generated as a ring by $Tr(\sigma)$. Consider $\lambda \in k$ and $(v_0, v_1) \in R_{0,1}$. Then λ can be expressed in terms of elements of $Tr(\sigma)$ i.e. for there exist $\theta_i \in Tr(\sigma)$, $1 \leq i \leq n$ such that for example :

$$\lambda = \sum_{j=1}^{n} \theta_{1,j} \ldots \theta_{i(j),j},\ \theta_{k,j} \in Tr(\sigma).$$

Denote as before $S_{(v_0,v_1)} = \langle u((\infty); (0,0), (v_0, v_1)), m_\mu | \mu \in Tr(\sigma) \rangle$.
Using property (3.24) we find a $\psi_\lambda \in S_{(v_0,v_1)}$ such that :

$$\begin{aligned}
\beta(\psi_\lambda((v_0, v_1))) &= \beta((v_0\lambda, \lambda\bar{v}_1\lambda^{\sigma})) \\
&= (\beta(v_0)\lambda', \lambda\bar{v}_1\lambda^{\sigma'}) \\
&= \beta((v_0\lambda, \lambda\bar{v}_1\lambda^{\sigma})) \\
&= (\beta(v_0\lambda), \beta^{v_0\lambda}(\lambda\bar{v}_1\lambda^{\sigma}))
\end{aligned}$$

with $(v_0, \bar{v}_1) \in R_{0,1}$ and $(w_0', \bar{w}_1') \in R_{0,1}'$ and $\lambda' = \sum_{j=1}^{n} \beta(\theta_{1,1}^{j}) \ldots \beta(\theta_{1,i(j)}^{j})$
This implies :

$$\beta(v_0\lambda) = \beta(v_0)\lambda',\ \forall v_0 \in V_0. \tag{3.33}$$

Remark that we can not use Lemma 54 in this case to show that β induces a semi-linear transformation from V_0 to V_0' as $dim(V_0)$ can be 1. Therefore we will have to proceed in another way.
Define the map α from k to k' in the following way. If $\mu \in k$, we choose an expression of μ in terms of elements of $Tr(\sigma)$ i.e. for example :

$$\mu = \sum_{j=1}^{m} \bar{\theta}_{1,j} \ldots \bar{\theta}_{i(j),j},\ \bar{\theta}_{k,j} \in Tr(\sigma).$$

Subsequently we define :

$$\mu^{\alpha} = \sum_{j=1}^{m} \beta(\bar{\theta}_{1,j}) \ldots \beta(\bar{\theta}_{i(j),j}).$$

We show that α is well defined i.e. that it is independent of the expression of μ in terms of elements of $Tr(\sigma)$. This will be done is we prove that every expression of 0 in terms of elements of $Tr(\sigma)$ is sent over by α to 0.
Consider an arbitrary expression of elements of $Tr(\sigma)$ i.e. for example $\sum_{j=1}^{l}$ $\rho_{1,j}\ldots\rho_{i(j),j}$, $\rho_{k,j}\in Tr(\sigma)$. Choose $v_0\neq 0$ in V_0.
Then we know that :

$$\beta((v_0(\sum_{j=1}^{l}\rho_{1,j}\ldots\rho_{i(j),j}))) = \beta(v_0)(\sum_{j=1}^{l}\beta(\rho_{1,j})\ldots\beta(\rho_{i(j),j}).$$

Hence :

$$\sum_{j=1}^{l}\rho_{1,j}\ldots\rho_{i(j),j} = 0 \Leftrightarrow \sum_{j=1}^{l}\beta(\rho_{1,j})\ldots\beta(\rho_{i(j),j}) = 0.$$

This shows that α is a well defined bijection from k to k'. By construction we see that α defines a field isomorphism from k to k' satisfying :

$$\beta(v_0\lambda) = \beta(v_0)\lambda^{\alpha},\ \forall v_0\in V_0, \lambda\in k.$$

and :

$$\lambda^{\sigma\alpha} = \lambda^{\alpha\sigma'},\ \forall,\ \forall\lambda\in k.$$

This prove that β defines a semi-linear transformation from V_0 to V_0' with associated field isomorphism α.
Let $(v_0,v_1)\in R_{0,1}$.
Then :

$$\begin{aligned}\beta s_{(0,1)}(v_0,v_1) &= (-\beta(v_0v_1^{-1}), \beta^{-v_0v_1^{-1}}(-v_1^{-1}))\\ &= (-\beta(v_0)(v_1^{-1})^{\alpha}, \beta^{-v_0v_1^{-1}}(-v_1^{-1}))\\ &= (-\beta(v_0)(\beta^{v_0}(v_1))^{-1}, (-\beta^{v_0}(v_1)^{-1})\\ &= s_{(0,1)}(\beta(v_0),\beta^{v_0}(v_1)).\end{aligned}$$

yields :

$$\begin{aligned}(\beta^{v_0}(v_1))^{-1} &= (v_1^{-1})^{\alpha}\\ -(\beta^{v_0}(v_1))^{-1} &= \beta^{-v_0v_1^{-1}}(-v_1^{-1}).\end{aligned}$$

The first of these equations shows $\beta^{v_0}(v_1)$ is independent of v_0 and :

$$\beta^{v_0}(v_1) = v_1^{\alpha}.$$

Therefore we have :

$$\beta(v_0, v_1) = (\beta(v_0), v_1^\alpha),\ \forall(v_0, v_1) \in R_{0,1}. \tag{3.34}$$

Let (v_0, v_1), $(w_0, w_1) \in R_{0,1}$. As $\beta((v_0, -g(v_0, v_0))) = (\beta(v_0),(-g(v_0, v_0))^\alpha)$ and $(\beta(v_0),-g'(\beta(v_0), \beta(v_0)) \in R'_{0,1}$ we find :

$$g(v_0, v_0)^\alpha = g'(\beta(v_0), \beta(v_0)) + k'_{\sigma',-1},\ \forall v_0 \in V_0.$$

Moreover equation (3.34) implies :

$$\begin{aligned}
&(\beta(v_0) + \beta(w_0), v_1^\alpha + w_1^\alpha - (f(v_0, w_0))^\alpha)\\
&= \beta((v_0, v_1) \oplus (w_0, w_1))\\
&= \beta((v_0, v_1)) \oplus \beta((w_0, w_1))\\
&= (\beta(v_0) + \beta(w_0), v_1^\alpha + w_1^\alpha - f'(\beta(v_0), \beta(w_0))).
\end{aligned}$$

Therefore :

$$(f(v_0, w_0))^\alpha = f'(\beta(v_0), \beta(w_0)),\ \forall v_0, w_0 \in V_0.$$

Define the semi-linear transformation φ from V to V' by :

$$\varphi(e_{-1}x_{-1} + x_0 + e_1x_1) = e'_{-1}x_{-1}^\alpha + \beta(x_0) + e'_1x_1^\alpha,\ \forall x_{-1}, x_1 \in k, \forall x_0 \in V_0.$$

We check φ preserves the forms.
For $x = e_{-1}x_{-1} + x_0 + e_1x_1 \in V$ we have :

$$q(x) = -x_{-1}^\sigma x_1 + q(x_0).$$

Applying α to this equation gives :

$$\begin{aligned}
(q(x))^\alpha &= -x_{-1}^{\sigma\alpha}x_1^\alpha + (g(x_0, x_0))^\alpha + k'_{\sigma',-1}\\
&= -x_{-1}^{\alpha\sigma'}x_1^\alpha + q'(\beta(x_0))\\
&= q'(e'_{-1}x_{-1}^\alpha + \beta(x_0) + e'_1x_1^\alpha)\\
&= q'(\varphi(x)).
\end{aligned}$$

Let $e_{-1}x_{-1} + x_0 + e_1x_1$, $e_{-1}y_{-1} + y_0 + e_1y_1 \in V$ then :

$$f(x, y) = -x_{-1}^\sigma y_1 + x_1^\sigma y_{-1} + f(x_0, y_0).$$

Applying α to this equation gives :

$$\begin{aligned}
(f(x, y))^\alpha &= (-x_{-1}^\sigma y_1 + x_1^\sigma y_{-1} + f(x_0, y_0))^\alpha\\
&= -x_{-1}^{\alpha\sigma'}y_1^\alpha + x_1^{\alpha\sigma'}y_{-1}^\alpha + f(\beta(x_0), \beta(y_0)\\
&= f'(\varphi(x), \varphi(y)).
\end{aligned}$$

Thus φ is a semi-linear transformation from V_0 to V_0' with associated field isomorphism α satisfying :

$$\begin{aligned}
\beta(\langle x\rangle) &= \langle\varphi(x)\rangle,\ \forall\langle x\rangle \in \mathcal{M}U(V,q,k,\sigma)\\
(q(x))^{\alpha} &= q'(\varphi(x)),\ \forall x\in V\\
(f(x,y))^{\alpha} &= f'(\varphi(x),\varphi(y)),\ \forall x,y\in V.
\end{aligned}$$

Throughout the proof we assumed that $\beta((0,1)) = (0,1)$. This might involve a possible multiplication of q' with a certain constant. Namely let $\mathcal{M}U(V,q,k,\sigma)$ and $\mathcal{M}U'(V',q',k',\sigma')$ be two unitary Moufang sets isomorphic under β. Choose coordinatizations of both sets. In order to assure that $\beta((0,1)) = (0,1)$ one can choose can choose $a \in Fix(\sigma)$ and $a' \in Fix(\sigma')$ such that $\psi_{a'}\beta\,\psi_a^-(0,1) = (0,1)$, where ψ_a is the isomorphism from $\mathcal{M}U(V,q,k,\sigma)$ to $\mathcal{M}U(V,aq,k,\sigma^a)$ and $\psi_{a'}$ the isomorphism from $\mathcal{M}U(V',q',k',\sigma')$ to $\mathcal{M}U(V'$, $a'q'$, k', $\sigma'^{a'}$) (as defined in section 3.12.3). By what is already proved we find a bijective semi-linear transformation φ with associated field isomorphism α such that :

$$\begin{aligned}
\psi_{a'}^{-1}\beta\psi_a(\langle x\rangle) &= \langle\varphi(x)\rangle,\ \forall\langle x\rangle \in \mathcal{M}U(V,aq,k,\sigma^a)\\
(aq(x))^{\alpha} &= a'q'(\varphi(x)),\ \forall x\in V\\
(af(x,y))^{\alpha} &= a'f'(\varphi(x),\varphi(y)),\ \forall x,y\in V\\
\lambda^{\sigma^a\alpha} &= \lambda^{\alpha\sigma'^{a'}},\ \forall\lambda\in k.
\end{aligned}$$

Let $a'^{-1}a^{\alpha} = c'$. We find :

$$\begin{aligned}
\beta(\langle x\rangle) &= \langle\varphi(x)\rangle,\ \forall\langle x\rangle \in \mathcal{M}U(V,q,k,\sigma)\\
c'(q(x))^{\alpha} &= q'(\varphi(x)),\ \forall x\in V\\
c'(f(x,y))^{\alpha} &= f'(\varphi(x),\varphi(y)),\ \forall x,y\in V\\
c'\lambda^{\sigma\alpha}c'^{-1} &= \lambda^{\alpha\sigma'}
\end{aligned}$$

c' meets the requirements of the Proposition.

Conversely suppose φ is a bijective semi-linear transformation from V to

V' with associated field isomorphism α such that that there exists a constant $c' \in k'$ with :

$$\begin{aligned} c'(q(x))^\alpha &= q'(\varphi(x)),\ \forall x \in V \\ c'(f(x,y))^\alpha &= f'(\varphi(x),\varphi(y)),\ \forall x \in V \\ c'\lambda^{\sigma\alpha}c'^{-1} &= \lambda^{\alpha\sigma'},\ \forall \lambda \in k. \end{aligned}$$

Then Lemma 102 shows that the map β defined by

$$\beta(\langle x\rangle) = \langle \varphi(x)\rangle,\ \forall \langle x\rangle \in \mathcal{M}U(V,q,k,\sigma)$$

defines a Moufang set isomorphism.

□

Proposition 130 *Let $\mathcal{M}U(V,q,k,\sigma)$ be a unitary Moufang set with commutative root groups. Then $\mathcal{M}U(V,q,k,\sigma)$ is isomorphic to a projective, orthogonal, hermitian or indifferent Moufang set $(X',(U_{x'})_{x'\in X'})$ if and only if :*
(i) $(X,(U_{x'})_{x'\in X'})$ is a projective Moufang set of the form $\mathcal{P}(k')$, k is a generalized quaternion algebra with standard involution σ, $\mathcal{M}U(V,q,k,\sigma) \cong \mathcal{P}(Z(k))$ and $Z(k) \cong k'$,
(ii) $(X',(U_{x'})_{x'\in X'})$ is an orthogonal Moufang set of the form $\mathcal{M}O(V',q',k')$ and $\mathcal{M}\ U(V,q,k,\sigma)$ is an extended polar line defined over a generalized quaternion algebra k with standard involution σ isomorphic to $\mathcal{M}O(V',k',q')$.
(iii) $(X',(U_{x'})_{x'\in X'})$ is a hermitian Moufang set of the form $\mathcal{M}H(V',q',k',\sigma')$, k is a generalized quaternion algebra with standard involution σ, $\mathcal{M}U(V,q,k,\sigma) \cong \mathcal{P}(Z(k))$, $\mathcal{M}H(V',q',k',\sigma') \cong \mathcal{P}(Fix(\sigma'))$ and $Z(k) \cong Fix(\sigma')$,
(iv) $(X',(U_{x'})_{x'\in X'})$ is an indifferent Moufang set of the form $\mathcal{P}(\bar{k}',l';k')$, $\mathcal{M}U(V,q,k,\sigma)$ is an extended polar line defined over a generalized quaternion algebra isomorhic to $\mathcal{P}(\bar{k}',l';k')$. Moreover if $char(k) \neq 2$, $dim(V)=2$ and $Z(k) \cong l' = k'$.

First case : $(X',(U_{x'})_{x'\in X'})$ is a projective Moufang set $\mathcal{P}(k')$.
We refer to Propositions 125 and 126.
Second case : $(X',(U_{x'})_{x'\in X'})$ is an orthogonal Moufang set $\mathcal{M}O(V',q',k')$.
In this case we refer to Proposition 127.

Third case : $(X', (U_{x'})_{x'\in X'})$ is a hermitian Moufang set $\mathcal{M}H(V', q', k', \sigma')$. For this case we refer to Proposition 128.

Fourth case : $(X', (U_{x'})_{x'\in X'})$ is an indifferent Moufang set $\mathcal{P}(\bar{k}'; l', k')$. As $\mathcal{P}(\bar{k}', l'; k')$ is commutative and has commutative root groups Lemmas 104 and 123 imply that k is a generalized quaternion algebra with standard involution σ and $\mathcal{M}U(V, q, k, \sigma) \cong \mathcal{P}(Z(k))$. Proposition 126 yields then that $\mathcal{M}U(V, q, k, \sigma) \cong \mathcal{P}(\bar{k}', l'; k')$ if and only if $Z(k) \cong l' = k'$.

3.15.6 The isomorphism problem for indifferent Moufang sets.

Proposition 131 *An indifferent Moufang set $\mathcal{P}(\bar{k}, l; k)$ is isomorphic under β to a classical or indifferent Moufang set $(X', (U_{x'})_{x'\in X'})$ if and only if one of the following occurs :*
(i) $(X, (U_{x'})_{x'\in X'})$ is a projective Moufang set $\mathcal{P}(k')$ with $l = k \cong k'$,
(ii) $(X', (U_{x'})_{x'\in X'})$ is an orthogonal Moufang set $\mathcal{M}O(V', k', q')$ and one of the following subcases occurs :
(ii.a) $dim(V') = 3$, $l = k \cong k'$, or
(ii.b) $dim(V') = 4$, $codim(Rad(f')) \neq 2$ and $l = k \cong k''$, with k'' the quadratic Galois extension of k'
determined by $\mathcal{M}O(V', q', k')$.
(ii.c) $codim(Rad(f')) = 2$, β induces a bijection φ from l to $\{q'(w') \mid w' \in Rad(f')\}$, there exist constants $c \in k$, $c' \in k'$ such that $1 \in c'q'(w') \mid w' \in Rad(f')\}$ and an isomorphism α from k to the field generated by $\{c'q'(w') \mid w' \in Rad(f')\}$ such that :

$$c'\varphi(v) = (cv)^{\alpha},\ \forall v \in l,$$

(iii) $(X', (U_{x'})_{x'\in X'})$ is a hermitian Moufang set $\mathcal{M}(H(V', q', k', \sigma'))$ with $dim(V') = 2$ and $l = k \cong Fix(\sigma')$,
(iv) $(X', (U_{x'})_{x'\in X'})$ is a unitary Moufang set $\mathcal{M}U(V', q', k', \sigma')$ defined over a generalized quaternion algebra k' with standard involution σ', $dim(V') = 2$ and $k = l \cong Z(k')$,
(v) $(X, (U_{x'})_{x'\in X'})$ is an indifferent Moufang set $\mathcal{P}(\bar{k}', l'; k')$, there exists a

field isomorphism from k to k', a constant $c' \in k'$ such that :

$$\beta((x)) = (c'x^{\alpha}),\ \forall (x) \in \mathcal{P}(\bar{k}, l; k).$$

proof :

First case : $(X'(U_{x'})_{x' \in X'})$ is a projective Moufang set $\mathcal{P}(k)$.
As $\mathcal{P}(\bar{k}, l; k)$ as commutative Lemma 119 implies that k is a field. Using Proposition 126 we see that in this case $\mathcal{P}(\bar{k}, l; k)$ is isomorphic to $\mathcal{P}(k)$ if and only if $l = k \cong k'$,

Second case : $(X'(U_{x'})_{x' \in X'})$ is an orthogonal Moufang set $\mathcal{MO}(V', q', k', \sigma')$.
In this case we refer to Proposition 127.

Third case $(X'(U_{x'})_{x' \in X'})$ is a hermitian Moufang set $\mathcal{MH}(V', q', k', \sigma')$.
As $\mathcal{P}(\bar{k}, l; k)$ has commutative root groups Lemma 118 implies that $dim(V')$ $= 2$ and $\mathcal{MH}(V', q', k', \sigma') \cong \mathcal{P}(Fix(\sigma'))$. Proposition 126 implies then that $\mathcal{P}(\bar{k}, l; k)$ is isomorphic to $\mathcal{MH}(V', q', k', \sigma')$ if and only if $l = k \cong Fix(\sigma')$.

Fourth case $(X'(U_{x'})_{x' \in X'})$ is a unitary Moufang set $\mathcal{MU}(V', q', k', \sigma')$.
Without loss of generality we can assume that q' is a $(\sigma, -1)$-quadratic form such that $1 \in Tr(\sigma)$. As $\mathcal{P}(\bar{k}, l; k)$ is commutative with commutative root groups Lemma 104 and Corollary 122 imply that $\mathcal{MU}(V', q', k', \sigma')$ is an extended polar line defined over a generalized quaternion algebra k' with standard involution σ'. Lemma 123 implies that $dim(V') = 2$, and $\mathcal{MU}(V', q', k', \sigma') \cong \mathcal{P}(Z(k'))$. Using Proposition 125 we find that $\mathcal{P}(\bar{k}, l; k')$ $\cong \mathcal{MU}(V, q, k, \sigma))$ if and only if $l = k \cong Z(k)$.

Fifth case : $(X'(U_{x'})_{x' \in X'})$ is an indifferent Moufang set of the form $\mathcal{P}(\bar{k}', l'; k')$.
choose for both $\mathcal{P}(\bar{k}, l; k)$ and $\mathcal{P}(\bar{k}', l'; k')$ coordinatizations. After a possible re coordinatization of $\mathcal{P}(\bar{k}', l'; k')$ we can assume that $\beta((0)) = ((0))$, $\beta(\infty) = (\infty)$ and $\beta((1)) = (1)$. As β defines a bijection between the points of $\mathcal{P}(\bar{k}, l; k)$ and $\mathcal{P}(\bar{k}', l'; k')$ it induces a bijection from l to l' which we also denote by β if we set :

$$\beta((v)) = (\beta(v)),\ \forall v \in l.$$

As $u((\infty); (0), (v))((w)) = (v + w)$ we deduce $\beta(v + w) = \beta(v) + \beta(w)$. Hence β defines an additive morphism from l to l'

The assumptions on l and l' show that $k^2 = l^2$ and $(k')^2 = (l')^2$.
The equations $s_{(0,1)}(v) = (v^{-1})$ and $s_{(0,v)}\ (1) = (v^{-2})$ show that $\beta((v^{-1})) = ((\beta(v))^{-1})$ and $\beta((v^2)) = ((\beta(v))^2)$, $\forall v \in l$. Using these equations one easily shows that β defines a bijection from l^2 to $(l')^2$ preserving the additive group structure, squares and inverse. The proof of Proposition 124 yields that β defines an isomorphism from k^2 to $(k')^2$. Thus if we define the map α from k to k' by :

$$\alpha(x) = \sqrt{\beta(x^2)},\ \forall x \in k$$

we find that α defines a field isomorphism from k to k' such that

$$\beta((v)) = (v^\alpha),\ \forall v \in l.$$

Remark that we the assumption that $\beta((1)) = (1)$ can require a possible re coordinatization. This means that with respect to the original coordinate system we find a $c' \in k'$ such that

$$\beta((v)) = (c'v^\alpha),\ \forall v \in l.$$

Conversely let α be a field isomorphism from k to k' such that there exists a constant c' such that :

$$c'x^\alpha \in l',\ \forall x \in l.$$

Define the bijection β from $\mathcal{P}(\bar{k}, l; k)$ to $\mathcal{P}(\bar{k}', l'; k')$ by :

$$\beta((x)) = (c'x^\alpha),\ \forall x \in l.$$

Using Lemma 41 we check that β defines a Moufang set isomorphism. In order to use this lemma we have to show that the map $\beta_{(\infty)}$ defined by :

$$\beta_{(\infty)}(u_{(\infty)}) = \beta \circ u_{(\infty)} \circ \beta^{-1},\ \forall u_{(\infty)} \in U_{(\infty)}$$

defines a bijection from the root elations fixing (∞) in $\mathcal{P}(\bar{k}, l; k)$ to the set of root elations in $\mathcal{P}(\bar{k}', l'; k')$ fixing (∞) and similarly the map $\beta_{(0)}$ defined by :

$$\beta_{(0)}(u_{(0)}) = \beta \circ u_{(0)} \circ \beta^{-1},\ \forall u_{(0)} \in U_{(0)}$$

defines a bijection from the root elations in $\mathcal{P}(\bar{k}, l; k)$ fixing (0) to the root elations of $\mathcal{P}(\bar{k}', l'; k')$ fixing (0).

Let $(v') \in l'$ we calculate :

$$\begin{aligned} \beta u((\infty);(0),(t))\beta^{-1}((v')) &= \beta u((\infty);(0),(t))(((c'^{-1}v')^{\alpha^{-1}})) \\ &= \beta(((c'^{-1}v')^{\alpha^{-1}}+t)) \\ &= ((v'+c't^{\alpha})) \\ &= u((\infty);(0),\beta(t))((v')) \\ \beta u((\infty);(0),(t))\beta^{-1}((\infty)) &= (\infty) \end{aligned}$$

showing that $\beta_{(\infty)}= (u((\infty);(0),(t)))\ u((\infty);(0),\beta((t))))$.
Let $v' \in l' \setminus \{0\}$.
We find :

$$\begin{aligned} \beta u((0);(\infty),(t))\beta^{-1}((v')) &= \beta u((0);(\infty),(t))(((c'^{-1}v')^{\alpha^{-1}})) \\ &= \beta(((((c'v'^{-1})^{\alpha^{-1}}+t^{-1})^{-1})) \\ &= ((v'^{-1}+(c't^{\alpha})^{-1})^{-1}) \\ &= u((0);(\infty),\beta(t))((v')) \\ \beta u((0);(\infty),(t))\beta^{-1}((0)) &= (0) \end{aligned}$$

showing that $\beta_{(0)}\ (u((0);(\infty),(t))) = u((0);(\infty),\beta((t)))$.
Using Lemma 41 we thus find that β defines a Moufang set isomorphism.

□

3.16 Local characterizations of some classical Moufang sets

Main aim of this section is to generalize the techniques developed in the previous paragraphs and come to a more abstract theory. To simplify the notations and calculations we will assume that we always work with $(\sigma,-1)$-quadratic forms with $1 \in Tr(\sigma)$ if $\sigma \neq 1$. As explained in Corollary 92 and section 3.12.3 this does not put any restrictions on the forms we consider.

Theorem 132 *Let $(X,(U_x)_{x\in X})$ be a Moufang set with non-commutative root groups. Then $(X,(U_x)_{x\in X})$ is isomorphic to a Moufang set of the form $\mathcal{M}(V,q,k,\sigma)$ where $dim(V)\geq 5$, and if k is a generalized quaternion algebra σ is a non standard involution if and only if there exists a family of proper Moufang subsets $(Y_i)_{i\in I}$ of $(X,(U_x)_{x\in X})$ and two points y_1 and y_2 such that :*

(i) *For every* $i \in I$, Y_i *is isomorphic to a Moufang set of the form* $\mathcal{M}(V_i, q_i, k_i, \sigma_i)$ *where if* k_i *is a generalized quaternion algebra* σ_i *is a non standard involution. All* Y_i *have the same type. If the* Y_i *are hermitian Moufang sets* $dim(V_i) > 4$, $\forall i \in I$,

(ii) y_1, $y_2 \in Y_i$, $\forall \in I$ *and every three points* x_1, x_2 *and* x_3 *are contained in some* Y_i,

(iii) *If the* Y_i *are orthogonal Moufang sets the following condition holds : for every* i, $j \in I$ *the Moufang set* $Y_i \cap Y_j$ *is non-commutative with :*

$$Z(Fix_{TY_i}\{y_1, y_2\}) = Z(Fix_{TY_j}\{y_1, y_2\}),\ \forall i, j \in I.$$

If $char(k_j) = 2$, $\forall j \in I$, $\varphi_i^{-1}(Y_j)$ *is a Moufang set* $\mathcal{M}(V_{ij}, q_{ij}, k_j, \sigma_j)$, *where* V_{ij} *is a subspace of* V_j *and* $q_{ij} = q_i|_{V_{ij}}$.

(iv) *If the* Y_i *are not orthogonal Moufang sets the following condition holds :*

$$Z(Stab_{U_{y_1}}(Y_i)) = Z(Stab_{U_{y_1}}(Y_j)),\ \forall i, j \in I,$$

(v) *If the* Y_i *are hermitian Moufang sets there exists a Moufang subset* Y_0 *of the family such that* $Y_0 \cap Y_i \cap Y_j$ *is a Moufang set with non-commutative root groups for every couple* i, $j \in I$.

proof :

Suppose $(X, (U_x)_{x\in X})$ is a Moufang set $\mathcal{M}(V, q, k, \sigma)$ such that $dim(V) \geq 5$ and if k is a generalized quaternion algebra σ is a non standard involution. Choose a coordinatization of $\mathcal{M}(V, a, k, \sigma)$ associated to a decomposition $V = e_{-1}k \oplus V_0 \oplus e_1 k$. Set $H_0 = \{ W_0^i \mid W_0^i$ is a subspace op V and $codim(W_0^i) = 3 \}$. By construction we can consider for every $W_0^i \in H_0$ the Moufang set $Y_i = \mathcal{M}(e_{-1}k \oplus W_0^i \oplus e_1 k,\ q,\ k,\ \sigma)$. One easily checks that if we set the family (Y_i) satisfies the conditions of the Theorem.

Conversely suppose $(X, (U_x)_{x\in X})$ and Y_i are Moufang sets as in the theorem. If all Moufang sets (Y_i) are orthogonal we put $\epsilon = 1$ and in the other cases we set $\epsilon = -1$. For $i \in I$ we will assume that q_i (σ_i, ϵ)-quadratic form with associated sesquilinear form f_i. Moreover we will denote the isomorphism from $\mathcal{M}(V_i, q_i, k_i, \sigma_i)$ to Y_i by φ_i. Choose for every $i \in I$ a coordinatization

of $\mathcal{M}(V_i,q_i,k_i,\sigma_i)$ associated to a decomposition $e^i_{-1}k \oplus V^i_0 \oplus e^i_1$ with labelling set $R^i_{0,1} = \{(v^i_0, v^i_1) \in V^i_0 \times k_i \,|q_i(v^i_0) + v^i_1 = 0\}$ such that $\varphi_i\ ((\infty)) = y_1$ and $\varphi_i((0,0)) = y_2$.
Remark that the conditions (iii), (iv) and (v) of the theorem imply that $Y_i \cap Y_j \neq \emptyset$, $\forall\ i,\ j \in I$. For $i \in I$ we have $char(k_i) = ord(u_{y_1})$, $\forall u_{y_1} \in U_{y_1}$ and it follows that $char(k_i) = char(k_j)\ \forall\ i,\ j \in I$.
Throughout this proof we will use the following notation which was introduced in Chapter 1. If $(X,(U_x)_{x\in X})$ and $(Y,(U_y)_{y\in Y})$ are two Moufang sets and φ is an injection from $(X,(U_x)_{x\in X})$ into Y. Then we will denote for $u \in TX$, the automorphism $\varphi \circ u \circ \varphi^{-1}$ of $(Y,(U_y)_{y\in Y})$ as u^{φ}.
Also important to mention is that by Lemma 47 the conditions on k_i and σ_i yield that $Tr(\sigma_i)$ generates k_i as a ring if $Z(k_i) \neq k_i$.
If $\sigma_i \neq 1$, $\forall \in I$ we introduce the following notations :
Condition (iv) of the theorem yields that for $i,\ j \in I$:

$$\{\varphi_i((a^i_0, a^i_1))|(a^i_0, a^i_1) \in Z(R^i_{0,1}, \oplus)\} = \{\varphi_j((a^j_0, a^j_1))|(a^j_0, a^j_1) \in Z(R^j_{0,1}, \oplus)\} \quad (3.35)$$

If $i \in I$ we will denote in the sequel $L_i = \{a^i_1 \in R^i_1\ |a^i_1 = q(a^i_0),\ a^i_0 \in Rad(f_i)\}$ $= \{a^i_1\ |(a^i_0, a^i_1) \in Z(R^i_{0,1}, \oplus)$ for some $a^i_0 \in V^i_0\}$. Remark that for $a^i_1 \in L_i$ the element $a^i_0 \in V^i_0$ such that $q_i(a^i_0) = a^i_1$ is uniquely determined by a^i_1. (If there would be another $b^i_0 \in Rad(f_i)$ with $q_i(b^i_0) = a^i_1$ the equation $q(a^i_0 + b^i_0) = 0$ implies that $a^i_0 = b^i_0$.) Moreover as $(0,1) \in R^i_{0,1}$, $\forall i \in I$, by assumption we can introduce the following notation. For $\theta^i \in L_i$ we set :

$$m_{\theta^i} = s_{(x^i_0,\theta^i)}s_{(0,1)},$$

with x^i_0 the unique vector in $Rad(f_i)$ satisfying $q(x^i_0) = \theta^i$.
As a first step we show that for every $i,\ j \in I$ the set $\varphi_i^{-1}(Y_j)$ is a Moufang subset of $\mathcal{M}(V_i,q_i,k_i,\sigma_i)$ of the form $\mathcal{M}(V_{ij},q_{ij},k_i,\sigma_i)$ where V_{ij} is a subspace of V_i and $q_{ij} = q_i|_{V_{ij}}$.
Remark that if (v^i_0, v^i_1), $(w^i_0, w^i_1) \in \varphi_i^{-1}(Y_j)$ also $(v^i_0, v^i_1) \oplus (w^i_0, w^i_1) \in \varphi_i^{-1}(Y_j)$.
1. The Y_l are orthogonal Moufang sets $\forall l \in I$.
If $char(k) = 2$ it follows by assumption that $\varphi_i^{-1}(Y_j)$ is of the form $\mathcal{M}(V_{ij}, q_{ij}, k_i, \sigma_i)$ where V_{ij} is a subspace of V_i and $q_{ij} = q_i|_{V_{ij}}$.
Remains the case where $char(k) \neq 2$.
Let $(v^i_0, v^i_1) \in \varphi_i^{-1}(Y_j)$ and $\lambda^i \in k_i$. By Lemma 109 we know that $s_{(v^i_0\lambda^i, v^i_1(\lambda^i)^2)}$ $s_{(v_0,v_1)} \in Z(Fix_{TY_i}\ \{y_1,\ y_2\})$. As $Z(Fix_{TY_i}\ \{y_i,\ y_j\}) = Z(Fix_{TY_j}\ \{y_1,\ y_2\})$ we find that $s^{\varphi_i}_{(v^i_0,v^i_1)}\ s_{(v^i_0\lambda^i, v^i_1(\lambda^i)^2)} \in Stab_{TX}\ (Y_i \cap Y_j)$. In particular $s_{(v^i_0\lambda^i, v^i_1(\lambda^i)^2)}$

$s_{(v_0^i,v_1^i)}$ $((v_0^i,v_1^i)) = (v_0^i(\lambda^i)^2, v_1^i{\lambda^i}^4) \in \varphi_i^{-1}(Y_j)$.
Moreover as :

$$(v_0^i(1+\lambda^i)^2, v_1^i(1+\lambda^i)^4) \quad = \quad (v_0^i,v_1^i) \oplus 2(v_0^i\lambda^i, v_1^i(\lambda^i)^2) \oplus (v_0^i(\lambda^i)^2, v_1^i{\lambda^i}^4)$$

and $(v_0(\lambda^i)^2, v_1{\lambda^i}^4) \in \varphi_i^{-1}$ (Y_j) we see that $2(v_0^i\lambda^i,\ v_1^i(\lambda^i)^2) \in \varphi_i^{-1}(Y_j)$. Hence $(v_0^i\lambda^i$,v_1^i ${\lambda_i}^2) \in \varphi_i^{-1}(Y_j)$.

2. The Y_l are hermitian Moufang sets such that $dim(V_l) \geq 4$, $\forall l \in I$.

Let $(v_0^i, v_1^i) \in \varphi_i^{-1}(Y_j)$. Equation (3.35) shows that $\{(0,\theta^i)\ |\theta^i \in Fix(\sigma_i)\} \subset \varphi_i^{-1}(Y_j)$. Therefore we find for $\mu^i \in k_i$ that $u((\infty);(0,0),(0,\mu^i)$ and m_{μ^i} stabilize $\varphi_i^{-1}(Y_j)$. This means that $\{(v_0^i, v_1^i+\theta^i)\ |\theta^i \in Fix(\sigma_i)\} \subset \varphi_i^{-1}(Y_j)$ and $\{(v_0\mu^i, \mu^i v_1\mu^i + \theta)\ |\theta^i,\ \mu^i \in Fix(\sigma_i)\} = \{m_{\mu^i}(v_0^i, v_1^i+\theta^i)\ |\theta^i,\ \mu^i \in Fix(\sigma_i)\}$ $\subset \varphi_i^{-1}(Y_j)$.

By assumption we have $(0,1) \in \varphi_i^{-1}$ (Y_j). Hence φ_i^{-1} (Y_j) is stabilized by $s_{(0,1)}$. This implies that also the set $\{(v_0^i{v_1^i}^{-1}\mu^i$ $,\mu^i{v_1^i}^{-1}\mu^i$ $+\theta^i)\ |\theta^i,\ \mu^i$ $\in Fix(\sigma_i)\}$ is contained in $\varphi_i^{-1}(Y_j)\}$. As $v_1^i \notin Fix(\sigma_i) = Tr(\sigma_i)$, the field k_i equals $Fix(\sigma_i)({v_1^i}^{-1})$. We thus find that $\varphi_i^{-1}(Y_j)$ is of the form $\mathcal{M}(V_{ji}, q_{ji}, k_i, \sigma_i)$, where V_{ij} is the space spanned by the vectors in $\{(\varphi_i^{-1}(y_j))_0\ |y_j \in Y_j \cap Y_i\}$ and $q_{ij} = q_i|_{V_{ij}}$.

3. The Y_l are unitary Moufang sets with such that if k_l is a generalized quaternion algebra, σ_l is not its standard involution. $\neq Z(k_l)$, $\forall l \in I$.

Let $(v_0^i, v_1^i) \in \varphi_i^{-1}(Y_j)$. Formula (3.35) shows that $\{(0,\theta^i)\ |\theta^i \in Tr(\sigma_i)\} \subset \varphi_i^{-1}(Y_j)$. Hence $u((\infty);(0,0)$,$(0,\theta_i))$ stabilizes $\varphi_i^{-1}(Y_j)$, $\forall\theta_i \in Tr(\sigma_i)$. This means that $\{(v_0^i, v_1^i+\theta^i)|\theta^i \in Tr(\sigma_i) \in \varphi_i^{-1}(Y_j)\}$. As also $(0,1) \in \varphi_i^{-1}(Y_j)$ the elements $m_{\theta_i} = s_{(0,\theta_i)}$ $s_{(0,1)}$ for $\theta_i \in k_i$ will stabilize $\varphi_i^{-1}(Y_j)$.

Let $\lambda^i \in k_i$. Denote as in the the second subcase of the proof of Theorem 129 $S_{(v_0^i,v_1^i)}$ $\langle u((\infty);(0,0),(v_0^i,v_1^i)),\ m_{\theta^i}\ |\theta^i \in Tr(\sigma_i)\rangle$. Then we proved that there exists a $\psi_{\lambda^i} \in S_{(v_0^i,v_1^i)}$ such that $\varphi_{\lambda^i}((v_0^i,v_1^i)) = (v_0^i\lambda^i,\ {\lambda^i}^\sigma\bar{v}_1^i\ \lambda^i)$, with $(v_0^i,\bar{v}_1^i)$ $\in R_{0,1}^i$. By definition of $S_{(v_0^i,v_1^i)}$ it follows that $S_{(v_0^i,v_1^i)} \in Stab(\varphi_i^{-1}(Y_j)$. This means that $(v_0^i\lambda^i,\ {\lambda^i}^\sigma\bar{v}_1^i\ \lambda^i) \in \varphi_i^{-1}(Y_j)$ and $\{(v_0^i\lambda^i,\ {\lambda^i}^\sigma\ v_1^i\lambda^i+\theta^i)\ |\theta^i \in Tr(\sigma_i)\}$ $\subset \varphi_i^{-1}(Y_j)$.

This shows that $\varphi_i^{-1}(Y_j)$ will be the Moufang set of the form $\mathcal{M}(V_{ij}, q_{ij}, k_i, \sigma_i)$ where V_{ij} is the space spanned by $\{(\varphi_i^{-1}(y_j))_0\ |y_j \in Y_i\ \cap Y_j\}$ and $q_{ij} = q_i|_{V_{ij}}$.

As a next step we show that for any every $i,\ j \in I$ there exists a unique element $c_{i,j} \in k_j$, and a unique bijective semi-linear transformation β_{ij} from V_0^i to V_0^j and a field isomorphism α_{ij} from k_i to k_j such that :

$$\varphi_i((v_0^i,v_1^i)) \quad = \quad (\beta_{ij}(v_0^i), c_{ij}{v_1^i}^{\alpha_{ij}}),\ \forall (v_0^i,v_1^i) \in \varphi_i^{-1}(Y_j) \qquad (3.36)$$

$$c_{ij}(\lambda^i)^{\sigma_i \alpha_{ij}} c_{ij}^{-1} = \lambda^{\alpha_{ij}\sigma_j}, \forall \lambda_i \in k_i. \quad (3.37)$$

Firstly we show the unicity of c_{ij}, β_{ij} and α_{ij}. Suppose that there exist $d_{ij} \in Fix(\sigma) \cap Z(k)$, a semi-linear transformation δ_{ij} with field isomorphism γ_{ij} such that a formula similar to (3.37) hold. We find :

$$\begin{aligned} \beta_{ij}(v_0^i) &= \delta_{ij}(v_0^i) \\ c_{ij}{v_1^i}^{\alpha_{ij}} &= d_{ij}{v_1^i}^{\gamma_{ij}} \end{aligned}$$

for $(v_0^i, v_1^i) \in \varphi_i^{-1}\ (Y_j)$. The first equation shows that $\beta_{ij} = \delta_{ij}$ while the second equation implies that $c_{ij}, = d_{ij}$.
If the Y_l orthogonal or hermitian Moufang sets the assumptions of the theorem imply that there exists at least one $(v_0^i, v_1^i) \in \varphi_i^{-1}(Y_j)$ with $v_0 \neq 0$. In this case the first equation shows $\alpha_{ij} = \gamma_{ij}$.
If the Y_l are unitary Moufang sets and $\varphi_i^{-1}(Y_j) = \{(0, \theta^i)\ |\theta^i \in Tr(\sigma^i)$, the second equation shows that α_{ij} equals γ_{ij} as $Tr(\sigma^i)$ generates k_i as a ring.
Firstly we prove formula (3.37) if $Y_i \cap Y_j$ contains a $z \in X$ such that $u(y_1; y_2, z) \notin Z(Stab_{U_{y_1}}(Y_i)) = Z(Stab_{U_{y_1}}(Y_j))$.
Remark that this is only possible if all Y_k are hermitian or unitary.
If $\mathcal{M}(V_i, q_i, k_i, \sigma_i)$ is hermitian we choose $z_i \in Y_i$, $z_j \in Y_j$ such that $(\varphi_i^{-1}(z_i))_0$ and $(\varphi_i^{-1}(z))_0$ are linearly independent and similarly $(\varphi_j^{-1}(z_j))_0$ and $(\varphi_j^{-1}(z))_0$ are linearly independent.
If $\mathcal{M}(V_i, q_i, k_i, \sigma_i)$ is unitary we choose $z_i \in Y_i$ and $z_j \in Y_j$ such that :

$$[u(y_1; y_2, z_i), u(y_1; y_2, z)] \neq 1 \text{ and } [u(y_1; y_2, z), u(y_1; y_2, z_j] \neq 1.$$

Let Y_l be a Moufang set of the family containing z_i, z_j and z.
The permutation $\varphi_l^{-1}\varphi_i$ of $Y_i \cap Y_l$ defines by assumption an automorphism of the Moufang set $Y_i \cap Y_l$. The Propositions 128 and 129 show that there exists a constant $c_{il} \in k_l$ and a bijective semi-linear transformation β_{il} from V_0^i to V_0^l with associated field isomorphism α_{il} such that :

$$\begin{aligned} c_{il}\lambda^{i^{\sigma_i \alpha_{il}}} c_{il}^{-1} &= \lambda^{i^{\alpha_{il}\sigma_j}}, \forall \lambda^i \in k_i \\ c_{lj}\varphi_l^{-1}\varphi_i((v_0^i, v_1^i))c_{lj}^{-1} &= (\beta_{il}(v_0^i), c_{il}{v_1^i}^{\alpha_{il}}), \forall (v_0^i, v_1^i) \in \varphi_i^{-1}(Y_l) \end{aligned}$$

By similar arguments there exists a constant $c_{lj} \in k_j$ and a semi-linear transformation β_{lj} from V_0^l to V_0^j with associated field isomorphism α_{lj} such that :

$$\begin{aligned} c_{lj}\lambda^{l^{\sigma_l \alpha_{lj}}} c_{lj}^{-1} &= \lambda^{l^{\alpha_{lj}\sigma_j}}, \forall \lambda^l \in k_l \\ \varphi_j^{-1}\varphi_l((v_0^l, v_1^l)) &= (\beta(v_0^l), c_{lj}{v_1^l}^{\alpha_{lj}}), \forall (v_0^l, v_1^l) \in \varphi_l^{-1}(Y_j). \end{aligned}$$

In particular :

$$\begin{aligned}\varphi_j^{-1}\varphi_i((0,\theta^i)) &= \varphi_j^{-1}\varphi_l(\varphi_l^{-1}\varphi_i((0,\theta^i)))\\ &= \varphi_j^{-1}\varphi_l((0,c_{il}\theta^{i\alpha_{il}}))\\ &= (0,c_{lj}c_{il}^{\alpha_{lj}}\theta^{i\alpha_{il}\alpha_{lj}})\\ &= (0,c_{ij}\theta^{i\alpha_{ij}})\end{aligned}$$

if we set $c_{ij} = c_{il}^{\alpha_{lj}} c_{lj}$ and $\alpha_{ij} = \alpha_{il}\alpha_{lj}$ we find $c_{ij} \in k_j$ and $c_{ij}\ \lambda^{i\sigma_i\alpha_{ij}}\ c_{ij}^{-1}$ $= \lambda^{i\alpha_{ij}\sigma_j}$, $\forall\lambda^i \in k_i$.

If $\mathcal{M}(V_i, q_i, k_i, \sigma_i)$ is unitary we leave it as an exercise for the reader to check that one can proceed as in the proof of the second subcase of Theorem 129 to see that formula (3.37) holds.

If $\mathcal{M}(V_i, q_i, k_i, \sigma_i)$ is hermitian, the assumption that $Y_i \cap Y_j$ has non-commutative root groups implies $dim(\langle\ (\varphi_i^{-1}(y_j))_0\ |y_j \in Y_i \cap Y_j\rangle) \geq 1$. We consider two subcases.

First subcase : $dim(\langle(\varphi_i^{-1}(y_j))_0\ |y_j \in Y_i \cap Y_j\ \rangle) > 1$.

Then one can proceed as in the proof of $(ii.c)$ of Proposition 128 to see that formula (3.37) holds.

Second subcase : $dim(\langle(\varphi_i^{-1}(y_j))_0\ |y_j \in Y_i \cap Y_j\ \rangle = 1$. In this case we have $Y_i \cap Y_j \subset Y_l$. Applying Proposition 128 to $\varphi_l^{-1}\ \varphi_i$ and $\varphi_j^{-1}\ \varphi_l$ proves that also in this case formula (3.37) holds.

Remains to prove formula (3.37) whenever $Stab_{U_{y_1}}(Y_i \cap Y_j) \subset Z(Stab_{U_{y_1}}(Y_i)$. We distinguish three cases.

If the Y_l are orthogonal, $\varphi_i^{-1}(Y_j)$ and $\varphi_j^{-1}(Y_i)$ are non-commutative orthogonal Moufang sets. As $\varphi_j^{-1}\ \varphi_i$ defines an isomorphism from $\varphi_i^{-1}(Y_j))$ to $\varphi_j^{-1}(Y_i)$ formula (3.37) follows from Proposition 127.

If the Y_l are hermitian assumption (v) of the theorem imply that in this case $Stab_{U_{y_1}}(Y_i \cap Y_j)$ cannot be contained in $Z(Stab_{U_{y_1}}(Y_i))$. Hence this case cannot occur.

If the Y_l are unitary Moufang sets the assumption on $Stab_{U_{y_1}}$ implies that φ $\{(a_0^i, a_1^i)\ |(a_0^i, a_1^i) \in Rad(f_i)\} = Y_i \cap Y_j \subset \bigcap_{k\in I} Y_k$. Consider in this case a Moufang set Y_l such that $Y_i \cap Y_l$ contains an element z_i with $u(y_1; y_2, z_i)$ $\notin Z(Stab_{U_{y_1}}(Y_i)$ and similarly $Y_l \cap Y_j$ contains a z_l with $u(y_1; y_2, z_l) \notin$ $Z(Stab_{U_{y_1}}(Y_l)$. Then we already know that formula (3.37) holds for $\varphi_l^{-1}\varphi_i$ and $\varphi_j^{-1}\varphi_l$. As $Y_i \cap Y_j \subset Y_l$ formula (3.37) will also holds for $\varphi_j^{-1}\varphi_i = \varphi_j^{-1}\varphi_l$ $\varphi_l^{-1}\varphi_i$.

To proceed we choose an initial Moufang subset Y_0, where Y_0 is arbitrarily if

the Y_i are orthogonal or unitary and is as in condition (v) of the theorem if the Y_i are hermitian.
Using Y_0 we define the following binary relation (denoted by $\overset{v}{\sim}$) on X.
Let x, $x' \in X$. We set :

$$x \overset{v}{\sim} x' \Leftrightarrow u(y_1; y_2, z)(x) = x', \text{ for some } z \in \varphi_0\{(v_0, v_1) \in R^0_{0,1} \cap \{0\} \times k_0\}.$$

Remark that if all Y_i are orthogonal we find for v, $v' \in X$, $v \overset{v}{\sim} v'$ if and only if $v = v'$.
If $\sigma_0 \neq 1$ we have :

$$x \overset{v}{\sim} x' \Leftrightarrow u((\infty); (0,0), (0,\theta))^{\varphi_0}(x) = (x'), \text{ for some } \theta^0 \in Tr(\sigma_0).$$

We check that $\overset{v}{\sim}$ is an equivalence relation on X.
If the Y_i are orthogonal this is clear. Remains to consider the case where $\sigma_i \neq 1$, $\forall i \in I$.
(a) $\overset{v}{\sim}$ is reflexive.
This is clear as for any $x \in X$, $u(y_1; y_2, y_2)(x) = x$.
and $\varphi_0((0,0)) = y_2$.
(b) $\overset{v}{\sim}$ is symmetric.
If $x \overset{v}{\sim}$ there exists a $\theta^0 \in Tr(\sigma_0)$ with $u((\infty); (0,0), (0,\theta^0))^{\varphi_0}(x) = x'$.
Equivalently :

$$\begin{aligned}(u((\infty); (0,0), (0,\theta^0))^{\varphi_0})^{-1}(x') &= u((\infty); (0,0), (0,-\theta^0))^{\varphi_0}(x') \\ &= x,\end{aligned}$$

and we find $x' \overset{v}{\sim} x$.
(c) $\overset{v}{\sim}$ is transitive.
Let $x \overset{v}{\sim} x'$ and $x' \overset{v}{\sim} x''$.
This means that there exist θ^0, $\theta'^0 \in Tr(\sigma_0)$ such that :

$$\begin{aligned}u((\infty); (0,0), (0,\theta^0))^{\varphi_0}(x) &= x' \\ u((\infty); (0,0), (0,\theta'^0))^{\varphi_0}(x') &= x''.\end{aligned}$$

Hence :

$$\begin{aligned}u((\infty); (0,0), (0,\theta^0))^{\varphi_0} u((\infty); (0,0), (0,\theta'^0))^{\varphi_0}(x) &= u((\infty); (0,\theta^0 + \theta'^0))^{\varphi_0}(x) \\ &= x''\end{aligned}$$

and $x \overset{v}{\sim} x''$.
In the sequel we will denote for $x \in X$ its equivalence class with respect to $\overset{v}{\sim}$ as x_0 and we set $V_0 = \{x_0 | x \in X\}$.
As a next step we show that there exists an addition and scalar multiplication with elements of k_0 on V_0 turning it into a right k_0-vector space.

1. Addition :
Let x_0, $y_0 \in V_0$. Choose $x \in x_0$, $y \in y_0$ and set :

$$x_0 + y_0 = (u(y_1; y_2, x)(y))_0.$$

We show that this is well defined i.e. independent of the representatives we choose for x_0 and y_0. Let $x' \in x_0$. Then we have to show that :

$$(u(y_1; y_2, x)(y))_0 = (u(y_1; y_2, x')(y))_0.$$

If the Y_i are orthogonal Moufang sets this is clear as for every $x \in X$, $x_0 = \{x\}$. Remains the case where the Y_i are not orthogonal.
Let Y_i be a Moufang subset as in the theorem containing x and y with $\varphi_i^{-1}(x) = (v_0^i, v_1^i)$ and $\varphi_i^{-1}(y) = (w_0^i, w_1^i)$. By assumption on x and x' there exists a $\theta^0 \in Tr(\sigma_0)$ such that $u((\infty); (0,0), (0, \theta^0))^{\varphi_0}(x) = x'$.
Formula (3.37) shows that $\varphi_0(0, \theta^0) = \varphi_i((0, \theta^i))$ for some $\theta^i \in Tr(\sigma_i)$. It follows that $x' \in Y_i$, $u((\infty); (0,0), (0, \theta^0))^{\varphi_0} = u((\infty); (0,0), (0, \theta^i))^{\varphi_i}$ and $\varphi_j^{-1}(x') = (v_0^i, v_1^i + \theta^i)$.
Therefore we find :

$$\begin{aligned} u(y_1; y_2, x')(y) &= (u((\infty); (0,0), (v_0^i, v_1^i + \theta^i))^{\varphi_i}(y) \\ &= u((\infty); (0,0), (0, \theta^i))^{\varphi_i} u((\infty); (0,0), (v_0^i, v_1^i))^{\varphi_i}(y) \\ &= u((\infty); (0,0), (0, \theta^0))^{\varphi_0}(u(y_1; y_2, x)(y)). \end{aligned}$$

Hence $(u(y_1; y_2, x')(y))_0 = (u(y_1; y_2, x)(y))_0$.

We prove that $(V_0, +)$ is an abelian group.

> $+$ is associative on V_0.
> Let x_0, y_0, $z_0 \in V_0$, $x \in x_0$, $y \in y_0$, $z \in z_0$ and Y_i a Moufang subset as in the theorem containing x, y and z. Associativity follows from the associativity of $\oplus$ on $R_{0,1}^i$ using the isomorphism φ_i.

$(y_2)_0$ is neutral element for $+$.
This follows from the fact that $\varphi_i((0,0)) = y_2$, $\forall i \in I$.

Every x_0 has an inverse for $+$.
If $x_0 \in V_0$ and $(\varphi_i((v_0^i, v_1^i))_0 = x_0$ one checks that the inverse if x_0 is given by $(\varphi_i((-v_0^i, -v_1^{-i} - f(v_0^i, v_0^i)))_0$.

$+$ is commutative on V_0.
Let x_0, $y_0 \in V_0$, $x \in x_0$, $y \in y_0$ and Y_i a Moufang subset as in the Theorem containing x and y.
Consider the equations :

$$\begin{aligned}
u(y_1; y_2, y)(x) &= u((\infty); (0,0), (w_0^i, w_1^i))^{\varphi_i}(x) \\
&= \varphi_i(u((\infty); (0,0), (w_0^i, w_1^i))((v_0^i, v_1^i))) \\
&= \varphi_i((w_0^i, w_1^i) \oplus (v_0^i, v_1^i)) \\
&= \varphi_i((0, f(v_0^i, w_0^i) - f(w_0^i, v_0^i)) \oplus (v_0^i, v_1^i) \oplus (w_0^i, w_1^i)) \\
&= \varphi_i(u((\infty); (0,0), (0, f(v_0^i, w_0^i) - f(w_0^i, v_0^i)) \\
&\quad u((\infty); (0,0), (v_0^i, v_1^i))(w_0^i, w_1^i)) \\
&= u((\infty); (0,0), (0, f(v_0^i, w_0^i) - f(w_0^i, v_0^i))^{\varphi_i} \\
&\quad u((\infty); (0,0), (v_0^i, v_1^i))^{\varphi_i}(y) \\
&= u((\infty); (0,0), (0, f(v_0^i, w_0^i) - f(w_0^i, v_0^i))^{\varphi_i} \\
&\quad u(y_1; y_2, x)(y)
\end{aligned}$$

as $u((\infty); (0,0), (0, f(v_0^i, w_0^i) - f(w_0^i, v_0^i))^{\varphi_i} = u((\infty); (0,0), (a_0^i, a_1^i))$ for some $(a_0^i, a_1^i) \in R_{0,1}^i \cap \{0\} \times k_i$. this shows that $(u(y_1; y_2, x)(y))_0 = (u(y_1; y_2, y)(x))_0$ and thus $x_0 + y_0 = y_0 + x_0$.

2. Scalar multiplication :
Let $\lambda \in k_0$ and $x_0 \in V_0$. Define $x_0\lambda$ in the following way. Choose a $x \in x_0$ and Y_i such that $x \in Y_i$ then we set :

$$x_0\lambda = (\varphi((\varphi_i^{-1}(x))_0\lambda^{\alpha_{0i}}, \lambda^{\alpha_{0i}\sigma_i}(\varphi_i^{-1}(x))_1\lambda^{\alpha_{0i}}))_0.$$

We check that this multiplication is well defined i.e. independent from the representative x of x_0 and independent ofthe Y_i containing x.
Suppose $x' \in x_0$ and $x' \neq x$. This means that there exists a $\theta^i \in Tr(\sigma)$ such that $\varphi_i^{-1}(x') = ((\varphi_i^{-1}(x))_0,\ (\varphi_i^{-1}(x))_1 + \theta^i)$.
We find :

$$\begin{aligned}
&((\varphi_i^{-1}(x'))_0\lambda^{\alpha_{0i}}, \lambda^{\alpha_{0i}\sigma_i}(\varphi_i^{-1}(x'))_1\lambda^{\alpha_{0i}}) \\
= \ &((\varphi_i^{-1}(x))_0\lambda^{\alpha_{0i}}, \lambda^{\alpha_{0i}\sigma_i}((\varphi_i^{-1}(x))_1 + \theta^i)\lambda^{\alpha_{0i}}).
\end{aligned}$$

Therefore we find that $(\varphi_i((\varphi_i^{-1}(x))_0\lambda^{\alpha_{0i}},\ \lambda^{\alpha_{0i}\sigma_i}\ (\varphi_i^{-1}(x))_1\lambda^{\alpha_{0i}}))_0 =$ $(\varphi_i((\varphi_i^{-1}(x'))_0\ \lambda_{\alpha_{0i}},\ \lambda^{\alpha_{0i}\sigma_i}\ (\varphi_i^{-1}(x'))_1\ \lambda^{\alpha_{0i}}))_0$. Showing that the definition of multiplication with elements of k is independant from the representative x we choose for x_0.

Remains to show that the definition of scalar multiplication is independent of the initial Moufang set Y_i containing x we choose. To prove this we suppose $x \in Y_i \cap Y_j$.

Let c_{ij}, β_{ij} and α_{ij} be as in formula (3.37). For the following calculation we distinguish two cases.

First case : The Y_l are hermitian or orthonal Moufang sets.

Conditions (iii) and (v) of the theorem imply that there exists a (v_0^i, v_1^i) $\in \varphi_0^{-1}(Y_i) \cap \varphi_0^{-1}(Y_j)$ with $v_0^i \neq 0$.

Using formula (3.37) we find for $\lambda \in k_0$ the equations :

$$\begin{aligned}
\varphi_0((v_0^i\lambda, \lambda^\sigma v_1^i\lambda)) &= \varphi_j((\beta_{0j}(v_0^i)\lambda^{\alpha_{0j}}, c_{0j}\lambda^{\alpha_{0j}\sigma}v_1^{i\,\alpha_{0j}}\lambda^{\alpha_{0j}})) \\
&= \varphi_i((\beta_{0i}(v_0^i)\lambda^{\alpha_{0i}}, c_{0i}\lambda^{\alpha_{0i}\sigma}v_1^{i\,\alpha_{0i}}\lambda^{\alpha_{0i}})) \\
&= \varphi_j((\beta_{ij}\beta_{0i}(v_0^i)\lambda^{\alpha_{0i}\alpha_{ij}}, c_{ij}c_{0i}^{\alpha_{ij}}\lambda^{\alpha_{0i}\alpha_{ij}\sigma}v_1^{i\,\alpha_{0i}\alpha_{ij}}\lambda^{\alpha_{0i}\alpha_{ij}})).
\end{aligned}$$

This yields :

$$\begin{aligned}
c_{0i}^{\alpha_{ij}}c_{ij} &= c_{0j} \\
\lambda^{\alpha_{0i}\alpha_{ij}} &= \lambda^{\alpha_{0j}},\ \forall\lambda \in k_0
\end{aligned} \tag{3.38}$$

Second case : The Y_i are unitary Moufang sets.

The equations :

$$\begin{aligned}
\varphi_0((0,\theta^0)) &= \varphi_j((0, c_{0j}\theta^{0^{\alpha_{0j}}})) \\
&= \varphi_i((0, c_{0i}\theta^{0^{\alpha_{0i}}})) \\
&= \varphi_j((0, c_{ij}c_{0i}^{\alpha_{ij}}\theta^{0^{\alpha_{0i}\alpha_{ij}}}))
\end{aligned}$$

show that :

$$\begin{aligned}
c_{ij}c_{0i}^{\alpha_{ij}} &= c_{0j} \\
\theta^{0^{\alpha_{0i}\alpha_{ij}}} &= \theta^{0^{\alpha_{0j}}},\ \forall\theta^0 \in Tr(\sigma_0).
\end{aligned} \tag{3.39}$$

As $Tr(\sigma_0)$ generates k_0 as a ring this implies :

$$\lambda^{\alpha_{0i}\alpha_{ij}} = \lambda^{\alpha_{0j}},\ \forall\lambda \in k_0. \tag{3.40}$$

We return to the general case.
By formula (3.37) we have :

$$\varphi_j^{-1}(x) = (\beta_{ij}((\varphi_i^{-1}(x))_0), c_{ij}((\varphi_i^{-1}(x))_1)^{\alpha_{ij}})$$

yielding :

$$\begin{aligned} (\varphi_j^{-1}(x))_0 &= (\beta_{ij}((\varphi_i^{-1}(x)))_0 \\ (\varphi_j^{-1}(x))_1 &= c_{ij}(\varphi_i^{-1}(x))_1 \end{aligned}$$

Using equations (3.38), (3.39) and (3.40) we have :

$$\begin{aligned} & \varphi_i(((\varphi_i^{-1}(x))_0\lambda^{\alpha_{0i}}, \lambda^{\alpha_{0i}\sigma_i}(\varphi_i^{-1}(x))_1\lambda^{\alpha_{0i}})) \\ = & \varphi_j((\beta_{ij}((\varphi_i^{-1}(x))_0)\lambda^{\alpha_{0i}\alpha_{ij}}, c_{ij}\lambda^{\alpha_{0i}\alpha_{ij}\sigma_i}(\varphi_i^{-1}(x))_1\lambda^{\alpha_{0i}\alpha_{ij}})) \\ = & \varphi_j((\beta_{ij}((\varphi_i^{-1}(x))_0)\lambda^{\alpha_{0i}\alpha_{ij}}, \lambda^{\alpha_{0i}\alpha_{ij}}c_{ij}(\varphi_i^{-1}(x))_1\lambda^{\alpha_{0i}\alpha_{ij}} \\ = & \varphi_j((\varphi_j^{-1}(x))_0\lambda^{\alpha_{0j}}, \lambda^{\alpha_{0j}\sigma_i}(\varphi_j^{-1}(x))_1\lambda^{\alpha_{0j}}. \end{aligned}$$

In this way we can conclude that the definition of multiplication is independent ofthe Y_i containing x we choose. Hence the definition of multiplication of vectors in V_0 with elements of k_0 is well defined. By construction one easily checks that the multiplication determines a scalar multiplication on V_0 turning V_0 into a right k_0-vector space.

Remains to define a quadratic form on V_0. As for $i,\ j \in I$, $\varphi_i((0, c_{0i})) = \varphi_j((0, c_{0j})) = \varphi_0((0, 1))$ we deduce that $s^{\varphi_i}_{(0,c_{0i})} = s^{\varphi_j}_{(0,c_{0j})}$.
Let $x_0 \in V_0$. We show that $s^{\varphi_0}_{(0,1)}$ defines a permutation on V_0 sending (x_0) to a vector of $\langle x_0 \rangle$. To prove this we choose a representative $x \in x_0$ and Y_i containing x.
We have :

$$\begin{aligned} s^{\varphi_0}_{(0,1)}(x) &= s^{\varphi_i}_{(0,c_{0i})}(x) \\ &= \varphi_i((-(\varphi_i^{-1}(x))_0((\varphi_i^{-1}(x))_1)^{-1}c_{0i}, -c_{0i}((\varphi_i^{-1}(x))_1)^{-1}c_{0i}) \qquad (3.41) \\ &\in x_0(-(((\varphi_i^{-1}(x))_1)^{-1})^{\alpha_{0i}^{-1}}c_{0i}^{\alpha_{0i}^{-1}}). \end{aligned}$$

As for any $\theta^i \in Tr(\sigma_i)$:

$$s^{\varphi_0}_{(0,1)}\varphi_i((0, \theta^i)) = \varphi_i((0, -c_{0i}\theta^i c_{0i})),$$

we find that :

$$s^{\varphi_0}_{(0,1)}(x_0) = x_0\lambda_{x_0}, \text{ for some } \lambda_{x_0} \in k_0 \text{ depending on } x_0.$$

Therefore the following definition of q makes sense. For x_0 we set :

$$q(x_0) = -\lambda_{x_0}^{-1}, \forall x_0 \in V_0$$

Using the form q, we define the form f on V_0 by :

$$q(x_0 + y_0) = q(x_0) + q(y_0) + f(x_0, y_0), \forall x_0, y_0 \in V_0.$$

The fact that for $i \in I$, $s^{\varphi_0}_{(0,1)} = s^{\varphi_i}_{(0,c_{0i})}$ and equation (3.41) show that $q(x_0)$ can be calculated in the following way. Choose a representative $x \in x_0$ and a Y_i containing x.
We have :

$$q(x) = (c_{0i}^{-1})^{\alpha_{0i}^{-1}}(q_i((\varphi_i^{-1}(x))_0))^{\alpha_{0i}^{-1}}.$$

We show that q is an anisotropic $(\sigma_0, -1)$ quadratic form on V_0 with associated $(\sigma_0, -1)$-sesquilinear form f.
Let x_0, y_0, z_0 and $\lambda \in k_0$. Choose representatives $x \in x_0$, $y \in y_0$, $z \in z_0$ and a Moufang subset Y_i as in the theorem containing x, y and z.
By definition of addition and scalar multiplication we have $((\varphi_i^{-1}(x))_0 + (\varphi_i^{-1}(y))_0,\ (\varphi_i^{-1}(x))_1 + (\varphi_i^{-1}(y))_1\ -f((\varphi_i^{-1}(x))_0,\ (\varphi_i^{-1}(y))_0) \in x_0 + y_0$ and $((\varphi_i^{-1}(x))_0\lambda^{\alpha_{0i}},\ \lambda^{\alpha_{0i}\sigma_i}\ (\varphi_i^{-1}(x))_1\ \lambda^{\alpha_{0i}}) \in x_0\lambda$.
We find :

$$\begin{aligned}
q(x_0\lambda) &= (c_{0i}^{-1})^{\alpha_{0i}^{-1}}(q_i((\varphi_i^{-1}(x)_0\lambda^{\alpha_{0i}}))^{\alpha_{0i}^{-1}} \\
&= (c_{0i}^{-1})^{\alpha_{0i}^{-1}}\lambda^{\alpha_{0i}\sigma_i\lambda_{0i}^{-1}}(q_i((\varphi_i^{-1}(x))_0))^{\alpha_{0i}^{-1}}\lambda \\
&= \lambda^{\sigma_0}(q_i((\varphi_i^{-1}(x))_0))^{\alpha_{0i}^{-1}}\lambda \\
&= \lambda^{\sigma_0}q(x_0)\lambda.
\end{aligned}$$

where we used the fact that by formula (3.37), $\lambda^{\alpha_{0i}}(c_{0i}^{-1})^{\alpha_{0i}^{-1}} = (c_{0i}^{-1})^{\alpha_{0i}^{-1}}$ $\lambda^{\alpha_{0i}\sigma_i\alpha_{0i}^{-1}}$, $\forall \lambda \in k_0$.
Moreover the equations :

$$\begin{aligned}
q(x_0 + y_0) &= (c_{0i}^{-1})^{\alpha_{0i}^{-1}}(q_i((\varphi_i^{-1}(x))_0 + (\varphi_i^{-1}(y))_0))^{\alpha_{0i}^{-1}} \\
&= (c_{0i}^{-1})^{\alpha_{0i}^{-1}}(q_i(x_0))^{\alpha_i^{-1}} + (c_{0i}^{-1})^{\alpha_{0i}^{-1}}(q_i(y_0))^{\alpha_i^{-1}} + \\
&\quad (c_{0i}^{-1})^{\alpha_{0i}^{-1}}(f_i((\varphi_i^{-1}(x))_0, (\varphi_i^{-1}(y))_0))^{\alpha_{0i}^{-1}},
\end{aligned}$$

show that :

$$f(x_0,y_0)=(c_{0i}^{-1})^{\alpha_{0i}^{-1}}(f_i((\varphi_i^{-1}(x))_0,(\varphi_i^{-1}(y))_0))^{\alpha_{0i}^{-1}}.$$

As :

$$\begin{array}{lcl}
f(x_0+y_0,z_0) & = & (c_{0i}^{-1})^{\alpha_{0i}^{-1}}(f_i((\varphi_i^{-1}(x))_0+(\varphi_i^{-1}(y))_0,(\varphi_i^{-1}(z))_0))^{\alpha_{0i}^{-1}} \\
& = & (c_{0i}^{-1})^{\alpha_{0i}^{-1}}(f_i((\varphi_i^{-1}(x))_0,(\varphi_i^{-1}(z))_0))^{\alpha_{0i}^{-1}} \\
& & +(c_{0i}^{-1})^{\alpha_{0i}^{-1}}(f_i((\varphi_i^{-1}(y))_0,(\varphi_i^{-1}(z))_0))^{\alpha_{0i}^{-1}} \\
& = & f(x_0,z_0)+f(y_0,z_0) \\
f(x_0\lambda,y_0) & = & (c_{0i}^{-1})^{\alpha_{0i}^{-1}}(f_i((\varphi_i^{-1}(x))_0\lambda^{\alpha_{0i}},(\varphi_i^{-1}(y))_0))^{\alpha_{0i}^{-1}} \\
& = & \lambda^{\sigma_0}(c_{0i}^{-1})^{\alpha_{0i}^{-1}}(f_i((\varphi_i^{-1}(x))_0,(\varphi_i^{-1}(y))_0))^{\alpha_i^{-1}} \\
& = & \lambda^{\sigma_0}f(x_0,y_0) \\
f(y_0,x_0\lambda) & = & (c_{0i}^{-1})^{\alpha_{0i}^{-1}}(f_i((\varphi_i^{-1}(y))_0,(\varphi_i^{-1}(x))_0\lambda^{\alpha_{0i}^{-1}}))^{\alpha_{0i}^{-1}} \\
& = & (c_{0i}^{-1})^{\alpha_{0i}^{-1}}(f_i((\varphi_i^{-1}(y))_0,(\varphi_i^{-1}(x))_0))^{\alpha_{0i}^{-1}}\lambda \\
& = & f(y_0,x_0)\lambda \\
f(x_0,y_0) & = & (c_{0i}^{-1})^{\alpha_{0i}^{-1}}(f_i((\varphi_i^{-1}(x))_0,(\varphi_i^{-1}(y))_0))^{\alpha_{0i}^{-1}} \\
& = & \epsilon(c_{0i}^{-1})^{\alpha_{0i}^{-1}}(f_i((\varphi_i^{-1}(y))_0,(\varphi_i^{-1}(x))_0))^{\alpha_{0i}^{-1}} \\
& = & \epsilon f(y_0,x_0)
\end{array}$$

we conclude that q is a (σ_0,ϵ)-quadratic form with associated $(\sigma_0,-1)$ sesquilinear form f. The fact that q_i is anisotropic on V_0^i for every $i\in I$ yields that q is anisotropic on V_0.
In order to construct a unitary Moufang set isomorphic to $(X,(U_x)_{x\in X})$ we consider the vector space $e_{-1}k_0\oplus V_0\oplus e_1k_0$ where $e_{-1}=y_1$ and $e_1=y_2$. Define the forms q and f on V by :

$$\begin{array}{ll}
& q(e_{-1}x_{-1}+x_0+e_1x_1) \\
= & \epsilon x_{-1}^{\sigma}x_1+q(x_0) \\
& f(e_{-1}x_{-1}+x_0+e_1x_1,e_{-1}z_{-1}+z_0+e_1z_1) \\
= & \epsilon x_{-1}^{\sigma}z_1+x_1^{\sigma}z_{-1}+f(x_0,z_0).
\end{array}$$

As q is anisotropic on V_0, q will be a (σ_0,ϵ)-quadratic form on V of Witt index 1 with associated (σ_0,ϵ)-sesquilinear form f. Therefore we can consider the Moufang set $\mathcal{M}U(V,q,k_0,\sigma_0)$. Choose the coordinatization of $\mathcal{M}U(V,q,k_0,\sigma_0)$ associated to the decomposition $V=e_{-1}k_0\oplus V_0\oplus e_1k_0$. Define the map γ from $(X,(U_x)_{x\in X})$ in the following way.
Let $x\in X$. Choose a Y_i such that $x\in Y_i$ and define :

$$\gamma(x)=(x_0,(c_{0i}^{-1})^{\alpha_{0i}^{-1}}((\varphi_i^{-1}(x))_1)^{\alpha_{0i}^{-1}}).$$

Using Lemma 41 we check that β determines a Moufang set isomorphism.
1. γ is well defined map from X to $\mathcal{M}U(V,q,k,\sigma)$ i.e. γ maps elements of X to points of $\mathcal{M}U(V,q,k,\sigma)$ and for $x \in X$, $\gamma(x)$ is independent of the Moufang subset Y_i containing x we choose.
Let $x \in X$ and choose Y_i with $x \in Y_i$. Following the definition of q on V_0 we find :

$$\begin{aligned} q(x_0) &= (c_{0i}^{-1})^{\alpha_{0i}^{-1}}(q_i((\varphi_i^{-1}(x))_0))^{\alpha_{0i}^{-1}} \\ &= -(c_{0i}^{-1})^{\alpha_{0i}^{-1}}((\varphi_i^{-1}(x))_1)^{\alpha_{0i}^{-1}}. \end{aligned}$$

This shows that $\gamma(x) \in \mathcal{M}U(V,q,k,\sigma)$.
Remains to prove that γ is well defined. Suppose $x \in Y_i \cap Y_j$. Using formulas (3.38), (3.39) and (3.40) we deduce :

$$(\varphi_j^{-1}(x))_1 = c_{0j}(c_{0i}^{-1})^{\alpha_{0i}^{-1}\alpha_{0j}}((\varphi_i^{-1})(x))_1)^{\alpha_{0i}^{-1}\alpha_{0j}}.$$

Equivalently :

$$(c_{0i}^{-1})^{\alpha_{0i}^{-1}}((\varphi_i^{-1}(x))_1)^{\alpha_{0i}^{-1}} = (c_{0j}^{-1})^{\alpha_{0j}^{-1}}((\varphi_j^{-1}(x))_1)^{\alpha_{0j}^{-1}},$$

showing that γ is well defined.

2. γ defines a bijection from X to the points of $\mathcal{M}U(V,q,k,\sigma)$.
Let $(v_0,v_1) \in \mathcal{M}U(V,q,k,\sigma)$. Choose a $v \in v_0$ and Y_i containing v. Using the definition of q on V we find :

$$\begin{aligned} q(v) &= (c_{0i}^{-1})^{\alpha_{0i}^{-1}}(q_i((\varphi_i^{-1}(v))_0)^{\alpha_{0i}^{-1}} \\ &= v_1. \end{aligned}$$

We find that $((\varphi_i^{-1}(v))_0,\, c_{0i}v_1^{\alpha_{0i}}) \in \varphi_i^{-1}(Y_i)$. Remark that by definition of V_0, $(\varphi_i^{-1}(v))_0 = (\varphi_i^{-1}(v'))_0$, if $v' \in v_0$. This means that (v_0,v_1) determines the unique point $x = \varphi(((\varphi_i^{-1}(v))_0, c_{0i}v_1^{\alpha_{0i}}))$ such that $\gamma(x) = (v_0,v_1)$.
3. γ induces bijections $\gamma_{(\infty)}$ from U_{y_1} to $U_{(\infty)}$ and $\gamma_{(0,0)}$ from U_{y_2} to $U_{(0,0)}$ defined by :

$$\begin{aligned} \gamma_{(\infty)}(u(y_1;y_2,x)) &= \gamma \circ u(y_1;y_2,x) \circ \gamma^{-1} \\ \gamma_{(0,0)}(u(y_2;y_1,x)) &= \gamma \circ u(y_2;y_1,x) \circ \gamma^{-1}. \end{aligned}$$

Let $u(y_1;y_2,x) \in U_{y_1}$, $(z_0,z_1) \in \mathcal{M}U(V,q,k,\sigma)$. Choose a Y_i such that x, $\gamma^{-1}(z_0,z_1) \in Y_i$ with $\varphi_i((\bar{x}_0,\bar{x}_1)) = x$, $\varphi_i((\bar{z}_0, c_{0i}z_1^{\alpha_{0i}})) = \gamma^{-1}\ (z_0,z_1)$.
We find :

$$\begin{aligned}
\gamma u(y_1;y_2,x)\gamma^{-1}((z_0,z_1)) &= \gamma u(y_1;y_2,x)\varphi_i((\bar{z}_0, c_{0i}z_1^{\alpha_{0i}}) \\
&= \gamma\varphi_i(u((\infty);(0,0),(\bar{x}_0,\bar{x}_1)))((\bar{z}_0, c_{0i}z_1^{\alpha_{0i}})) \\
&= \gamma\varphi_i((\bar{x}_0+\bar{z}_0, \bar{x}_1 + c_{0i}z_1^{\alpha_{0i}} - f_i(\bar{x}_0,\bar{z}_0))) \\
&= (x_0+z_0, (c_{0i}^{-1})^{\alpha_{0i}^{-1}}\bar{x}_1 + z_1 \\
&\quad -(c_{0i}^{-1})^{\alpha_{0i}^{-1}}(f_i(\bar{x}_0,\bar{z}_0))^{\alpha_{0i}^{-1}}) \\
&= (x_0+z_0, (c_{0i}^{-1})^{\alpha_{0i}^{-1}}\bar{x}_1 + z_1 - f(x_0,z_0)) \\
&= u((\infty);(0,0),(x_0,(c_{0i}^{-1})^{\alpha_{0i}^{-1}}\bar{x}_1))((z_0,z_1)) \\
&= u((\infty);(0,0),\gamma(x))((z_0,z_1)).
\end{aligned}$$

This set of equations clearly yields that $\gamma_{(\infty)}$ defines a bijection from U_{y_1} to $U_{(\infty)}$.
Similar calculations show that $\gamma_{(0,0)}$ defines a bijection from U_{y_2} to $U_{(0,0)}$.
By Lemma 41 it follows that γ defines a Moufang set isomorphism. □

Chapter 4

Existence and non-existence

4.1 Introduction

In the standard reference [32], J. Tits sketches a possible outline leading to a classification of twin buildings. Having already working with Moufang buildings in the past B. Müherr took on the subject with Tits' approach as starting point. After a while he managed to write a concrete classification program for 2-spherical twin buildings down. To this end a lot of techniques and theorems of algebraic group theory (as described for example in [33]) had been extended. Especially the theory on Galois cohomology. For a detailed description we refer to [20] more particularly to Chapter 7 of this work.
To complete the classification program B. Mühlherr still needed a classification of 3 types of geometries, namely twin buildings of type $\tilde{A}_2$, $\tilde{B}_2$ and 443. One of these types consists of the class of $\tilde{B}_2$ twin buildings. The $\tilde{B}_2$ case seemed to be a crucial case where a lot of work was around. One of the main aims of this thesis was therefore to describe all possible $\tilde{B}_2$ twin buildings where the rank 2 residues are one of the quadrangles as described in Chapter 3.
To simplify notations and theory we will not give a concrete list of all these buildings as twin buildings. Instead we give give a description of all Moufang

buildings of type $\tilde{B}_2$. As under some restrictions on the residues 2-spherical twin buildings and Moufang buildings are the same objects we have therefore also a classification of $\tilde{B}_2$ twin buildings with some restrictions on the residues. (For a proof of the fact that in almost all cases 2-spherical twin buildings and Moufang buildings are the same objects we refer to Chapter 2 and [18]). Moreover the techniques used for the $\tilde{B}_2$ case also applies to the 443 case leading to existence condition for twin buildings of type 443. We start by recalling some known theorems and lemmas on isomorphism and automorphisms which will be useful later on.

4.1.1 Isomorphism and automorphisms of some quadrangles

In this section we rephrase some isomorphism and automorphisms of some of the quadrangles described in Chapter 3. Most of the result in this paragraph can be found in or derived from [37] or in Chapter 8 in [29]. For sake of completeness and as some results are not presented in [37] or [29] in the form we want, we will give in most cases explicit proofs.

Theorem 133 *Let $Q(E,q,k,\sigma)$ be a quadrangle defined by a (σ,ϵ)-quadratic form q of Witt index 2. Let f be the (σ,ϵ)-hermitian form associated to q. Suppose that $dim(E) \geq 5$ if $\sigma = 1$ and $\epsilon = 1$, and if k is a generalized quaternion algebra, σ is not its standard involution.*
A permutation g of points and lines of $Q(E,q,k,\sigma)$ is an automorphism of Q (E,q,k,σ) if and only if there exists a constant $c \in k$, a semi-linear transformation φ with associated field automorphism α such that :

$$\begin{aligned} g(\langle x,y\rangle) &= \langle \varphi(x),\varphi(y)\rangle,\ \forall\langle x,y\rangle \in Q(E,q,k,\sigma) \\ c(q(x))^\alpha &= q(\varphi(x)),\ \forall x \in E \\ c(f(x,y))^\alpha &= f(\varphi(x),\varphi(y)),\ \forall x,y \in E. \end{aligned}$$

proof :

Let g be a permutation of the points and lines of $Q(E,q,k,\sigma)$ preserving incidence. Theorem 8.6 in [29] implies that there exists a constant $c \in k$, a semi-linear transformation φ with associated field automorphism α such that :

$$g(\langle x,y\rangle) = \langle \varphi(x),\varphi(y)\rangle,\ \forall\langle x,y\rangle \in Q(E,q,k,\sigma)$$

$$\begin{aligned} c(q(x))^\alpha &= q(\varphi(x)),\ \forall x \in E \\ c(f(x,y))^\alpha &= f(\varphi(x),\varphi(y)),\ \forall x,y \in E. \end{aligned}$$

Conversely let φ be a semi-linear transformation with associated field isomorphism α such that :

$$\begin{aligned} c(q(x))^\alpha &= q(\varphi(x)),\ \forall x \in E \\ c(f(x,y))^\alpha &= f(\varphi(x),\varphi(y)),\ \forall x,y \in E. \end{aligned}$$

Then we can define the permutation g of $Q(E,q,k,\sigma)$ by :

$$g(\langle x,y\rangle) = \langle \varphi(x),\varphi(y)\rangle, \forall \langle x,y\rangle \in Q(E,q,k,\sigma).$$

As incidence in $Q(E,q,k,\sigma)$ is completely defined in terms of f one easily checks that g defines an automorphism of $Q(E,q,k,\sigma)$. □

Proposition 134 *Let $W(k)$ be a symplectic quadrangle defines over the field k. Then $W(k)$ is dually isomorphic to an orthogonal quadrangle $QO(E,q,k)$ with $dim(E) = 5$. Conversely every orthogonal quadrangle $QO(E,q,k)$ with $dim(E) = 5$ is dually isomorphic to the symplectic quadrangle $W(k)$.*

proof :

Let $W(k)$ be the symplectic quadrangle defined over k. We use the coordinatization of $W(k)$ as described in section 3.5.2. To construct $QO(E,q,k)$ we reason as follows. Let $E = e_{-2}k \oplus e_{-1}k \oplus e_0k \oplus e_1k \oplus e_2k$. Define the forms f and q on E by setting ($x = e_{-2}x_{-2} + e_{-1}x_{-1} + e_0\lambda + e_1x_1 + e_2x_2$ and $y = e_{-2}y_{-2} + e_{-1}y_{-1} + e_0\mu + e_1y_1 + e_2y_2$)

$$\begin{aligned} q(x) &= \lambda^2 + x_{-2}x_2 + x_{-1}x_1 \\ f(x,y) &= 2\lambda\mu + x_{-2}y_2 + x_2y_{-2} + x_{-1}y_1 + x_1y_{-1}. \end{aligned}$$

One easily checks that q defines a quadratic form on E of Witt index 2 such that $q(x+y)= q(x) + q(y) + f(x,y)$, $\forall x,y \in E$. We can thus consider the quadrangle $QO(E,q,k)$. Coordinatize $QO(E,q,k)$ using the decomposition $E = e_{-2}k \oplus e_{-1}k + \oplus\, e_0k \oplus e_1k \oplus e_2k$. With respect to the coordinatization we define the bijection β from $W(k)$ to $QO(E,q,k)$ as follows :

$$\begin{aligned}
\beta((\infty)) &= [\infty] \\
\beta((x)) &= [(e_0x,-x^2)] \\
\beta((v,y)) &= [v,(e_0y,-y^2)] \\
\beta((x,w,x')) &= [(e_0x,-x^2),w,(e_0x',-x'^2)] \\
\beta([\infty]) &= (\infty) \\
\beta([v]) &= (v) \\
\beta([x,w]) &= ((e_0x,-x^2),w) \\
\beta([v,y,v']) &= (v,(e_0y,-y^2),v').
\end{aligned}$$

By construction β defines a bijection from the point set of $W(k)$ to the line set of $QO(E,q,k)$ and from the line set of $W(k)$ to the point set of $QO(E,q,k)$. As β preserves incidence we see that $W(k)$ is dually isomorphic to $QO(E,q,k)$ under β.

Conversily let $QO(E,q,k)$ be an orthogonal quadrangle defined in the vector space E such that $dim(E)=5$. Choose a coordinatization of $QO(E,q,k)$ associated to a decomposition $E=e_{-2}k\oplus\ e_{-1}k\ \oplus V_0\ \oplus e_1k\ \oplus e_2k$ with labelling set $R_{0,1}$. Without loss of generalitiy we can then assume that there exists a vector e_0 with $q(e_0)=1$. Remarkt that this implies that $R_{0,1}=\{(e_0\lambda,-\lambda^2)\ |\lambda\in k\}$. Consider the symplectic quadrangle $W(k)$ coordinatizated as explained in section 3.5.2. Define the bijection β from $QO(E,q,k)$ to $W(k)$ by :

$$\begin{aligned}
\beta((\infty)) &= [\infty] \\
\beta((v)) &= [v] \\
\beta(((e_0x,-x^2),w)) &= [x,w] \\
\beta((v,(e_0y,-y^2),v')) &= [v,y,v'] \\
\beta([\infty]) &= (\infty) \\
\beta([(e_0x,-x^2)]) &= (x) \\
\beta([v,(e_0y,-y^2)]) &= (v,y) \\
\beta([(e_0x,-x^2),w,(e_0x',-x'^2)]) &= (x,w,x').
\end{aligned}$$

Then one easily checks that β defines a duality from $QO(E,q,k)$ to $W(k)$.

□

Proposition 135 *Let $QO(E, q, k)$ be an orthogonal quadrangle with $dim(E) = 6$. Then $QO(E, q, k)$ is dually isomorphic to a hermitian quadrangle $QH(E', q', k', \sigma')$ where k' is a quadratic Galois extension of k. Conversely every hermitian quadrangle $QH(E, q, k, \sigma)$ with $dim(E) = 5$ is dually isomorphic to an orthogonal quadrangle $QO(E', q', Fix(\sigma))$ with $dim(E') = 6$.*

proof :

We refer to Proposition 3.4.9 of [37]. □

Proposition 136 *Let $QO(E, q, k)$ be an orthogonal quadrangle defined by a quadratic form q with associated linear form f such that $codim(Rad(f)) = 2$. Then $QO(E, q, k)$ is isomorphic to an indifferent quadrangle $Q(k, k'; k, l')$. Conversely every indifferent quadrangle $Q(k, k'; k, l')$ is isomorphic to an orthogonal quadrangle $Q(\bar{E}, \bar{q}, k)$ and every indifferent quadrangle $Q(k, k'; l, k')$ is dually isomorphic to an orthogonal quadrangle $QO(\bar{E}, \bar{q}, k)$.*

proof :

Let $QO(E, q, k)$ be as in the proposition. Consider a coordinatization of the set associated to a decomposition $E = e_{-2}k \oplus e_{-1}k \oplus V_0 \oplus e_1 k \oplus e_2 k$ with labelling set $R_{0,1}$. Remark that then $Rad(f) = V_0$. Let $(e_0, c^{-1}) \in R_{0,1}$, define the set l' by $l' = \{cq(w) \,|w \in Rad(f)\} = \{cq(w_0) \,|w_0 \in V_0\}$ and denote the subfield of k generated as ring by l' as k'. Clearly l' satisfies :
(i) $1'$ is an additive subgroup of k',
(ii) $l'^{-1} = l'$
(iii) $l'^{-1} = l'$,
(iv) l' generates k' as a ring.
(v) l' is a vector space over k'^2
Let $W(k)$ be the symplectic quadrangle defined over k coordinatized as explained in section 3.5.2. The conditions on l' ensure that we can consider the indifferent quadrangle $Q(k, k'; k, l')$ by restricting the coordinates in the coordinatization table for $W(k)$ (cfr. section 131). Define the bijection β from $\mathcal{M}O(V, q, k)$ to $Q(k, k'; k, l')$ as follows (x, x', $y \in k$ and (v_0, v_1), (v_0', v_1'), $(w_0, w_1) \in R_{0,1}$) :

$$\beta((\infty)) \;\; = \;\; (\infty)$$

$$\begin{array}{rcl}
\beta((x)) & = & (x) \\
\beta(((v_0, v_1), y)) & = & (cv_1, y) \\
\beta((x, (w_0, w_1), x')) & = & (x, cw_1, x') \\
\beta([\infty]) & = & [\infty] \\
\beta([(v_0, v_1)]) & = & [cv_1] \\
\beta([x, (w_0, w_1)]) & = & [x, cw_1] \\
\beta([(v_0, v_1), y, (v'_0, v'_1)]) & = & [cv_1, y, cv'_1].
\end{array}$$

Clearly β defines in this way an isomorphism from $QO(E, q, k)$ to $Q(k, k'; k, l')$. Conversily consider an indifferent quadrangle $Q(k, k'; k, l')$. Let $\{e_0^i \mid i \in I\}$ be a base of l', where l' is seen as a k^2-vector space. Put $\bar{E} = \bar{e}_{-2}k \oplus \bar{e}_{-1}\, k \oplus \bar{V}_0 \oplus \bar{e}_1 k \oplus \bar{e}_2\, k$ where $\bar{V}_0$ is a k-vector space with base $\{e_0^i \mid i \in I\}$. Remark that the construction of $\bar{V}_0$ implies that we can define a bijection γ from l' to $\bar{V}_0$ in the following way. Let $\bar{v}_0 \in l'$. Then $\bar{v}_0$ can be written in a unique way as a sum $\sum_j e_0^j x_j^2$, $x_j \in k$. In the sequel we put :

$$\gamma(\bar{v}_0) = \sum e_0^j v_j.$$

Define the forms $\bar{q}$ and $\bar{f}$ on $\bar{E}$ in the following way :

$$\begin{array}{l}
\bar{q}(\bar{e}_{-2}x_{-2} + \bar{e}_{-1}x_{-1} + \sum_j e_0^j v_j + \bar{e}_1 x_1 + \bar{e}_2 x_2) \\
= x_{-2}x_2 + x_{-1}x_1 + \sum e_0^j v_j^2 \\
\bar{f}(\bar{e}_{-2}x_{-2} + \bar{e}_{-1}x_{-1} + \sum_j e_0^j v_j + \bar{e}_1 x_1 + \bar{e}_2 x_2, \\
\bar{e}_{-2}y_{-2} + \bar{e}_{-1}y_{-1} + \sum_j e_0^j w_j + \bar{e}_1 x_1 + \bar{e}_2 x_2 \\
= x_{-2}x_2 + x_{-1}x_1
\end{array}$$

In this way we get a quadratic form q on $\bar{E}$ of Witt index 2 with associated form $\bar{f}$. Therefore we can consider the orthogonal quadrangle $QO(\bar{E}, \bar{q}, k)$. By construction we have clearly that $codim(Rad(\bar{f})) = 2$. Define the bijection β from $Q(k, k'; k, l')$ to $QO(\bar{E}, \bar{q}, k)$ in the following way :

$$\begin{array}{rcl}
\beta((\infty)) & = & (\infty) \\
\beta((x)) & = & (x) \\
\beta(v, y) & = & ((\gamma(v), v), y) \\
\beta((x, w, x')) & = & (x, (\gamma(w), w), x') \\
\beta([\infty]) & = & [\infty]
\end{array}$$

$$\begin{aligned}\beta([v]) &= [(\gamma(v), v)]\\ \beta([x,w]) &= [x, (\gamma(w), w)]\\ \beta([v,y,v']) &= [(\gamma(v), v), y, (\gamma(v'), v')].\end{aligned}$$

Then one easily checks that β defines bijection from points and lines of $Q(k,k';k,l')$ to points and lines of $QO(\bar{E},\bar{q},k)$ preserving incidence. Hence $Q(k,k';k,l')$ is isomorphic to $QO(\bar{E},\bar{q},k)$.
For a indifferent quadrangle of the form $Q(k,k';l,k')$ we find by Proposition 3.4.4 in [37] that $Q(k,k';l,k')$ is dually isomorphic to $Q(k',k^2;k',l^2)$. By what we already proved we know that $Q(k',k^2;k',k^2)$ is isomorphic to an orthogonal quadrangle $QO(\bar{E},\bar{q},k')$. Hence $Q(k',k;l,k)$ is dually isomorphic to $QO(\bar{E},\bar{q},k')$. □

Corollary 137 *If $char(k) = 2$ the symplectic quadrangle $W(k)$ is isomorphic to an orthogonal quadrangle $QO(E,q,k)$.*

proof :

Let $W(k)$ be as in the corollary and consider $W(k)$ as an indifferent quadrangle $Q(k,k;k,k)$. Proposition 136 shows then that $Q(k,k;k,k)$ is isomorphic to an orthogonal quadrangle $QO(E,q,k)$. Hence $W(k)$ is also isomorphic to $QO(E,q,k)$. □

Proposition 138 *Let $Q(E,q,k,\sigma)$ be a quadrangle defined by a $(\sigma,-1)$-quadratic form such that $dim(E) = 4$, $k$is a generalized quaternion algebra with standard involution σ. Then $Q(E,q,k,\sigma)$ is dually isomorphic to an orthogonal quadrangle $QO(E',q',Z(k))$ with $dim(E') = 8$ such that $\mathcal{M}_l(QO(E',q',Z(k))$ consists of non-commutative orthogonal Moufang sets.*

Let $Q(E,q,k,\sigma)$ be as in the theorem. As k is a generalized quaternion algebra there exist (cfr [7] p73) i, $j \in k$ such that $k = Z(k) \oplus iZ(k) \oplus jZ(k) \oplus jiZ(k)$ with if $char(k) \neq 2$, $i^2 = \alpha_0$, $j^2 = \beta_0$, and if $char(k) = 2$, $i^2 = i+\alpha_0$, $j^2 = \beta_0$, $ij = ij + j$, with α_0, $\beta_0 \in Z(k) \setminus Z(k)^2$. The norm function on k is denoted by N. Define the 8-dimensional$Z(k)$-vector space E' by :

$$E' = e'_{-2}Z(k) \oplus e'_{-1}Z(k) \oplus V'_0 \oplus e'_1 Z(k) \oplus e'_2 Z(k),$$

with $V_0' = {e_0'}^1 Z(k) \oplus {e_0'}^2 Z(k) \oplus {e_0'}^3 Z(k) \oplus {e_0'}^4 Z(k)$. Suppose that the forms g', f' and q' on E' are defines as follows. If $x' = e'_{-2}x'_{-2} + e'_{-1}x'_{-1} + x'_0 + e'_1 x'_1 + e'_2 x'_2$ with $x'_0 = {e_0'}^1 z'_1 + {e_0'}^2 z'_2 + {e_0'}^3 z'_3 + {e_0'}^4 z'_4$, $\lambda' = z'_1 + iz'_2 + jz'_3 + lz'_4$, $y' = e'_{-2}y'_{-2} + e'_{-1}y'_{-1} + y'_0 + e'_1 y'_1 + e'_2 y'_2$ with $y'_0 = {e_0'}^1 u'_1 + {e_0'}^2 u'_2 + {e_0'}^3 u'_3 + {e_0'}^4 u'_4$ and $\mu' = u'_1 + iu'_2 + ju'_3 + lu'_4$ we set :

$$\begin{aligned} g'(x',x') &= x'_{-2}x'_2 + x'_{-1}x'_1 + N(\lambda') \\ f'(x',y') &= x'_{-2}y'_2 + x'_2 y'_{-2} + x'_{-1}y'_1 + x'_1 y'_{-1} + \lambda'^{\sigma}\mu' + \mu'^{\sigma}\lambda' \\ q(x') &= g(x',x') + Tr(\sigma) \\ &= g(x',x') + Z(k) \end{aligned}$$

One easily checks that q' defines a quadratic form on E' of Witt index 2. Therefore we can consider the quadrangle $QO(E', q', Z(k))$. Choose a coordinatization of $Q(E, q, k, \sigma)$ associated to a decomposition $E = e_{-2}k \oplus e_{-1}k \oplus V_0 \oplus e_1 k \oplus e_2 k$ and coordinatize $QO(E', q', Z(k))$ via the decomposition $E' = e'_{-2}Z(k) \oplus e'_{-1}Z(k) \oplus V'_0 \oplus e'_1 Z(k) \oplus e'_2 Z(k)$. Define the map β from $Q(E, q, k, \sigma)$ to $QO(E', q', Z(k))$ in the following way : ($x = x_1 + ix_2 + jx_3 + jix_4$, $x' = x'_1 + ix'_2 + jx'_3 + jix'_4$ and $y = y_1 + iy_2 + jy_3 + jiy_4$) :

$$\begin{aligned} \beta((\infty)) &= [\infty] \\ \beta((x)) &= [({e_0'}^1 x'_1 + {e_0'}^2 x'_2 + {e_0'}^3 x_3 + {e_0'}^4 x_4, -N(x))] \\ \beta((0,v_1),y) &= [v_1, ({e_0'}^1 y_1 + {e_0'}^2 y_2 + {e_0'}^3 y_3 + {e_0'}^4 y_4, -N(y))] \\ \beta((x,(0,w_1),x')) &= [({e_0'}^1 x'_1 + {e_0'}^2 x'_2 + {e_0'}^3 x_3 + {e_0'}^4 x_4, -N(x)), w_1, \\ &\quad ({e_0'}^1 y_1 + {e_0'}^2 y_2 + {e_0'}^3 y_3 {e_0'}^4 y_4, -N(y))] \\ \beta([\infty]) &= (\infty) \\ \beta([(0,v_1)]) &= (v_1) \\ \beta([x,(0,v_1)]) &= (({e_0'}^1 x_1 + {e_0'}^2 x_2 + {e_0'}^3 x_3 + {e_0'}^4 x_4, -N(x)), v_1) \\ \beta([(0,v_1),y,(0,v'_1)]) &= (v_1, ({e_0'}^1 y_1 + {e_0'}^2 y_2 + {e_0'}^3 y_3 + {e_0'}^4 y_4, -N(y)), v'_1) \end{aligned}$$

By construction we see that β defines a bijection from the point set of $Q(E, q, k, \sigma)$ to the line set of $QO(E', q', Z(k))$ and from the line set of $Q(E, q, k, \sigma)$ to the point set of $QO(E', q', Z(k))$ preserving incidence. This proves that $Q(E, q, k, \sigma)$ is dually isomorphic to $QO(E', q', Z(k))$. That $\mathcal{M}_l$ $QO(E', q', Z(k)))$ is not commutative follows from the fact that $\mathcal{M}_l(QO(E', q', Z(k)) = \mathcal{M}_p(Q(E, q, k, \sigma))$ where $\mathcal{M}_p(Q(E, q, k, \sigma))$ is the isomorphism class containing $\mathcal{P}(k)$. $\square$

4.2 Moufang foundations

4.2.1 Integrable Moufang foundations

Let $M = (m_{ij})_{i,j \in I}$ be a Coxeter matrix and (Δ, W, S, d) be a Moufang building of type M with root groups $(U_\alpha)_{\alpha \in \Phi}$, where Φ is a root system of type M. Suppose the standard apartment in Δ is given by Σ_0 and the isomorphism from W to Σ_0 by γ_0. Let $\gamma_0(1) = c_+ \in \Sigma_0$. Consider the tuple $((R_{ij}(c_+))_{\{i,j\} \in E(M)}, (c_{ij})_{\{i,j\} \in E(M)}, (\beta_{ijk})_{\{i,j\},\{j,k\} \in E(M)})$ with $(c_{ij})_{\{i,j\}} = c_+$, $\beta_{ijk} = 1$, $\forall$ $\{i,j\}$, $\{j,k\} \in E(M)$. It follows by the definition that $((R_{ij}(c_+))_{\{i,j\} \in E(M)}, (c_{ij})_{\{i,j\} \in E(M)}, (\beta_{ijk})_{\{i,j\},\{j,k\} \in E(M)})$ is a Moufang foundation. As for every Moufang building the automorphism group acts transitively on the chambers (cfr. Proposition 64 of Chapter 2) we see that the isomorphism class of $((R_{ij}(c_+))_{\{i,j\} \in E(M)}, (c_{ij})_{\{i,j\} \in E(M)}, (\beta_{ijk})_{\{i,j\},\{j,k\} \in E(M)})$ is independent from c_+. In the sequel we will therefore denote this isomorphism class as *MoFo(Δ)*.

Definition 139 Let $M = (m_{ij})_{i,j \in I}$ be a Coxeter matrix and $((\Delta_{ij})_{\{i,j\} \in E(M)}, (c_{ij})_{\{i,j\} \in E(M)}, (\beta_{ijk})_{\{i,j\}\{j,k\} \in E(M)})$ a Moufang foundation of type M. Then we say that $((\Delta_{ij})_{\{i,j\} \in E(M)}, (c_{ij})_{\{i,j\} \in E(M)}, (\beta_{ijk})_{\{i,j\}\{j,k\} \in E(M)})$ is *integrable* if there exist a Moufang building (Δ, W, S, d) such that $((\Delta_{ij})_{\{i,j\} \in E(M)}, (c_{ij})_{\{i,j\} \in E(M)}, (\beta_{ijk})_{\{i,j\}\{j,k\} \in E(M)})$ belongs to $MoFo(\Delta)$.

A first result on the integrability of Moufang foundations is the following theorem. But first we give a definition.

Definition 140 A generalized Moufang polygon Γ is called *semi-pappian* if $\mathcal{M}_p(\Gamma)$ or $\mathcal{M}_l(\Gamma)$ is isomorphic to a commutative projective Moufang set.

The question of integrability of Moufang foundations in the case where the polygons involved are semi-pappian is solved by theorem 7.2.6 of [20]. We restate this theorem without proof. For the details we refer to [20].

Theorem 141 *Let M be an irreducible, 2-spherical, locally finite Coxeter matrix. Let F be a Moufang foundation of type M with the property that every polygons of the Moufang foundation is semi-pappian. Then F is integrable.*

proof :

See Theorem 7.2.6 in [20].

□

4.2.2 Moufang foundations and property (Ind)

The following notion will be useful for proving integrability of Moufang foundations.

Definition 142 Suppose that $F = ((\Delta_{ij})_{\{i,j\}\in E(M)}, (c_{ij})_{\{i,j\}\in E(M)}, (\beta_{ijk})_{\{i,j\}\{j,k\}\in E(M)})$ is a Moufang foundation. Let $\varphi = (\varphi_{ij})_{\{i,j\}\in E(M)}$, where for every $\{i,j\} \in E(M)$, φ_{ij} defines an isomorphism from Δ_{ij} to a Moufang generalized polygon Δ'_{ij} preserving the Moufang structure. Then we denote the Moufang foundation $((\Delta'_{ij})_{\{i,j\}\in E(M)}, (\varphi_{ij}(c_{ij}))_{\{i,j\}\in E(M)}, (\varphi_{jk}\beta_{ijk}\varphi_{ij}^{-1})_{\{i,j\}\{j,k\}\in E(M)})$ as $\varphi(F)$.

An important remark concerning this definition is the following lemma.

Lemma 143 *Let* $F = ((\Delta_{ij})_{\{i,j\}\in E(M)}, (c_{ij})_{\{i,j\}\in E(M)}, (\beta_{ijk})_{\{i,j\}\{j,k\}\in E(M)})$ *be a Moufang foundation and* $\varphi = (\varphi_{ij})_{\{i,j\in E(M)\}}$, *where for every* $\{i,j\} \in E(M)$, φ_{ij} *defines an isomorphism from* Δ_{ij} *to a Moufang polygon* Δ'_{ij} *preserving the Moufang structure. Then* $\varphi(F)$ *is isomorphic to* F.

proof :

If F and φ are as in the Lemma an isomorphism from F to $\varphi(F)$ is given by (φ_{ij}, Id). □

The following definition is motivated by the theory exposed in [22].

Definition 144 Let $\Gamma = (\mathcal{P}, \mathcal{L}, I)$ be a Moufang generalized n-gon with $n < \infty$ and $x \in \mathcal{P} \cup \mathcal{L}$. Then we say that Γ satisfies condition (Ind) on $\Gamma(x)$ if every automorphism of the induced Moufang set $\mathcal{M}_{\Gamma(x)}(\Gamma)$ extends to an automorphism of Γ. If condition (Ind) is satisfied for every panel of Γ we say that Γ satisfies condition (Ind).

The importance of condition (Ind) is illustrated in the following lemma.

Lemma 145 *Let* M *be a* 2*-spherical Coxeter matrix such that* $G(M)$ *is a tree. Suppose* $\{\Delta_{ij} | \{i,j\} \in E(M)\}$ *is a set of Moufang polygons,* c_{ij} *a chamber in* Δ_{ij}, $\forall \{i,j\} \in E(M)$. *Assume that for every* $\{i,j\} \in E(M)$ *such that* *there exists a $k \in I$ with $\{j,k\} \in E(M)$, Δ_{ij} satisfies condition (Ind) on every j-panel. Then all Moufang foundations of the form $\varphi((\Delta_{ij})_{\{i,j\}\in E(M)}, (c_{ij})_{\{i,j\}\in E(M)}, (\beta_{ijk})_{\{i,j\}\{j,k\}\in E(M)})$ with $\varphi = (\varphi_{ij})_{\{i,j\}\in E(M)}$, where every φ_{ij} defines an isomorphism from Δ_{ij} to a Moufang polygon Δ'_{ij}, are isomorphic.*

proof :

As every Moufang foundation F of the form $((\Delta_{ij})_{\{i,j\}\in E(M)}, (c_{ij})_{\{i,j\}\in E(M)}, (\beta_{ijk})_{\{i,j\}\{j,k\}\in E(M)})$ and $\varphi((\Delta_{ij})_{\{i,j\}\in E(M)}, (c_{ij})_{\{i,j\}\in E(M)}, (\beta_{ijk})_{\{i,j\}\{j,k\}\in E(M)})$,with $\varphi = (\varphi_{ij})_{\{i,j\}\in E(M)}$ as in the lemma, are isomorphic the lemma will be proved if we show that all Moufang foundations $((\Delta_{ij})_{\{i,j\}\in E(M)}, (c_{ij})_{\{i,j\}\in E(M)}, (\beta_{ijk})_{\{i,j\}\{j,k\}\in E(M)})$ are isomorphic. Suppose $((\Delta_{ij})_{\{i,j\}\in E(M)}, (c_{ij})_{\{i,j\}\in E(M)}, (\beta_{ijk})_{\{i,j\}\{j,k\}})$ and $((\Delta_{ij})_{\{i,j\}\in E(M)}, (c_{ij})_{\{i,j\}\in E(M)}, (\beta'_{ijk})_{\{i,j\}\{j,k\}})$ are two Moufang foundations involving Δ_{ij}. Let γ the identity on M. Let i, j, $k \in I$ with $\{i,j\} \in E(M)$ and $\{j,k\} \in E(M)$. The conditions on the Δ_{ij} yield that every Moufang set isomorphism ${\beta'_{ijk}}^{-1}\ \beta_{ijk}$ can be extended to an isomorphism of Δ_{ij} which we will denote by $({\beta'_{ijk}}^{-1}\ \beta_{ijk})^*$ Similarly $\beta_{ijk}\ {\beta'_{ijk}}^{-1}$ can be extended to an isomorphism of Δ_{jk} which we denote by $(\beta_{ijk}\ {\beta'_{ijk}}^{-1})^*$. If we put $(\gamma_{ij}, \gamma_{jk}) = (Id, (\beta_{ijk}{\beta'_{ijk}}^{-1})^*)$ or $(\gamma_{ij}, \gamma_{jk}) = (({\beta'_{ijk}}^{-1}\beta_{ijk})^*, Id)$, γ_{ij} and γ_{jk} clearly satisfy :

$$\gamma_{jk}^{-1}\beta'_{ijk}\gamma_{ij} = \beta_{ijk}.$$

One checks that for every $\{i,j\}$ $\{j,k\} \in E(M)$ we can choose the γ_{ij} and γ_{jk} out the two possibilities described above in such a way that $((\gamma_{ij})_{\{i,j\}\in E(M)}), \gamma)$ defines an isomorphism from $((\Delta_{ij})_{\{i,j\}\in E(M)}, (c_{ij})_{\{i,j\}\in E(M)}, (\beta_{ijk})_{\{i,j\}\{j,k\}})$ to $((\Delta_{ij})_{\{i,j\}\in E(M)}, (c_{ij})_{\{i,j\}\in E(M)}, (\beta'_{ijk})_{\{i,j\}\{j,k\}})$. □

To end this section we give some Proposition concerning property (Ind) in the generalized quadrangles which were studied in Chapter 3.

Proposition 146 *Let $W(k)$ and $W(k')$ be two symplectic quadrangles. Choose $x \in W(k)$, $x' \in W(k')$. Then every isomorphism from $\mathcal{M}_{\Gamma(x)}(W(k))$ to $\mathcal{M}_{\Gamma(x')}(W(k'))$ extends to an isomorphism from $W(k)$ to $W(k')$. In particular $W(k)$ satisfies condition (Ind).*

proof :

Let $W(k)$, $W(k')$ be symplectic quadrangle as described in sections 3.5.1 and 3.5.2. Then we saw that for every point P and every line L in $W(k)$,

$\mathcal{M}_{\Gamma(P)}(W(k)) \cong \mathcal{M}_{\Gamma(L)}(W(k)) \cong \mathcal{P}(k)$. We prove that the proposition holds if x and x' are both point rows. The other cases can be proved in a similar way. We use the coordinatizations of $W(k)$ and $W(k')$ as described in section 3.5.2. Choose as generic point rows $\Gamma([0])$ and $\Gamma([0])$. Let β be an isomorphism from $\mathcal{M}_{\Gamma([0])}(W(k))$. to $\mathcal{M}_{\Gamma([0])}(W(k'))$. By proposition 124 we see that we can assume that without loss of generality β induces a field isomorphism α from k to k' such that $\beta((0,t)) = (0,t^\alpha)$, $\forall t \in k$. Define the map β^* from $W(k)$ to $W(k')$ by :

Elements of $W(k)$	Image under β^*
(∞)	(∞)
(x)	(x^α)
(v,y)	(v^α, y^α)
(x,w,x')	$(x^\alpha, w^\alpha, x'^\alpha)$
$[\infty]$	$[\infty]$
$[v]$	$[v^\alpha]$
$[x,w]$	$[x^\alpha, w^\alpha]$
$[v,y,v']$	$[v^\alpha, y^\alpha, v'^\alpha]$

Then one easily checks that β^* defines an isomorphism from $W(k)$ to $W(k')$ which extends β.
This implies in particular that for $x \in W(k)$ every automorphism β of $\mathcal{M}_{\Gamma(x)}(W(k))$ extends to an automorphism of $W(k)$. Thus in this way we see that $W(k)$ satisfies conidition (Ind). □

We remark that the fact that every symplectic quadrangle satsisfies condition (Ind) can already be found in [22] (cfr. Proposition 1 of loc. cit.). In fact it is proved in this paper that every finite generalized polygon satisfies condition (Ind).

Proposition 147 *Let $Q(E,q,k,\sigma)$ be a generalized quadrangle defined by a (σ,ϵ)-quadratic form and $Q(E',q',k',\sigma')$ be a generalized quadrangle defined by a (σ',ϵ')-quadratic form q' such that of the following occurs :*

(i) $\mathcal{M}_l(Q(E,q,k,\sigma))$ and $\mathcal{M}_l(Q(E',q',k',\sigma')$ consist of non-commutative orthogonal Moufang sets,

(ii) $\mathcal{M}_l(Q(E,q,k,\sigma))$ *and* $\mathcal{M}_l(Q(E',q',k',\sigma'))$ *consist of hermitian Moufang sets and* $dim(E) > 5$,

(iii) $\mathcal{M}_l(Q(E,q,k,\sigma))$ *and* $\mathcal{M}_l(Q(E',q',k',\sigma'))$ *consist of unitary Moufang sets with non-commutative root groups such that* $Rad(f) = 0$ *if* $char(k) = 2$, k *is a generalized quaternion algebra with standard involution* σ, *where* f *is the* (σ,ϵ)*-hermitian the form associated to* q.

Suppose p *is a point in* $Q(E,q,k,\sigma))$ *and* p' *a point in* $Q(E',q',k',\sigma')$. *Then every isomorphism* β *from* $\mathcal{M}_{\Gamma(p)}(Q(E,q,k,\sigma))$ *to* $\mathcal{M}_{\Gamma(p')}$ $(Q(E',q',k',\sigma'))$ *can be extended to an isomorphism from* $Q(E,q,k,\sigma)$ *to* $Q(E',q',k',\sigma')$. *In particular under the conditions of the proposition both* $Q(E,q,k,\sigma)$ *and* $Q(E',q',k',\sigma')$ *satisfy condition* (Ind) *on their line pencils.*

proof :

Suppose $Q(E,q,k,\sigma)$ and $Q(E',q',k',\sigma')$ satisfy (i), (ii) or (iii), p a point in $Q(E,q,k,\sigma)$ and p' a point in $Q(E',q',k',\sigma')$. Throughout this proof we will use the coordinatizations of $Q(E,q,k,\sigma)$ and $Q(E',q',k',\sigma')$ as described in section 3.5.4. Assume that these coordinatizations are associated to decompositions $E = e_{-2}k \oplus e_{-1}k \oplus E_0 \oplus e_1k \oplus e_2k$ and $E' = e'_{-2}k' \oplus e'_{-1}k' \oplus E'_0 \oplus e'_1k' \oplus e'_2k'$ with labelling sets $R_{0,1}$ and $R'_{0,1}$. Let β be an isomorphism from $\mathcal{M}_{\Gamma(p)}(Q(E,q,k,\sigma))$ to $\mathcal{M}_{\Gamma(p')}(Q(E',q',k',\sigma'))$ Without loss of generality we can assume $p = (0)$, $p' = (0)$, $\beta([\infty])$ $[\infty]$ and $\beta([0,(0,0)]) = [0,(0,0)]$. Denote $V = e_{-2}k \oplus E_0 \oplus e_2k$ and $V' = e'_{-2}k' \oplus E'_0 \oplus e'_2k'$. We use the identification of $\mathcal{M}(V,q,k,\sigma)$ and $\mathcal{M}_{\Gamma((0))}Q(E,q,k,\sigma)$ and $\mathcal{M}(V',q',k',\sigma')$ with $\mathcal{M}_{\Gamma(0)}$ $Q(E',q',k',\sigma')$ as described in Lemma 99 of Chapter 3. Propositions 127, 128 and 129 show that there exists a semi-linear transformation φ from V to V' with associated field isomorphism α and a constant $c \in k$ such that :

$$\begin{aligned}
\beta([0,(v_0,v_1)]) &= \langle \varphi(e_{-2}v_1 + v_0 + e_2)\rangle,\ \forall (v_0,v_1) \in R_{0,1} \\
c(q(x))^\alpha &= q(\varphi(x)),\ \forall x \in V \\
c(f(x,y))^\alpha &= f(\varphi(x),\varphi(y)),\ \forall x,y \in V, \\
c\lambda^{\sigma\alpha}c^{-1} &= \lambda^{\alpha\sigma},\ \forall \lambda \in k \\
c\epsilon^\alpha &= c^\sigma\epsilon. \qquad (4.1)
\end{aligned}$$

Define the semi-linear transformation φ^* with associated field isomorphism α from E to E' in the following way :

$$\begin{aligned}
&\varphi^*(e_{-2}x_{-2} + e_{-1}x_{-1} + x_0 + e_1x_1 + e_2x_2) \\
&= e_{-1}x_{-1}^\alpha + e_1cx_1^\alpha + \varphi(e_{-2}x_{-2} + x_0 + e_1x_1).
\end{aligned}$$

Let $x = e_{-2}x_{-2} + e_{-1}x_{-1} + x_0 + e_1x_1 + e_2x_2$ and $y = e_{-2}y_{-2} + e_{-1}x_{-1} + y_0 + e_1y_1 + e_2y_2$.
We find :

$$\begin{aligned}&c(q(x))^{\alpha}\\&= c(q(e_{-2}x_{-2} + x_0 + e_2x_2))^{\alpha} + c(x_{-1}^{\sigma}x_1)^{\alpha}\\&= q(\varphi(e_{-2}x_{-2} + x_0 + e_2x_2)) + c(x_{-1}^{\sigma\alpha}x_1^{\alpha})\\&= q(\varphi(e_{-2}x_{-2} + x_0 + e_2x_2)) + x_{-1}^{\alpha\sigma}cx_1^{\alpha}\\&= q(\varphi^*(x))\end{aligned}$$

and :

$$\begin{aligned}&c(f(x,y))^{\alpha}\\&= c(f(e_{-2}x_{-2} + x_0 + e_2x_2, e_{-2}y_{-2} + y_0 + e_2y_2))^{\alpha} + c(x_{-1}^{\sigma}y_1 + x_1^{\sigma}\epsilon y_{-1})^{\alpha}\\&= f(\varphi(e_{-2}x_{-2} + x_0 + e_2x_2), \varphi(e_{-2}y_{-2} + y_0 + e_2y_2)) + cx_{-1}^{\sigma\alpha}y_1^{\alpha} + cx_1^{\sigma\alpha}\epsilon^{\alpha}y_{-1}^{\alpha}\\&= f(\varphi(e_{-2}x_{-2} + x_0 + e_2x_2), \varphi(e_{-2}y_{-2} + y_0 + e_2y_2)) + x_{-1}^{\alpha\sigma}cy_1^{\alpha} + x_1^{\alpha\sigma}c\epsilon^{\alpha}y_{-1}^{\alpha}\\&= f(\varphi(e_{-2}x_{-2} + x_0 + e_2x_2), \varphi(e_{-2}y_{-2} + y_0 + e_2y_2)) + x_{-1}^{\alpha\sigma}cy_1^{\alpha} + x_1^{\alpha\sigma}c^{\sigma}\epsilon y_{-1}^{\alpha}\\&= f(\varphi(x), \varphi(y)),\end{aligned}$$

where we used the properties of c and α and φ as described in formula (4.1). Therefore we can define the bijection β^* from $Q(E,q,k,\sigma)$ to $Q(E',q',k',\sigma')$ if we set :

$$\begin{aligned}\beta^*(\langle x\rangle) &= \langle\varphi(x)\rangle,\ \forall\langle x\rangle \in Q(E,q,k,\sigma)\\ \beta^*(\langle x,y\rangle) &= \langle\varphi(x),\varphi(y)\rangle, \forall\langle x,y\rangle \in Q(E,q,k,\sigma) \text{ such that } x \notin \langle y\rangle.\end{aligned}$$

As incidence in $Q(E,q,k,\sigma)$ and $Q(E',q',k',\sigma')$ are inherited from their embedding in $PG(E)$ and $PG(E')$, we see that β^* defines an isomorphism from $Q(E,q,k,\sigma)$ to $Q(E',q',k',\sigma')$.

□

Proposition 148 *Let $Q(E,q,k,\sigma)$ be a quadrangle defined by $(\sigma,-1)$-quadratic form of Witt index 2 such that k is a generalized quaternion algebra with standard involution σ and $dim(V) = 4$. Then $Q(E,q,k,\sigma)$ satisfies condition (Ind) on its point rows.*

Let $Q(E,q,k,\sigma)$ be a quadrangle as in the proposition. By Proposition 138 we know that $Q(E,q,k,\sigma)$ is dually isomorphic to an orthogonal quadrangle $QO(E',q',Z(k))$ such that $\mathcal{M}_l(QO(E',q',Z(k))$ consists of a non-commutative orthogonal Moufang set.

Proposition 147 implies that $QO(E', q', Z(k))$ satisfies condition (Ind) on its line pencils. Hence $Q(E, q, k, \sigma)$ satisfies condition (Ind) on its point rows. □

Proposition 149 *Let $Q(E, q, k, \sigma)$ be a quadrangle defined by a (σ, ϵ)- quadratic form q of Witt index 2 such that $Z(k) \neq k$ and if k is a generalized quaternion algebra σ is not the standard involution. Suppose that the (σ, ϵ)-hermitian form associated to q is given by f. If $Q(E, q, k, \sigma)$ satisfies condition (Ind) on its point rows, k admits no anti-automorphism and for every automorphism γ of k there exists a constant $c \in k$ such that :*

$$\begin{aligned} c\gamma^{\sigma\gamma}c^{-1} &= \lambda^{\gamma\sigma} \\ c\epsilon^{\gamma} &= c^{\sigma}\epsilon \end{aligned}$$

proof :

Choose a coordinatization of $Q(E, q, k, \sigma)$ associated to a decomposition $E = e_{-2}k \oplus e_{-1}k \oplus V_0 \oplus e_1 k \oplus e_2\ k$. Suppose $Q(E, q, k, \sigma)$ satisfies condition (Ind) on its point rows. In particular this means that $Q(E, q, k, \sigma)$ should satisfy condition (Ind) on $\Gamma((0,0))$. Suppose k admits an anti-automorphism γ. Proposition 124 implies then that the permutation β of $\Gamma([(0,0)])$ defined by :

$$\begin{aligned} \beta((\infty)) &= [\infty] \\ \beta(((0,0),v)) &= [(0,0),v^{\gamma})],\ \forall v \in k \end{aligned}$$

defines a Moufang set isomorphism. Suppose that β can be extended to an automorphism of $Q(E, q, k, \sigma)$. But Theorem 133 implies that there exist an automorphism α of k such that $\gamma = \alpha$. This is only possible if $Z(k) = k$ contradicting the assumption on k. Hence k can only admit automorphisms. Suppose that γ is an automorphism of k. Then the permutation of $\Gamma([(0,0)])$ defined by :

$$\begin{aligned} \beta((\infty)) &= (\infty) \\ \beta(((0,0),v)) &= [(0,0),v^{\gamma}],\ \forall v \in k \end{aligned}$$

determines an automorphism of $\mathcal{M}_{\Gamma([(0,0)])}\ Q(E, q, k, \sigma)$ (cfr. Proposition 124). As $Q(E, q, k, \sigma)$ satisfies condition (Ind) on its point rows Theorem

133 implies that there exists a constant $c \in k$ and a semi-linear transformation with associated automorphism α of k such that :

$$\begin{aligned} c(f(x,y))^{\alpha} &= f(\varphi(x),\varphi(y)),\ \forall x,y \in E \\ c(q(x))^{\alpha} &= q(\varphi(x)),\ \forall x \in E \\ ((0,0),v^{\alpha}) &= ((0,0),v^{\gamma}). \end{aligned}$$

Thus we find $\alpha = \gamma$. Let x and $y \in E$ such that $f(x,y) \neq 0$. Then the first of these equations implies that for $\lambda \in k$:

$$\begin{aligned} c(f(x\lambda,y))^{\alpha} &= c\lambda^{\sigma\alpha}(f(x,y))^{\alpha} \\ &= c\lambda^{\sigma\alpha}c^{-1}(c(f(x,y))^{\alpha}) \\ &= f(\varphi(x)\lambda^{\alpha},\varphi(y)) \\ &= \lambda^{\alpha\sigma}f(\varphi(x),\varphi(y)) \end{aligned}$$

and we find :

$$c\lambda^{\sigma\gamma}c^{-1} = \lambda^{\alpha\sigma},\ \forall \lambda \in k.$$

Moreover we calculate for c :

$$\begin{aligned} (f(x,y))^{\alpha\sigma}c\epsilon^{\alpha} &= c(f(x,y))^{\sigma\alpha}\epsilon^{\alpha} \\ &= c(f(x,y)^{\sigma}\epsilon)^{\alpha} \\ &= c(f(y,x))^{\alpha} \\ &= f(\varphi(y),\varphi(x)) \\ &= f(\varphi(x),\varphi(y))^{\sigma}\epsilon \\ &= f(x,y))^{\alpha\sigma}c^{\sigma}\epsilon \end{aligned}$$

implying that $c\epsilon^{\alpha} = c^{\sigma}\epsilon$.

□

4.3 Integrability conditions for Moufang foundations of type $\tilde{B}_2$

4.3.1 Introduction

Definition 150 Let $M_{\tilde{B}_2}$ the Coxeter matrix defined over the set $I = \{1, 2, 3\}$ where $m_{12} = m_{23} = 4$ and $m_{13} = 2$. A Coxeter matrix M isomorphic to

$M_{\tilde{B}_2}$ will be said to be *of type* $\tilde{B}_2$.
A *root system of type* $\tilde{B}_2$ is defined as a root system of type $M_{\tilde{B}_2}$, whereas a *building of type* $\tilde{B}_2$ is a building of type $M_{\tilde{B}_2}$. A *Moufang foundation* is said to be *of type* $\tilde{B}_2$ if it is of type $M_{\tilde{B}_2}$.

Let (Δ, W, S, d) be a Moufang building of type $\tilde{B}_2$ with associated root groups system $(U_\alpha)_{\alpha\in\Phi_{\tilde{B}_2}})$ where $\Phi_{\tilde{B}_2}$ is a root system of type $\tilde{B}_2$. Choose a root base $\Lambda_{\tilde{B}_2} = \{\alpha_1, \alpha_2, \alpha_3\}$ in $\Phi_{\tilde{B}_2}$ with $\bar{m}_{\alpha_1\alpha_2} = \bar{m}_{\alpha_2\alpha_3} = 4$ and $\bar{m}_{\alpha_1\alpha_3} = 2$. We thus find that :

$$[U_{\pm\alpha_1}, U_{\pm\alpha_3}] = 1. \tag{4.2}$$

Using this equation we deduce the following necessary condition concerning integrability of Moufang foundations of type $\tilde{B}_2$.
Let $((\Delta_{ij})_{\{i,j\}\in E(M)}, (c_{ij})_{\{i,j\}\in E(M)}, (\beta_{ijk})_{\{i,j\}\{j,k\}\in E(M)})$ be a Moufang foundation of type $\tilde{B}_2$. For every $\{i,j\} \in E(M)$ we suppose that the root group structure on Δ_{ij} is given by $(U_{\alpha^k_{ij}})_{\alpha^k_{ij}\in\Phi_{ij}}$, where Φ_{ij} is a root system of type M_{ij}. Without loss of generality we can assume $\bar{m}_{\alpha^1_{12}\alpha^1_{12}} = \bar{m}_{\alpha^2_{23}\alpha^3_{23}} = 4$. Suppose that this Moufang foundation is integrable. This means that there exists a Moufang building (Δ, W, S, d) of type $\tilde{B}_2$ and a chamber $c_+ \in \Delta$ with $((\Delta_{ij})_{\{i,j\}\in E(M)}, (c_{ij})_{\{i,j\}\in E(M)}, (\beta_{ijk})_{\{i,j\},\{j,k\}\in E(M)}) \cong (R_{ij}(c_+), (\bar{c}_{ij})_{\{i,j\}\in E(M)}, (\bar{\beta}_{ijk})_{\{i,j\},\{j,k\}\in E(M)}$ if $\bar{c}_{ij} = c_+$ and $\bar{\beta}_{ijk} = Id$, $\forall\ \{i,j\}$, $\{j,k\} \in E(M)$. Then this implies in particular that after identification in Δ every element of $U_{\alpha^1_{12}}$ should commute with every element of $U_{\alpha^3_{23}}$ in the action on Δ. This means in particular that every $g \in \langle U_{-\alpha^1_{12}}, U_{\alpha^1_{12}}\rangle$ which stabilized $\mathcal{M}_{R_2(c_{12})}(\Delta_{12})$ should commute in its action on $\mathcal{M}_{R_2(c_{12})}(\Delta_{12})$ with every g' $\in \langle U_{-\alpha^3_{23}} U_{\alpha^3_{23}}\rangle$ which stabilizes $\mathcal{M}_{R_2(c_{23})}(\Delta_{23})$ in its action on $\mathcal{M}_{R_2(c_{12})}(\Delta_{12})$ after identification under β_{123}. Set $\beta_{123} = \beta$.
We thus find :

$$[g, \beta g' \beta^{-1}](z) = z,\ \forall z \in \mathcal{M}_{R_2(c_{12})}(\Delta_{12}), \tag{4.3}$$

$\forall g \in \langle U_{-\alpha^1_{12}}, U_{\alpha^1_{12}}\rangle \cap Stab(\mathcal{M}_{R_2(c_{12})}(\Delta_{12})$ and $g' \in \langle U_{-\alpha^3_{23}}, U_{\alpha^3_{23}}\rangle$ $\cap\ Stab(\mathcal{M}_{R_2(c_{23})}(\Delta_{23})$. We prove the following theorem.

Theorem 151 *Let* $M = (m_{ij})_{i,j\in I}$ *be a Coxeter matrix of type* $\tilde{B}_2$ *with* $m_{12} = 4$, $m_{23} = 4$ *and* $m_{13} = 2$ *and* Φ *a root system of type* $\tilde{B}_2$ *with root base* $\Lambda = \{\alpha_i | i \in I\}$, *such that* $\bar{m}_{\alpha_1,\alpha_2} = \bar{m}_{\alpha_2,\alpha_3} = 4$ *and* $\bar{m}_{\alpha_1,\alpha_3} = 2$. *Suppose* *that* $((\Delta_{ij})_{\{i,j\}\in E(M)}, (c_{ij})_{\{i,j\}\in E(M)} (\beta_{ijk})_{\{i,j\}\{j,k\}\in E(M)})$ *is a Moufang foundation of type* M *such that for every* $\{i,j\} \in E(M)$, $(U_{\alpha_{ij}^k})_{\alpha_{ij}^k \in \Phi_{\alpha_i,\alpha_j}}$ *defines a root group system for* Δ_{ij}. *If* Δ_{12} *is a unitary quadrangle* $Q(E,q,k,\sigma)$ *and* Δ_{23} *are unitary quadrangle* $Q(E',q',k',\sigma')$ *and the Moufang foundation is integrable one of the following possibilities occurs :*

(i) $\mathcal{M}_{R_2(c_12)}(\Delta_{12})$ *and* $\mathcal{M}_{R_2(c_{23})}(\Delta_{23})$ *are both point rows and one of the following subcases occurs :*
(i.a) β_{123} *induces a field anti-isomorphism from* k *to* k'.
(i.b) k *and* k' *are generalized quaternion algebras with standard involutions* σ *and* σ *and* β_{123} *defines a field isomorphism from* k *to* k',

(ii) $\mathcal{M}_{R_2(c_{12})}(\Delta_{12})$ *and* $\mathcal{M}_{R_2(c_{23})}(\Delta_{23})$ *are both line pencils,* k *and* k' *are generalized quaternion algebras with standard involutions* σ *and* σ', $dim(E) = dim(E') = 4$ *and* $\mathcal{M}_{R_2(c_{12})}(\Delta_{12}) \cong \mathcal{P}(Z(k)) \cong \mathcal{M}_{R_2(c_{23})}(\Delta_{23}) \cong \mathcal{P}(Z(k'))$.

If Δ_{12} *is a hermitian quadrangle* $Q(E,q,k,\sigma)$ *and* Δ_{23} *is a unitary quadrangle* $Q(E',q',k',\sigma')$ *one of the following possibilities occurs :*

(i) $\mathcal{M}_{R_2(c_{12})}(\Delta_{12})$ *is a point row,* $\mathcal{M}_{R_2(c_{23})}(\Delta_{23})$ *is a line pencil,* k' *is a generalized quaternion algebra with standard involution* σ' *and* $dim(E') = 4$ *such that* $k \cong Z(k')$,

(ii) $\mathcal{M}_{R_2(c_{12})}(\Delta_{12})$ *is a line pencil,* $\mathcal{M}_{R_2(c_{23})}(\Delta_{23})$ *is a line pencil,* k' *is a generalized quaternion algebra with standard involution* σ' *such that* $Fix(\sigma) \cong Z(k')$,

(iii) $\mathcal{M}_{R_2(c_{12})}(\Delta_{12})$ *and* $\mathcal{M}_{R_2(c_{23})}(\Delta_{23})$ *are both line pencils,* k' *is a generalized quaternion algebra with standard involution* σ' *such that* $k \cong Z(k')$ *and* $dim(E) = dim(E') = 4$

.

Without loss of generality we can assume using Corollary 3.12.3 and the results from section 3.12.2 that q is a $(\sigma,-1)$-quadratic form such that $1 \in Tr(\sigma)$ and similarly that q' is a $(\sigma',-1)$-quadratic form with $1 \in Tr(\sigma')$.

Choose a coordinatization of $Q(E,q,k,\sigma)$ associated to the decomposition $E = e_{-2}k \oplus e_{-1}k \oplus E_0 \oplus e_1k \oplus e_2k$ with labelling set $R_{0,1} = \{(e_0, e_1) \in E_0 \times k | q(e_0) + e_1 = 0\}$. Choose similarly a coordinatization for $Q(E', q', k', \sigma')$ associated to the decomposition $E' = e'_{-2}k' \oplus e'_{-1}k' \oplus E'_0 \oplus e'_1k' \oplus e'_2k'$ with labelling set $R'_{0,1} = \{(e'_0, e'_1) \in E'_0 \times k' | q'(e'_0) + e'_1 = 0\}$. Let B_0 be an ordered base of E_0 and B'_0 be an ordered base of E'_0. For the rest of this proof we will use the conventions and notations from paragraphs 3.5.4 and 3.8.2 and denote β_{123} shortly as β.

First case : Δ_{12} and Δ_{23} are unitary quadrangles.
Four possibilities occur.

1. $\mathcal{M}_{R_2(c_{12})}$ and $\mathcal{M}_{R_2(c_{23})}$ are both point rows.
Without loss of generality we can assume that in this case $\alpha^1_{12} = \{(\infty)$, $[(0,0)]$, $((0,0),0)$, $[(0,0),0,(0,0)]$, $(0,(0,0),0)$ $\}$, $\alpha^2_{12} = \{[0,(0,0)]$, $(0,(0,0),0)$, $[(0,0),0,(0,0)]$, $((0,0),0)$, $[(0,0)]\}$, $\alpha^1_{23} = \{(\infty)$, $[(0,0)]$, $((0,0),0)$, $[(0,0),$ $0,(0,0)]$, $(0,(0,0),0)$ $\}$, α^2_{23} $\{[0,(0,0)]$, $(0,(0,0),0)$, $[(0,0),0,(0,0)]$, $((0,0),0)$, $[(0,0)]\}$, $\mathcal{M}_{R_2(c_{12})}(\Delta_{12}) = \Gamma([(0,0)])$ and $\mathcal{M}_{R_2(c_{23})}$ $(\Delta_{23}) = \Gamma([(0,0)])$. Remark that by these assumptions β imply that β induces a bijection (also denoted by β) from k to k' if we set :

$$\beta(((0,0),v)) = ((0,0),\beta(v)),\ \forall v \in k.$$

Without loss of generality we can assume that $\beta((1)) = 1$. As mentioned in section 3.8.2 the bijections γ from $\mathcal{P}(k)$ to $\mathcal{M}_{\Gamma([(0,0)])}$ $Q(E,q,k,\sigma)$ and γ' from $\mathcal{P}(k')$ to $\mathcal{M}_{\Gamma([(0,0)])}$ with :

$$\begin{aligned} \gamma((v)) &= ((0,0),v),\ \forall v \in k \\ \gamma((\infty)) &= (\infty) \end{aligned}$$

and :

$$\begin{aligned} \gamma'((v')) &= ((0,0),v'),\ \forall v' \in k' \\ \gamma'((\infty)) &= (\infty) \end{aligned}$$

define a Moufang set isomorphisms from $\mathcal{P}(k)$ to $\mathcal{M}_{\Gamma([(0,0)])}Q(E,q,k,\sigma)$ and from $\mathcal{P}(k')$ to $\mathcal{M}_{\Gamma([(0,0)])}$ $Q(E',q',k',\sigma')$. Proposition 124 yields then that β defines a (anti)-isomorphism from k to k'.
Let $\theta \in Tr(\sigma)$ and consider the automorphism $s_{[0,(0,\theta)]}s_{[0,(0,1)]}$. Call it this g_θ.

One easily checks that g_θ has as matrix representation with respect to the ordered base $\{e_{-2},\, e_{-1},\, B_0,\, e_1,\, e_2\}$:

$$\begin{pmatrix} -\theta & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & I_{|B_0|} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -\theta^{-1} \end{pmatrix}$$

As :

$$\begin{aligned} g_\theta((0,0),v) &= ((0,0),-\theta v),\ \forall v \in k \\ g_\theta(\infty) &= (\infty) \end{aligned}$$

we see that $g_\theta \in \langle\ U_{-\alpha^1_{12}}\, U_{\alpha^1_{12}}\ \rangle \cap Stab(\mathcal{M}_{R_2(c_{12})}\ \Delta_{12})$.
In a complete analogous way we define for $\theta' \in Tr(\sigma')$ the automorphism $g_{\theta'}$ $= s_{[0,(0,\theta')]}\ s_{[0,(0,1)]}$ of $QU(E',q',k',\sigma')$ such that :

$$\begin{aligned} g_{\theta'}((0,0),v') &= ((0,0),-\theta' v'),\ \forall v' \in k' \\ g_{\theta'}(\infty) &= (\infty). \end{aligned}$$

We see that $g_{\theta'} \in \langle U_{-\alpha^1_{23}} U_{\alpha^1_{23}}\ \rangle \cap\ Stab(\mathcal{M}_{R_2(c_{23})}(\Delta_{23})$. Equation (4.3) implies that for $\theta \in Tr(\sigma)$ and $\theta' \in Tr(\sigma)$:

$$[g_\theta, \beta^{-1} g_{\theta'} \beta]((0,0),v) = ((0,0),v),\ \forall v \in k.$$

A brief calculation gives :

$$\begin{aligned} g_\theta \beta g_{\theta'} \beta^{-1}((0,0),v) &= ((0,0),\theta\beta^{-1}(\theta'\beta(v)) \\ \beta^{-1} g_{\theta'} \beta g_\theta((0,0),v) &= ((0,0),\beta^{-1}(\theta'\beta(\theta v)). \end{aligned}$$

Hence formula (4.3) is in this case equivalent to :

$$\theta\beta^{-1}(\theta'\beta(v)) = \beta^{-1}(\theta'\beta(\theta v)),\ \forall v \in k,\ \forall \theta, Tr(\sigma),\ \forall \theta' \in Tr(\sigma').$$

Two cases occur :

(a) β is an anti-isomorphism.

In this case we find $\theta\beta^{-1}(\theta'\beta(x)) = \beta^{-1}(\theta'\beta(\theta x))$ and equation (4.3) is satisfied.
(b) β is an isomorphism.
In this case the equation (4.3) is equivalent to requirement that :

$$\theta\beta^{-1}(\theta') = \beta^{-1}(\theta')\theta, \ \forall\theta \in Tr(\sigma), \theta' \in Tr(\sigma'). \tag{4.4}$$

Suppose that $Tr(\sigma) \nsubseteq Z(k)$. Lemma 8.13 in [29] and Lemma 47 imply that $Tr(\sigma)$ generates k as a ring and $Tr(\sigma')$ generates k' as a ring. But then the equation (4.4) implies that $Z(k) = k$, and $Z(k') = k'$ a contradiction. Hence by Lemma 8.13. in [29] the only possibility left is that both k and k' are generalized quaternion algebra's with standard involutions σ and σ'.

2. $\mathcal{M}_{R_2(c_{12})}(\Delta_{12})$ is a point row and $\mathcal{M}_{R_2(c_{23})}(\Delta_{23})$ is a line pencil.
Without loss of generality we can assume that in this case $\alpha_{12}^1 = \{[0,(0,0)], (0,(0,0),0), [(0,0),0,(0,0)], ((0,0),0), [(0,0)]\}$, $\alpha_{12}^2 = \{(\infty), [(0,0)], ((0,0),0), [(0,0),0,(0,0)], (0,(0,0),0)\}$, $\alpha_{23}^1 = \{(\infty), [(0,0)], ((0,0),0), [(0,0),0,(0,0)], (0,(0,0),0)\}$, $\alpha_{23}^2 = \{[0,(0,0)], (0,(0,0),0), ((0,0),0), [(0,0)], (0,(0,0),0)\}$ $R_2(c_{12}) = \Gamma([(0,0)])$ and $R_2(c_{23}) = \Gamma((0))$. In section 3.8.2 we saw that $\mathcal{M}_{\Gamma[(0,0)]}(Q(E,q,k,\sigma) \cong \mathcal{P}(k)$ and the proof of Lemma 99 shows that $\mathcal{M}_{\Gamma((0))}$ $(Q(E',q',k',\sigma')$ is isomorphic to $\mathcal{M}(V',q',k',\sigma')$ with $V' = e'_{-2}k' \oplus E'_0 \oplus e'_2k'$. As Proposition 125 shows that $\mathcal{P}(k)$ cannot be isomorphic to the unitary Moufang set $\mathcal{M}(V',k',q',\sigma')$ we see that this cannot occur.

3. $\mathcal{M}_{R_2(c_{12})}(\Delta_{12})$ is a line pencil and $\mathcal{M}_{R_2(c_{23})}(\Delta_{23})$ is a point row.
A similar proof as for when $\mathcal{M}_{R_2(c_{12})}(\Delta_{12})$ is a point row and $\mathcal{M}_{R_2(c_{23})}(\Delta_{23})$ is a line pencil shows that this cannot occur.

4. $\mathcal{M}_{R_2(c_{12})}(\Delta_{12})$ and $\mathcal{M}_{R_2(c_{23})}$ (Δ_{23}) are both line pencils.
Without loss of generality we can assume that in this case $\alpha_{12}^1 = \{[0,(0,0)], (0,(0,0),0), [(0,0),0,(0,0)], ((0,0),0), [(0,0)]\}$, $\alpha_{12}^2 = \{(\infty), [(0,0)], ((0,0),0), [(0,0),0,(0,0)], (0,(0,0),0)\}$, $\alpha_{23}^1 = \{[0,(0,0)], (0,(0,0),0), [(0,0),0,(0,0)], ((0,0),0), [(0,0)]\}$, $\alpha_{23}^2 = \{(\infty), [(0,0)], ((0,0),0), [(0,0),0,(0,0)], (0,(0,0),0)\}$. Without loss of generality we can assume that $\beta[0,(0,1)] = [0,(0,1)]$.
Denote for $z \in k$ the automorphism $s_{((0,0),z)}s_{((0,0),1)}$ as h_z. Using the coordinatization one calculates :

$$\begin{aligned} h_z([0,(v_0,v_1)]) &= [0,(-v_0z^\sigma, zv_1z^\sigma)], \ \forall(v_0,v_1) \in R_{0,1}, \\ h_z([\infty]) &= [\infty]. \end{aligned}$$

Hence $h_z \in \langle U_{-\alpha^1_{12}}, U_{\alpha^1_{12}}\rangle \cap Stab(\mathcal{M}_{R_2(c_{12})}(\Delta_{12}))$. In a complete similar way we denote for each $z' \in k'$ the automorphism $s_{((0,0),z')}\ s_{((0,0),1)}$ of $Q(E', q', k', \sigma')$ by $h_{z'}$.
We find :

$$\begin{aligned} h_{z'}([0,(v'_0,v'_1)]) &= [0,(-v'_0 z', -z' v'_1 z')],\ \forall (v'_0,v'_1) \in R'_{0,1}, \\ h_{z'}([\infty]) &= [\infty]. \end{aligned}$$

Hence we find that $h_{z'} \in \langle U_{-\alpha^1_{23}}, U_{\alpha^1_{23}}\ \rangle \cap Stab(\mathcal{M}_{R_2(c_{23})}(\Delta_{23}))$.
As β defines a Moufang set isomorphism we find that $\beta\{[0,(v_0,v_1)]\ |(v_0,v_1) \in Z(R_{0,1},\oplus)\} = \{[0,(v'_0,v'_1)]\ |(v'_0,v'_1) \in Z(R'_{0,1},\oplus)\ \}$. Denote $L = Z(R_{0,1},\oplus) \cap\{0\}\ \times k$ and similarly $L' = Z(R'_{0,1},\oplus) \cap\{0\}\ \times k'$. As for every $z_1 \in L$ the vector $z_0 \in V_0$ such that $q(z_0) = -z_1$ is uniquely determined by z_1 we see that β defines a bijection from L to L' (also denoted by β) and a bijection from $Rad(f)$ to $Rad(f')$ (also denoted by β) if we set :

$$\beta[0,(z_0,z_1)] = [0,(\beta(z_0),\beta(z_1))],\ \forall (z_0,z_1) \in Z(R_{0,1},\oplus).$$

If $z_1 \in L$ we consider $z_0 \in V_0$ such that $(z_0,z_1) \in R_{0,1}$ and denote the automorphism $s_{[0,(z_0,z_1)]}\ s_{[0,(0,1)]}$ of $Q(E,q,k,\sigma)$ as h_{z_1}. Using the descriptions of the s_x as described in section 3.13 shows that :

$$h_{z_1}([0,(v_0,v_1)]) = [0,(-v_0 z_1, z_1 v_1 z_1)].$$

By construction we have that $h_{z_1} \in \langle\ U_{-\alpha^1_{12}}, U_{\alpha^1_{12}}\ \rangle \cap Stab(\mathcal{M}_{R_2(c_{12})}(\Delta_{12})$. Moreover applying β to the explicit formula for h_{z_1} shows that for $z_1 \in L$

$$\begin{aligned} \beta([0,(-v_0 z_1, z_1 v_1 z_1)]) = [0,(-\beta(v_0)\beta(z_1), \beta(z_1)\beta(v_1)\beta(z_1))], \\ \forall (v_0,v_1) \in Z(R_{0,1},\oplus). \end{aligned} \tag{4.5}$$

Similarly one defines for $z'_1 \in L'$ the automorphism $h_{z'_1}$ as $s_{[0,(z'_0,z'_1)]}\ s_{[0,(0,1)]}$ with $(z'_0,z'_1) \in R'_{0,1}$. It follows that $h_{z'_1} \in \langle U_{-\alpha^1_{23}}, U_{\alpha^1_{23}}\ \rangle \cap Stab(\mathcal{M}_{R_2(c_{23})}(\Delta_{23})$. Let $z \in L$, $z' \in L'$. Formula (4.3) yields that :

$$[h_z, \beta^{-1}h_{z'}\beta]([0,(v_0,v_1)]) = [0,(v_0,v_1)],\ \forall (v_0,v_1) \in R_{0,1},\ \forall z \in L,\ \forall z' \in L'.$$

Or equivalently :

$$h_z\beta^{-1}h_{z'}\beta([0,(v_0,v_1)]) = \beta^{-1}h_{z'}\beta h_z([0,(v_0,v_1)]),\ \forall (v_0,v_1) \in R_{0,1}.$$

We calculate using formula (4.5) for $(a_0, a_1) \in Z(R_{0,1}, \oplus)$:

$$\begin{aligned} h_z\beta^{-1}h_{z'}\beta([0,(a_0,a_1)]) &= h_z\beta^{-1}([0,(-\beta(a_0)z', z'\beta(a_1)z')]) \\ &= [0,(a_0\beta^{-1}(z')z, z\beta^{-1}(z')a_1\beta^{-1}(z')z)] \end{aligned}$$

and by similarly we find :

$$\beta^{-1}h_{z'}\beta h_z([0,(a_0,a_1)]) = [0,(a_0 z\beta^{-1}(z'), \beta^{-1}(z')za_1z\beta^{-1}(z'))].$$

This means that formula (4.3) yields the following condition :

$$z\beta^{-1}(z')a_1\beta^{-1}(z')z = \beta^{-1}(z')za_1z\beta^{-1}(z'),\ \forall a_1, z \in L,\ \forall z' \in L'. \qquad (4.6)$$

But as $\beta^{-1}(L') = L$ this shows if the Moufang foundation is integrable the following should hold :

$$z_1z_2\theta z_2z_1 = z_2z_1\theta z_1z_2,\ \forall z_1, z_2, \theta \in L. \qquad (4.7)$$

If we set $\theta = 1$ it follows that :

$$z_1^{-1}z_2^{-1}z_1z_2 = (z_1z_2z_1^{-1}z_2^{-1})^{-1}$$

and equation (4.7) becomes :

$$[z_1, z_2]\theta[z_1, z_2]^{-1} = \theta,\ \forall z_1, z_2 \in L. \qquad (4.8)$$

Suppose that $Tr(\sigma) \not\subset Z(k)$. Then we know by Lemma 8.13 and Lemma 47 that as L contains $Tr(\sigma)$ it generates k as a ring and by the same reasoning it follows that L' generates k' as a ring. But then equation (4.6) shows that $[z_1, z_2] \in Z(k)\ \forall\ z_1,\ z_2 \in L$. By Lemma 49 we see that k is a generalized quaternion algebra with standard involution σ. Hence $Tr(\sigma)$ $= Z(k)$ a contradiction with the assumptions. The only possibility left is that $Tr(\sigma) \subset Z(k)$. But then Lemma 8.13 in [29] shows that k is a generalized quaternion algebra with standard involution σ.
Similar arguments show that k' is a generalized quaternion algebra with standard involution σ'.
As a next step we show that $Rad(f) = 0$ and $Rad(f') = 0$.
Suppose $Rad(f) \neq 0$. Then there exists a $\bar{x} \in L \setminus Tr(\sigma)$. Consider $L_{\bar{x}}$ $Z(k)(\bar{x})$. Suppose that there exists a $\lambda_0 \in k$ with $\lambda_0^\sigma\bar{x}\lambda_0 \notin L_{\bar{x}}$. Let $\lambda \in k$. As $\lambda_0^\sigma\bar{x}\lambda_0 \in L$ equation (4.8) implies that $[\bar{x}, \lambda^\sigma\bar{x}\lambda]$ commutes with $\bar{x}$ and $\lambda_0^\sigma\bar{x}\lambda_0$.

The choice of λ_0 implies that $\bar{x}$ and $\lambda_0^\sigma \bar{x}\ \lambda_0$ generate k seen as a $Z(k)$-algebra. But then it follows that $[\bar{x}, \lambda^\sigma \bar{x} \lambda] \in Z(k)$ and we find :

$$\lambda^\sigma \bar{x} \lambda \in L_{\bar{x}},\ \forall \lambda \in k.$$

The same reasoning as the one used to prove Lemma 123 leads to a contradiction. Hence $Rad(f) = \{0\}$. By similar arguments we find $Rad(f') = \{0\}$. As a next step we show that $V_0 = 0$ and $V_0' = 0$. Using Proposition 129 we see that β is induced by a semi-linear transformation φ with associated field isomorphism α such that :

$$\begin{aligned} \beta[0,(v_0,v_1)] &= [0,(\varphi(v_0), v_1^\alpha)],\ \forall (v_0,v_1) \in R_{0,1} \\ \lambda^{\sigma\alpha} &= \lambda^{\alpha\sigma'}. \end{aligned}$$

We reconsider for $z \in k$ and $z' \in k'$ the transformations h_z and $h_{z'}$. By formula (4.3) we know that

$$[h_z, \beta^{-1} h_{z'} \beta][0,(v_0,v_1)] = [0,(v_0,v_1)],\ \forall (v_0,v_1) \in R_{0,1}.$$

This leads to :

$$h_z \beta^{-1} h_{z'} \beta[0,(v_0,v_1)] = \beta^{-1} h_{z'} \beta h_z[0,(v_0,v_1)],\ \forall (v_0,v_1) \in R_{0,1}.$$

Using φ and α we have :

$$\begin{aligned} h_z \beta^{-1} h_{z'} \beta([0,(v_0,v_1)]) &= h_z \beta^{-1}[0,(-\varphi(v_0) z'^{\sigma'}, z' v_1^\alpha z'^{\sigma'})] \\ &= h_z([0,(-v_0 z'^{\sigma'\alpha^{-1}}, z'^{\alpha^{-1}} v_1 z'^{\sigma'\alpha^{-1}})]) \\ &= [0,(v_0 z'^{\sigma'\alpha^{-1}} z^\sigma, z z'^{\alpha^{-1}} v_1 z'^{\sigma'\alpha^{-1}} z^\sigma. \end{aligned}$$

Similarly :

$$\begin{aligned} \beta^{-1} h_{z'} \beta h_z([0,(v_0,v_1)]) &= \beta^{-1} h_{z'}([0,(-\varphi(v_0) z^{\sigma\alpha}, z^\alpha v_1^\alpha z^{\sigma\alpha})]) \\ &= \beta^{-1}([0,(-\varphi(v_0) z^{\sigma\alpha} z'^{\sigma'}, z' z^\alpha v_1^\alpha z^{\sigma\alpha} z'^{\sigma'})]) \\ &= [0,(v_0 z^\sigma z'^{\sigma'\alpha^{-1}}, z'^{\alpha^{-1}} z v_1 z^\sigma z'^{\sigma'\alpha^{-1}})]. \end{aligned}$$

If $V_0 \neq$ we can choose a $v_0 \neq 0 \in V_0$ and the above equation shows that :

$$z^\sigma z'^{\sigma'\alpha^{-1}} = z'^{\sigma'\alpha^{-1}} z^\sigma,\ \forall z \in k,\ \forall z' \in k'.$$

As $\sigma'\alpha^{-1}$ defines a field anti-isomorphism from k' to k this equation yields that $z^\sigma \in Z(k)$, $\forall z \in k$, hence $Z(k) = k$ a contradiction. Thus we find $V_0 = 0$. In a completely similar way one deduces that also $V_0' = 0$. But then Lemma 123 shows that $\mathcal{M}_{R_2(c_{12})}(\Delta_{12}) \cong \mathcal{P}(Z(k))$ and $\mathcal{M}_{R_2(c_{23})}(\Delta_{23}) \cong \mathcal{P}(Z(k'))$ and by Proposition 124 we find $Z(k) \cong Z(k')$.

<u>Second case</u> : Δ_{12} is hermitian and Δ_{34} is unitary :
Let $\Delta_{12} = Q(E, q, k, \sigma)$ and $\Delta_{23} = Q(E', q', k', \sigma')$. We distinguish as for the first case between four possibilities.
1. $\mathcal{M}_{R_2(c_{12})}(\Delta_{12})$ and and $\mathcal{M}_{R_2(c_{23})}(\Delta_{23})$ are both point rows.
This would imply that $\mathcal{P}(k) \cong \mathcal{P}(k')$ and hence by Proposition 124 that $k \cong k'$, a contradiction. This situation can thus no occur.

2. $\mathcal{M}_{R_2(c_{12})}(\Delta_{12})$ is a point row and $\mathcal{M}_{R_2(c_{23})}(\Delta_{23})$ is a line pencil.
Proposition 126 shows that k', is a generalized quaternion algebra with standard involution σ' and $dim(E') = 4$.

3. $\mathcal{M}_{R_2(c_{12})}(\Delta_{12})$ is a line pencil and $\mathcal{M}_{R_2(c_{23})}(\Delta_{23})$ is a point row.
Proposition 128 shows that $dim(E) = dim(E') = 4$, k' is a generalized quaternion algebra with standard involution σ' and $Fix(\sigma) \cong Z(k')$

4. $\mathcal{M}_{R_2(c_{12})}(\Delta_{12})$ and $\mathcal{M}_{R_2(c_{23})}(\Delta_{23})$ are both line pencils.
Proposition 128 implies that then $dim(E') = 6$, k' is a generalized quaternion algebra with standard involution σ'.
Suppose $dim(E) > 5$. Then we find by Propositon **??** that $dim(E') = 6$. Moreover we have that $Rad(f') = 0$ if f' is the form associated to q'. Using the property we can follow the proof from above in the case where Δ_{12} and Δ_{23} where both unitary and $\mathcal{M}_{R_2(c_{12})}(\Delta_{12})$ and $\mathcal{M}_{R_2(c_{23})}(\Delta_{23})$ are both line pencils. This then leads to $V_0' = 0$ a contradiction. Hence $dim(E) = 4$ and as in this case $\mathcal{M}_{R_2(c_{12})}(\Delta_{12}) \cong \mathcal{P}(Fix(\sigma))$ Proposition 126 implies that $dim(E') = 4$. □

4.4 Existence in $\tilde{B}_2$ case

In this section we will give a list of integrable Moufang foundations of type $\tilde{B}_2$ where the quadrangles involved are the ones described in Chapter 3. We adopt the notations as introduced in section 4.3.1. When working with these quadrangles we use the coordinatizations as introduced in Chapter 3. This means that for quadrangles the form $Q(E,q,k,\sigma)$ we fix a coordinatization associated to a decomposition $E = e_{-2}k \oplus e_{-1}k \oplus V_0 \oplus e_1 k \oplus e_2 k$ with labelling set $R_{0,1}$ as described in Chapter 3 section 3.5.4. For symplectic quadrangles $W(k)$ and indifferent quadrangles $Q(k,k';l,l')$ we use the coordinatization as described in sections 3.5.2 and 3.7 in Chapter 3. Moreover for quadrangles of the form $Q(E,q,k,\sigma)$, $\sigma \neq 1$ we will assume that q is a $(\sigma,-1)$ quadratic form. In view of Lemma 92 this will not put any restrictions. To make the list of integrable Moufang foundations we use the following strategy. We start with a classical or indifferent Moufang set $(X,(U_x)_{x\in X})$. Subsequently we make a list of all possible Moufang foundations $((\Delta_{ij})_{\{i,j\}\in E(M)}$, $(c_{ij})_{\{i,j\}\in E(M)}$, $(\beta_{ijk})_{\{i,j\}\{j,k\}\in E(M)})$ of type $\tilde{B}_2$ for which $\mathcal{M}_{R_2(c_{12})}(\Delta_{12})$ and $\mathcal{M}_{R_2(c_{23})}(\Delta_{23})$ are isomorphic to $(X,(U_x)_{x\in X})$. For every such Moufang foundation we investigate if condition (Ind) is satisfied on $\mathcal{M}_{R_2(c_{12})}(\Delta_{12})$ and $\mathcal{M}_{R_2(c_{23})}(\Delta_{23})$. If this is the case we know by Lemma 145 that there is up to isomorphism one Moufang foundation involving Δ_{12} and Δ_{23} such that $\mathcal{M}_{R_2(\Delta_{12})}(\Delta_{12})$ and $\mathcal{M}_{R_2(\Delta_{23})}(\Delta_{23})$ are isomorphic to $(X,(U_x)_{x\in X})$. This means that if we find a Moufang foundation $((\Delta_{ij})_{\{i,j\}\in E(M)}$, $(c_{ij})_{\{i,j\}\in E(M)}$, $(\tilde{\beta}_{ijk})_{\{i,j\}\{j,k\}\in E(M)})$ which is integrable then the original Moufang foundation $((\Delta_{ij})_{\{i,j\}\in E(M)}$,$(c_{ij})_{\{i,j\}\in E(M)}$, $(\beta_{ijk})_{\{i,j\}\{j,k\}\in E(M)})$ is also integrable. To construct integrable Moufang foundations we rely on the theory developed by B. Mühlherr and H. Van Maldeghem as exposed in [22] and [23].

Definition 152 Let $((\Delta_{ij})_{\{i,j\}\in E(M)}, (c_{ij})_{\{i,j\}\in E(M)}, (\beta_{ijk})_{\{i,j\}\{j,k\}\in E(M)})$ be a Moufang foundation of type $\tilde{B}_2$. If $\mathcal{M}_{R_2(c_{12})}(\Delta_{12})$ and $\mathcal{M}_{R_2(c_{23})}(\Delta_{23})$ are both point rows we will speak about a *gluing of type PP*. If $\mathcal{M}_{R_2(c_{12})}(\Delta_{12})$ and $\mathcal{M}_{R_2(c_{23})}(\Delta_{23})$ are both line pencils we will speak about a *gluing of type LL*. If $\mathcal{M}_{R_2(c_{12})}(\Delta_{12})$ is a point row and $\mathcal{M}_{R_2(c_{23})}(\Delta_{23})$ is a line pencil we will speak about a *gluing of type LP*. Finally if $\mathcal{M}_{R_2(c_{12})}(\Delta_{12})$ is a line pencil and $\mathcal{M}_{R_2(c_{23})}(\Delta_{23})$ is a point row we will speak about a *gluing of type PL*.

If Δ_{12} is $W(\bar{k})$ we make the following conventions.
If $\mathcal{M}_{R_2(c_{12})}(\Delta_{12})$ is point row we assume that it is $\mathcal{M}_{\Gamma([0])}(W(k))$.

If $\mathcal{M}_{R_2(c_{12})}(\Delta_{12})$ is a line pencil we assume that it equals $\mathcal{M}_{\Gamma((0))}(W(k))$.
If Δ_{12} is a generalized quadrangle $Q(E,q,k,\sigma)$ we make the following conventions.
If $\mathcal{M}_{R_2(c_{12})}(Q(E,q,k,\sigma)$ is a point row we assume it is $\mathcal{M}_{\Gamma([(0,0)])}(Q(E,q,k,\sigma))$.
If $\mathcal{M}_{R_2(c_{12})}(Q(E,q,k,\sigma)$ is a line pencil we assume it equals $\mathcal{M}_{\Gamma((0))}(Q(E,q,k,\sigma))$.
Finally for $\Delta_{12} = Q(k,k';l,l')$ we make the following conventions.
If $\mathcal{M}_{R_2(c_{12})}(\Delta_{12})$ is point row we assume that it is $\mathcal{M}_{\Gamma([0])}(Q(k,k';l,l')$.
If $\mathcal{M}_{R_2(c_{12})}(\Delta_{12})$ is a line pencil we assume that it equals $\mathcal{M}_{\Gamma((0))}(Q(k,k';l,l')$.
We make the same conventions for Δ_{23}.

Troughout the list we will always start with a Moufang foundation $((\Delta_{ij})_{\{i,j\}\in E(M)}, (c_{ij})_{\{i,j\}\in E(M)}, (\tilde{\beta}_{ijk})_{\{i,j\}\{j,k\}\in E(M)})$ of type $\tilde{B}_2$ that the Δ_{ij} are classical or indifferent quadrangles. The conventions made above on $\mathcal{M}_{R_2(c_{12})}$ and $\mathcal{M}_{R_2(c_{23})}$ imply that we need not explicitely to know c_{12} and c_{23}. To simplify notations we will therefore denote $((\Delta_{ij})_{\{i,j\}\in E(M)}, (c_{ij})_{\{i,j\}\in E(M)}, (\tilde{\beta}_{ijk})_{\{i,j\}\{j,k\}\in E(M)})$ in the sequel of the section as $((\Delta_{ij})_{\{i,j\}\in E(M)}, (\tilde{\beta}_{ijk})_{\{i,j\}\{j,k\}\in E(M)})$ or even as F.
We start this section by giving some usefull Propositions concerning integrability.

Proposition 153 *Every Moufang foundation F $((Q(E,q,k,\sigma), Q(E', q', k', \sigma'),\beta_{123})$ of type PP such that if $Z(k) \neq k$, β_{123} induces an anti-isomorphism from k to k' is integrable. In particular every Moufang foundation $((W(k), Q(E',q',k',\sigma'),\beta_{123})$ of type LP is integrable, every Moufang foundation $((W(k), W(k'),\beta_{123})$ of type LL is integrable and if $char(k) = 2$ every Moufang foundation $(W(k), Q(E, q, k, \sigma), \beta_{123})$ of type PP is integrable.*

The conventions on $Q(E,q,k,\sigma)$ and $Q(E',q',k',\sigma')$ show that β_{123} defines a Moufang set isomorphism from $\mathcal{M}_{\Gamma([(0,0)])}$ $Q(E,q,k,\sigma)$ to $\mathcal{M}_{\Gamma([(0,0)])}Q(E',q',k',\sigma')$. Without loss of generality we will assume that q and q' are $(\sigma,1)$-quadratic forms and $\beta_{123}((0,0),1) = ((0,0),1)$. If this is not the case we might have to consider for $c \in k$ and $c' \in k'$ such that the quadrangles $Q(E,cq,k,\sigma^c)$ with $\lambda^{\sigma^c} = c\lambda^{\sigma}c^{-1}$, $\forall\lambda \in k$ and $Q(E',c'q',k',\sigma'^{c'})$ with $\lambda'^{\sigma'^{c'}} = c'\ \lambda'^{\sigma'}\ c'^{-1}$, $\forall\lambda' \in k'$ are $(\sigma^c,1)$ and $(\sigma'^{c'},1)$-quadratic and $\beta_{123}((0,0),1) = ((0,0),1)$. Clearly the map φ_1 which induces the identity on points an lines defines an isomorphism from $Q(E,q,k,\sigma)$ to $Q(E,cq,k,\sigma)$ and a similar bijection defines an isomorphism from $Q(E',q',k',\sigma')$ to $Q(E',c'q',k',\sigma')$.

Set $\varphi = (\varphi_1, \varphi_2)$. As $F \cong \varphi(F)$ we can therefore consider $\varphi(F)$. By proposition 124 we know that there exists a (anti)-isomorphism α from k to k' such that :

$$\beta_{123}(((0,0),x)) = ((0,0),x^\alpha),\ \forall x \in k.$$

Let $B'_0 = \{{e'_0}^j\ j \in J\}$ be a base of V'_0. Define the generalized quadrangle $Q(\bar{E}$,$\bar{q}$,k,$\bar{\sigma})$ in the following way. If α is an isomorphism we set $\bar{E} = \bar{e}_{-2}k \oplus \bar{e}_{-1}k\ \bar{V}_0 \oplus \bar{e}_1 k \oplus \bar{e}_2 \bar{k}$ where $\bar{V}_0$ is the right k-vector space spanned by B'_0. If α defines an automorphism we set $\bar{E} = \bar{e}_{-2}k^{opp} \oplus e_{-1}k^{opp} \oplus \bar{V}_0 \oplus \bar{e}_1\ k^{opp} \oplus e_2 k^{opp}$ where $\bar{V}_0$ is the right k^{opp}-vector space spanned by B'_0. Define the (anti)-isomorphism γ from k' to k by $\gamma = \sigma\circ\ \alpha^{-1} \circ \sigma'$.
Then E' is clearly isomorphic to $\bar{E}$ under φ_γ if we set :

$$\begin{aligned}&\varphi_\alpha(e'_{-2}x'_{-2} + e'_{-1}x'_{-1} + \textstyle\sum_j {e'_0}^j {v'_0}^j + e'_1 x'_1 + e'_2 x'_2)\\ &= \bar{e}_{-2}{x'_{-2}}^\gamma + \bar{e}_{-1}{x'_1}^\gamma + \textstyle\sum_j {e'_0}^j({v'_0}^j)^\gamma + \bar{e}_1 {x'_1}^\gamma + \bar{e}_2 {x'_2}^\gamma\end{aligned}$$

where x'_i, ${v'_0}^j \in k'$.
Define $\bar{\sigma}$ by $x^{\bar{\sigma}} = x^{\gamma^{-1}\sigma'\gamma}$, $\forall x \in k$.
The forms $\bar{g}$, $\bar{f}$ and $\bar{q}$ are given by :

$$\begin{aligned}\bar{g}(\bar{x},\bar{y}) &= (g'(\varphi_\gamma^{-1}(\bar{x}),\varphi_\gamma^{-1}(\bar{y})))^\gamma,\ \forall \bar{x},\bar{y} \in \bar{E}\\ \bar{f}(\bar{x},\bar{y}) &= (f'(\varphi_\gamma^{-1}(\bar{x}),\varphi_\gamma^{-1}(\bar{y})))^\gamma,\ \forall \bar{x},\bar{y} \in \bar{E}\\ \bar{q}(\bar{x}) &= \bar{g}(\bar{x},\bar{x}) + k_{\bar{\sigma},1}.\end{aligned}$$

By construction it follows that $\bar{q}$ defines a $(\bar{\sigma},1)$-quadratic form on $\bar{E}$ of Witt index 2. This means that we can consider the quadrangle $Q(\bar{E}$,$\bar{q}$,k,$\bar{\sigma})$. Using Theorem 133 it is clear that φ_γ induces an isomorphism from $Q(E', q', k', \sigma')$ to $Q(\bar{E}$,$\bar{q}$,$\bar{k}$,$\bar{\sigma})$. In the sequel we will work with the coordinatization of $Q(\bar{E}, \bar{q}$, k,$\bar{\sigma})$ associated to the decomposition $\bar{E} = \bar{e}_{-2}k \oplus \bar{e}_{-1} \oplus \bar{V}_0 \oplus \bar{e}_1 k \oplus \bar{e}_2\ k$. Let $\varphi = (Id, \varphi_\gamma)$. Then the Moufang foundation $\varphi(F)$ consists of the Moufang foundation $(Q(E,q,k,\sigma)$, $Q(\bar{E},\bar{q},k,\bar{\sigma})$, $\bar{\beta}_{123})$ of type PP where $\bar{\beta}_{123}$ is given by :

$$\bar{\beta}_{123}((0,0),x) = ((0,0),x^{\sigma\bar{\sigma}}).$$

We rephrase the proof of [23] of the integrability of $\varphi(F)$. Firstly we define the division ring $k(t;\sigma,\bar{\sigma})$ as follows. Its elements are given by the rational functions with variable t. Multiplication is given by :

$$\begin{aligned}x.y &= xy,\ \forall x,y \in k\\ x^\sigma t &= t x^{\bar{\sigma}},\ \forall x \in k.\end{aligned}$$

By putting $t^\sigma = t$ we extend σ to an involution of $k(t;\sigma,\bar\sigma)$. Using V_0 and $\bar V_0$ we set $W_0^1 = k(t;\sigma,\bar\sigma) \otimes V_0$ and $W_0^2 = k(t;\sigma,\bar\sigma) \otimes \bar V_0$ and $W_0 = W_0^1 \oplus W_0^2$. Then this means that every element of w_0 W_0 can be written in a unique way as $\sum_j v_0^j t^j + \sum_l \bar v_0^l\, t^l$ where $v_0^j \in V_0$ and $\bar v_0^l \in \bar V_0$. Extend the forms g and $\bar g$ to a form g^* on W_0 by :

$$\begin{aligned} &g^*(\textstyle\sum_j v_0^j t^j + \sum_i \bar v_0^i t^i, \sum_j w_0^j t^j, \sum_i \bar w_0^i t^i) \\ &= \textstyle\sum_{i,j} t^i g(v_0^i, w_0^j) + \sum_{i,j} t^i \bar g(\bar v_0^i, \bar w_0^j) t^j. \end{aligned}$$

If we set $q^*(w_0) = g^*(w_0) + k(t;\sigma,\bar\sigma)_{(\sigma,1)}$, it follows from the results in [23] that q^* defines an anisotropic $(\sigma,1)$-quadratic form on W_0. Set W $e_{-1}^* k(t;\sigma,\bar\sigma)$ $\oplus e_{-1}^*\ k(t;\sigma,\bar\sigma)\ \oplus W_0 \oplus e_1^*\ k(t;\sigma,\bar\sigma) \oplus e_2^*\ k(t;\sigma,\bar\sigma)$ and extend g^* to a form on W as follows : $(u = e_{-2}^* x_{-2} + e_{-1}^* x_{-1} + u_0 + e_1^* x_1 + e_2^*\, x_2$ and $w = e_{-2}^* y_{-2}$ $+ e_{-1}^* y_{-1} + w_0 + e_1^* y_1 + e_2^* y_2$ with $x_i,\ y_i \in k(t;\sigma,\bar\sigma)$ and $u_0,\ w_0 \in W_0)$

$$g^*(u,w) = x_{-2}^\sigma y_2 + x_2^\sigma y_{-2} + x_{-1}^\sigma y_1 + x_1^\sigma x_{-1} + g^*(u_0,w_0).$$

Using g^* we extend q^* to W by setting :

$$q^*(w) = g(w,w) + (k(t;\sigma,\bar\sigma)_{\sigma,1},\ \forall w \in W.$$

In this way we see that q^* defines a $(\sigma,1)$-quadratic form on W of Witt index 2. Hence we can consider the quadrangle $Q(W,q^*,k(t;\sigma,\bar\sigma),\sigma)$. It is proved in [23] that $Q(W,q^*,k(t;\sigma,\bar\sigma),\sigma)$ is the quadrangle at ∞ of an affine Moufang building (Δ, W, S, d) of type $\tilde B_2$ such that $\varphi(F) = (Q(E,q,k,\sigma), Q(\bar E, \bar q, k, \bar\sigma)$, $\bar\beta_{123}) \in MoFo(\Delta)$. As $\varphi(F)$ is isomorphic to F this proves that F is integrable.

Let F $(W(k), Q(E',q',k',\sigma'), \beta_{123})$ be a Moufang foundation of type LP. By Proposition 134 and Lemma 143 we find that F will be isomorphic to a Moufang foundation $\bar F = (QO(E,q,k), Q(E',q',k',\sigma'),\ \bar\beta_{123})$ of type PP where $QO(E,q,k)$ is the orthogonal quadrangle dually isomorphic to $W(k)$ as desribed in the proof of Proposition 134. By what we already proved we know that $\bar F$ is integrable proving the integrability of F.

The integrability of a Moufang foundation $(W(k), W(k'), \beta_{123})$ of type LL follows by similar arguments.

Finally let $char(k) = 2$. Then we know by Proposition 137 that $W(k)$ is isomorphic to an orthogonal quadrangle $QO(E,q,k)$. Hence every foundation $F = (W(k),\ (Q(E',q',k',\sigma'), \beta_{123})$ of type is isomorphic to a foundation $\bar F = (QO(E,q,k), Q(E',q',k',\sigma'),\ \bar\beta_{123}))$ of type PP. As we already proved that $\bar{F}$ is integrable we find that every Moufang foundation $(W(k), Q(E', q', k', \sigma'),\beta_{123}))$ is integrable. □

Proposition 154 *Let $char(k) \neq 2$ then every Moufang foundation $(W(k), Q(E', q', k', \sigma'),\beta_{123})$ of type LL is integrable.*

Let F be a foundation as in the proposition. By Proposition 126 we can assume without loss of generality that β_{123} induces a field isomorphism from k to k'. Consider the symplectic quadrangle $W(k')$. Then $W(k')$ is clearly isomorphic to $W(k)$. Let $\varphi_{\alpha^{-1}}$ be the isomorphism from $W(k')$ to $W(k)$ by applying α^{-1} to the coordinates of elements of $W(k')$. Put $\varphi = (\varphi_{\alpha^{-1}}, Id)$. Let $\bar{F}$ be the Moufang foundation $((W(k'),\ Q(E', q', k', \sigma'),\ Id))$. The construction of $\varphi^{\alpha^{-1}}$ implies that $\varphi(\bar{F}) = F$. As $\bar{F}$ is integrable by the results in [23] we find that F is integrable. □

Proposition 155 *Suppose k and k' are fields. Then every Moufang foundation $F = (Q(E, q, k, \sigma),Q(E', q', k', \sigma'),\beta_{123})$ of type LL such that $\mathcal{M}_{\Gamma((0))}(Q(E, q, k, \sigma))$ is not-commutative is integrable. Moreover also every Moufang foundation $(W(k), Q(E', q', k', \sigma'), \beta_{123})$ of type PL and every Moufang foundation $(W(k),W(k'),\beta_{123})$ of type PP is integrable.*

<u>proof</u> :

We start by reducing the three cases to one.
1. Suppose firstly that $F = (Q(E, q, k, \sigma),Q(E', q', k', \sigma'),\beta_{123})$. The conditions on $\mathcal{M}_{\Gamma((0))}(Q(E, q, k, \sigma))$ yield that by Proposition 147, $Q(E, q, k, \sigma)$ satsisfies condition (Ind) on $\Gamma((0))$, $Q(E', q', k', \sigma')$ satsisfies condition (Ind) on $\Gamma((0))$ and $Q(E', q', k', \sigma')$ is isomorphic to $Q(E, q, k, \sigma)$. Suppose that the isomorphism from $Q(E, q, k, \sigma)$ to $Q(E', q', k', \sigma')$ is given by φ_{23}. Put $\varphi = (Id, \varphi_{23})$. Then we find by construction that the foundation $((Q(E, q, k, \sigma),Q(E', q', k', \sigma'),\beta_{123}))$ is isomorphic to $\varphi(F) = ((Q(E, q, k, \sigma),Q(E, q, k, \sigma),\varphi_{23}^{-1}\ \beta_{123}))$. As $Q(E, q, k, \sigma))$ satisfies condition (Ind) on $\Gamma((0))$, $\varphi(F)$ is isomorphic to the Moufang foundation $\bar{F} = (Q(E, q, k, \sigma),Q(E, q, k, \sigma),Id))$.

2. $F = (W(k), Q(E', q', k', \sigma')$ of type PL.

Similar techniques as the ones used in the proof of Proposition 153 show that F is isomorphic to the foundation $F' = (W(k'), Q(E', q', k', \sigma')$ of type PL. By Proposition 134 we know that $W(k')$ is dually isomorphic to an orthogonal quadrangle $QO(\bar{E}', \bar{q}', k')$. Denote the duality from $W(k')$ to $QO(\bar{E}', \bar{q}', k')$ as γ_2. Put $\gamma = (Id, \gamma_2)$. Then one easily checks that $\bar{F} = \gamma(F') = (QO(\bar{E}', \bar{q}', k'),\ Q(E', q', k', \sigma'), Id)$.

3. If $F = (W(k), W(k'), \beta_{123})$ we know by Proposition 134 that $W(k)$ is dually isomorphic to an orthogonal quadrangle $QO(E_1, q_1, k)$ and similarly $W(k')$ is dually isomorphic to an orthogonal quadrangle $QO(E_1', q_1', k')$. Suppose that γ_1 is a duality from $W(k)$ to $QO\ (E_1, q_1, k)$ and γ_1' a duality from $W(k')$ to $QO\ (E_1', q_1', k')$. Put $\gamma = (\gamma_1, \gamma_2)$. Then we can replace the F by $\gamma(F) = (Q(E_1, q_1, k), Q(E_1', q_1', k')\ \gamma_1'\beta_{123}\gamma_1^{-1})$, a Moufang foundation of type LL. Proposition 146 implies that $QO(E_1, q_1, k)$ and $QO(E_1', q_1', k')$ satisfy condition (Ind) on their line pencils and that $QO(E_1, q_1, k) \cong QO(E_1', q_1', k')$. Suppose that the isomorphism from $QO(E_1, q_1, k)$ is given by ψ_{23}. Set $\psi = (Id, \psi_{23})$. The we find by that F is isomorphic to $\psi\gamma(F) = (QO(E_1, q_1, k), QO(E_1', q_1', k'), \psi_{23}^{-1}\gamma_1'\ \beta_{123}\gamma_1^{-1})$. As $QO(E_1, q_1, k)$ and $QO(E_1, q_1, k)$ satisfy condition (Ind) on their line pencils we find that also in this case F is isomorphic to the Moufang foundation $\bar{F} = (QO(E_1, q_1, k),\ QO(E_1, q_1, k), Id)$.

In all three cases we find that F is integrable if and only if $\bar{F}$ is integrable. We rephrase a proof of the integrability of $\bar{F}$ given in [23].
Without loss of generality we will assume here that q is a $(\sigma, 1)$-quadratic form. In view of Lemma 92 this puts no restrictions on the form. (If q is a (σ, ϵ)-quadratic form then we can find a constant $c \in k$ such that cq is a $(\sigma, 1)$-quadratic form. As the foundation $F = (Q(E, q, k, \sigma), Q(E, q, k, \sigma), Id))$ is isomorphic to $F_c = (Q(E, cq, k, \sigma), Q(E, cq, k, \sigma), Id))$ we can work with F_c instead of F.) Firstly we define the division ring $k(t; \sigma, 1)$ in the following way. The elements of $k(t; \sigma, 1)$ are the rational functions in the variable t where multiplication is given by :

$$\begin{aligned} a.b &= ab, \forall t \in k \\ a^\sigma t &= ta, \forall a \in k \end{aligned}$$

We extend σ to $k(t; \sigma, 1)$ by setting $t^\sigma = t$. In this way we construct a skew field $k(t; \sigma, 1)$ with involution σ.

Subsequently we choose a coordinatization of $Q(E, q, k, \sigma)$ associated to a decomposition $E = e_{-2}k \oplus e_{-1}k \oplus V_0 \oplus e_1 k \oplus e_2 k$ with labelling set $R_{0,1}$. Suppose that g is a σ-sesquilinear form such that $q(x) = g(x) + k_{\sigma,1}$, $\forall x \in E$. Let $W_0 = V_0 \otimes k(t)$. Then every element of W_0 can be written uniquely as a sum $\sum v_0^i t^i$, $v_0^i \in V_0$.
We extend g and q to forms on W_0 by setting :

$$\begin{aligned} g(\sum v_0^i t^i, \sum w_0^j t^j) &= t^i g(v_0^i, w_0^j) t^j \\ q(\sum v_0^i t^i) &= g(\sum v_0^i t^i, \sum v_0^j t^j) + (k(t;\sigma,1))_{(\sigma,1)} \end{aligned}$$

with $\sum v_0^i t^i$ and $\sum w_0^j t^j \in W_0$. Then one easily checks that in this way g defines a σ sesquilinear form and q a $(\sigma, 1)$-quadratic form on W_0. Moreover it is proved in [23] that q is anisotropic on W_0.
Put $W = e_{-2}k(t) \oplus e_{-1}k \oplus W_0 \oplus e_1 k(t) \oplus e_2 k(t)$. We extend g and q to W in the following way :

$$\begin{aligned} & g(e_{-2}x_{-2} + e_{-1}x_{-1} + \sum v_0^i t^i + e_1 x_1 + e_2 x_2 \\ = \ & x_{-2}^\sigma x_2 + x_{-1}^\sigma x_1 + g(\sum v_0^i t^i),\ x_i \in k(t;\sigma,1), v_0^i \in k \\ & q(x) \\ = \ & g(x) + (k(t;\sigma,1))_{(\sigma,1)},\ \forall x \in k \end{aligned}$$

By construction q defines in this way a $(\sigma, 1)$-quadratic form of with index 2. Therefore we can consider the quadrangle $Q(W, q, k(t; \sigma, 1), \sigma)$. It is proved in [23] that $Q(W, q, k(t; \sigma, 1), \sigma)$ is the quadrangle at ∞ of an affine Moufang building (Δ, W, S, d) of type $\tilde{B}_2$ with $\bar{F} = (Q(E, q, k, \sigma), Q(E, q, k, \sigma), Id))$ $\in MoFo(\Delta)$. This means that $\bar{F}$ is integrable and hence F is integrable.

Proposition 156 *Every Moufang foundation* $F = (Q(k, k'; l, l'), Q(\bar{k}, \bar{k}', ; \bar{l}, \bar{l}'), \beta_{123})$ *is integrable.*

proof :

Let F be a Moufang foundation as in the proposition. As every indifferent quadrangle is dually isomorphic to an indifferent quadrangle we can assume without loss of generality that F is or type PP. Proposition 131 implies that β_{123} induces a field isomorphism from k to $\bar{k}$ such that for some $\bar{c} \in \bar{k}$:

$$\bar{c}\beta_{123}(l) = \bar{l}.$$

Upon a possible re coordinatization of $Q(\bar{k},\bar{k}',\bar{l},\bar{l}')$ we can assume that $\bar{c} = 1$. Let $Q(k,k'';l',l'')$ be the indifferent quadrangle obtained by applying α^{-1} to the coordinates of $Q(\bar{k},\bar{k}';\bar{l},\bar{l}')$. Then $Q(k,k'';l,l'')$ is isomorphic to $Q(\bar{k},\bar{k}';\bar{l},\bar{l}')$ by applying α to the coordinates of $Q(k,k'';l,l'')$. Denote this isomorphism by φ_α. Put $\varphi = (Id,\varphi_\alpha)$. By construction it then follows that $\bar{F} = \varphi^{-1}(F) = (Q(k,k';l,l'),\ Q(k,k'';l,l''),\ Id))$. As $\bar{F}$ is integrable by the results of [23] we find that F is integrable. □

4.4.1 Case I : $\mathcal{M}_{R_2(c_{12})}(\Delta_{12}) \cong \mathcal{P}(\bar{k})$.

$Z(\bar{k}) = \bar{k}$

In this section we assume that $\mathcal{M}_{R_2(c_{12})}(\Delta_{12}) \cong \mathcal{P}(\bar{k})$ with $\bar{k}$ a field. Remark that if in this case Δ_{12} or Δ_{23} is an indifferent quadrangle $Q(k,k';l,l')$ we find by Proposition 125 that $l = k \cong \bar{k}$. But then Proposition 136 shows that $Q(k,k';l,l')$ is isomorphic to an orthogonal quadrangle. Therefore we will not explicitly consider the cases where Δ_{12} or Δ_{23} are indifferent quadrangles.

Case I.1 $\Delta_{12} = W(k)$ and $\Delta_{23} = W(k')$.

Without loss of generality we can assume that $\bar{k} = k$ in view of Proposition 124.

The gluing is of type PP.
By proposition 124 we know that β induces a field isomorphism from k to k'. As both $W(k)$ and $W(k')$ satisfy condition (Ind) on their point rows (cfr. Proposition 146) there is up to isomorphism only one Moufang foundation $((\Delta_{ij})_{\{i,j\}\in E(M)},(c_{ij})_{\{i,j\}\in E(M)},\ (\beta_{ijk})_{\{i,j\}\{j,k\}\in E(M)})$ of type PP involving $W(k)$ and $W(k')$. The integrability of F follows from Proposition 155.

The gluing is of type PL, LL or LP.
A similar reasoning as for a gluing of type PP shows that the Moufang foundation $((\Delta_{ij})_{\{i,j\}\in E(M)},(c_{ij})_{\{i,j\}\in E(M)},(\beta_{ijk})_{\{i,j\}\{j,k\}\in E(M)})$ is integrable.

Case I.2 $\Delta_{12} = W(k)$ and $\Delta_{23} = QO(E', q', k')$.

The gluing is of type PP.
Proposition 124 implies that β induces a field isomorphism from k to k'. The integrability follows from Propositions 153 and 154.

The gluing is of type LP.
By Proposition we know that β induces a field isomorphism from k to k'. The integrability of the Moufang foundation follows from Proposition 153.

The gluing is of type PL.
Proposition 127 implies that one of the following cases occurs :
$dim(E') = 5$, $k \cong k'$. By Proposition 134 we have that $QO(E', q', k', \sigma')$ is dually isomorphic to $W(k')$. Hence the Moufang foundation F is isomorphic to a Moufang foundation $(W(k), W(k'), \bar{\beta}_{123})$ of type PP. Proposition 155 implies thus that in this case F is integrable.
$dim(E') = 6$, $k \cong k''$, where k'' is the quadratic Galois extension of k' determined by $\mathcal{M}O(V', q', k')$. Proposition 135 shows that $QO(E', q', k')$ is dually isomorphic to $QH(E'', q'', k'', \sigma'')$. Hence F is isomorphic to a Moufang foundation $\bar{F} = (W(k), QH(E'', q'', k'', \sigma'')$ of type PP. Propositions 153 and 154 yield the integrability of F.
$codim(Rad(f')) = 2$, there exists a constant $c' \in k'$ such that the set $\{c'q'(w') \mid w' \in Rad(f')\}$ is isomorphic to k. Proposition 136 implies that $QO(E', q', k')$ is isomorphic to an indifferent quadrangle. Hence F is isomorphic to a foundation $\bar{F}$ involving two indifferent quadrangles and the integrability of F follows from Proposition 156.

The gluing is of type LL.
Proposition 127 implies that one of the following cases occurs :
$dim(E') = 5$, $k \cong k'$ and hence $QO(E', q', k')$ is dually isomorphic to $W(k')$. The integrability of F follows from Proposition 153.
$dim(E'') = 6$ and $k \cong k''$ with k'' the quadratic Galois extension determined by $QO(E', q', k')$ as described in Lemma 112. In this case $QO(E', q', k')$ is dually isomorphic to a hermitian Moufang set $QH(E''$, q'', k'', $\sigma'')$. The integrability of the Moufang foundation follows from

Proposition 153. $codim(Rad(f')) = 2$ and there exists a constant c' $\in k'$ such that the set $\{c'q'(w') \mid w' \in Rad(f')\}$ is a field isomorphic to k. Proposition 136 shows that $QO(E', q', k')$ is isomorphic to an indifferent quadrangle. The integrability of F follows from Proposition 156.

Case I.3 $\Delta_{12} = W(k)$ and $\Delta_{23} = QH(E', q', k', \sigma')$.

The gluing is of type PP.
The integrability of F follows from Propositions 153 and 154.

The gluing is of type LP.
For the a proof of the integrability of F we refer to Proposition 153

The gluing is of type PL.
Proposition 155 shows that F is integrable in this case.

The gluing is of type LL.
Proposition 128 implies that $dim(E') = 4$ and $k \cong Fix(\sigma')$. Proposition 135 implies that then $QH(E', q', k', \sigma')$ is dually isomorphic to an orthogonal quadrangle $QO(\bar{E}', \bar{q}', k')$. Hence F is isomorphic to a Moufang foundation $(W(k), QO(\bar{E}', \bar{q}', k'))$ of type LP. Hence Proposition 153 implies that F is integrable.

Case I.4 $\Delta_{12} = W(k)$ and $\Delta_{23} = QU(E', q', k', \sigma')$.
Remark that in this case only gluings of type PL and LL are possible.

The gluing is of type PL.
Proposition 125 implies that $dim(E') = 4$, k' is a generalized quaternion algebra with standard involution σ'. Proposition 138 implies that $QU(E', q', k', \sigma')$ is dually isomorphic to an orthogonal quadrangle $QO(E'', q'', k'')$. The integrability of the Moufang foundation follows thus from the Propositions 153. and 154.

The gluing is of type LL.
The same conclusions holds as for case of a gluing of type PL.

Case I.5 $\Delta_{12} = QO(E, q, k)$ and $\Delta_{23} = QO(E', q', k')$.

The gluing is of type PP.
By Proposition we know that β induces a field isomorphism from k to k' in this case.
For the integrability of F we refer to Proposition 153.

The gluing is of type PL.
Proposition 125 shows that one of the following cases occurs:
$dim(E') = 5$ and $k \cong k'$.
Proposition 134 implies that $QO(E', q', k')$ dually isomorphic to $W(k')$. We can therefore refer to case $I.2$.
$dim(E') = 6$ and $k \cong k''$ where k'' is the quadratic Galois extension of k' determined by $QO(E', q', k')$. As in this case $QO(E', q', k')$ is dually isomorphic to a hermitian quadrangle $QH(E'', q'', k'', \sigma'')$ the Moufang foundation F is isomorphic to a Moufang foundation $\bar{F} = (QO(E, q, k), QH(E'', q'', k'', \sigma''), \bar{\beta}_{123})$ of type PP. The integrability of F follows from Proposition 153.
$codim(Rad(f')) = 2$, and there exists a constant $c' \in k'$ such that the set $\{c'q'(w') \mid w' \in Rad(f')\}$ is a field isomorphic to k.
Proposition 136 implies that $QO(E', q', k')$ is isomorphic to an indifferent quadrangle $Q(k, k''; l, l'')$. Proposition 3.4.4 in [37] shows that $Q(k, k''; l, l'')$ is dually isomorphic to $Q(k'', (k')^2; l'', (l')^2)$. Hence by Proposition 136 we see that $QO(E', q', k')$ is dually isomorphic to an orthogonal quadrangle $Q(E'', q'', k'')$. This implies that F is isomorphic to a foundation $\bar{F}$ of type PP involving two orthogonal quadrangles. The integrability of F therefore follows from Propostion 153.

The gluing is of type LL.
In this we find using Proposition 127 that $codim(Rad(f)) = codim(Rad(f')) = 2$ and that there exist constants $c \in k$ and $c' \in k'$ such that $\{cq(w) \mid w \in Rad(f)\}$ is a field isomorphic to the field $\{c'q'(w') \mid w' \in Rad(f')\}$. Lemma 136 implies that both Δ_{12} and Δ_{23} are indifferent Moufang sets. The integrability of the Moufang foundation follows from Proposition 156.

The gluing is of type LP.
This case is can be derived from the case where we consider a gluing of type PL.

<u>Case I.6</u> $\Delta_{12} = QO(E, q, k)$ and $\Delta_{23} = QH(E', q', k')$.

The gluing is of type PP.
Proposition 124 yields $k \cong k'$.
We refer to case I.3.

The gluing is of type LP.
Propositions 125, 134 and 135 imply that there are three possibilities.
$dim(E) = 5$ and $QO(E, q, k)$ is dually isomorphic to $W(k)$. The integrability follows from Proposition 153.
$dim(E) = 6$ and $QO(E, q, k)$ is dually isomorphic to a hermitian quadrangle $QH(E'', q'', k, \sigma'')$. The integrability follows from Proposition 153.
$codim(Rad(f)) = 2$, there exists a constant $c \in k$ such that the set $\{cq(w) \mid w \in Rad(f)\}$ is a field isomorphic to k'. Proposition 136 implies that $QO(E, q, k)$ is isomorphic to an indifferent quadrangle. Hence we refer to case I.5 for a discussion on the integrability of F.

The gluing is of type PL or LL.
Propositions 128 and 135imply that $dim(E') = 4$ and that $QH(E', q', k', \sigma')$ is dually isomorphic to an orthogonal quadrangle $QO(E'', q'', Fix(\sigma')$. Hence we can refer to Case I.5 for a discussion on the integrability of F.

Case I.7 $\Delta_{12} = QO(E, q, k)$ and $\Delta_{23} = QU(E', q', k', \sigma')$.
Proposition 125 shows that only gluings of type PL and LL are possible. But then Proposition 125 shows that $dim(E') = 4$ and by Proposition 138 we have that $QU(E', q', k', \sigma')$ is dually isomorphic to an orthogonal quadrangle $QO(E'', q'', Z(k'))$. This means that the Moufang foundation F is isomorphic to a Moufang foundation $\bar{F} = (QO(E, q, k), QO(E'', q'', Z(k')), \bar{\beta}_{123})$. For a discussion on the integrability of F we therefore refer to case I.5.

Case I.8 $\Delta_{12} = QH(E, q, k, \sigma)$ and $QH(E', q', k', \sigma')$.

The gluing is of type PP.
Proposition 124 implies $k \cong k'$.
The integrability of the Moufang foundation follows from Proposition 153.

The gluing is of type PL.
Propositions 125 and 135 imply that $dim(E') = 4$, $k \cong Fix(\sigma')$ and

that $QH(E', q', k', \sigma')$ is dually isomorphic to an orthogonal quadrangle $QO(E'', q'', Fix(\sigma'))$. The integrability of F then follows from Proposition 153.

The gluing is of type LL.
Propositions 125 and 135 shows that $dim(E) = dim(E') = 4$ and $Fix(\sigma) \cong Fix(\sigma')$, $QH(E, q, k, \sigma)$ is dually isomorphic to an orthogonal quadrangle $QO(\bar{E},\bar{q},Fix(\sigma))$ and $QH(E', q', k', \sigma')$ is dually isomorphic to an orthogonal quadrangle $QO(\bar{E}', \bar{q}', Fix(\sigma'))$. Hence F is isomorphic to a Moufang foundation $\bar{F}$ of type PP involving $QO(\bar{E}, \bar{q}, Fix(\sigma))$ and $QO(\bar{E}',\bar{q}',Fix(\sigma'))$. The integrability of the Moufang foundation follows from Propositions 153 and 154.

The gluing is of type LP.
We refer to the case of a gluing of type PL.

<u>Case I.9</u> $\Delta_{12} = QH(E, q, k, \sigma)$ and $\Delta_{23} = QU(E', q', k', \sigma')$.
Remark that Proposition 124 implies that in this case only gluings of type PL and LL are possible. Propositions 125 implies that $dim(E') = 4$. Therefore we have that $QU(E',q',k',\sigma'))$ is dually isomorphic to an orthogonal quadrangle $QO(E'',q'',Z(k'))$ by Proposition 138. For the integrability of the Moufang foundation we can thus refer to case I.6.

<u>Case I.10</u> $\Delta_{12} = QU(E, q, k, \sigma)$ and $\Delta_{23} = QU(E', q', k', \sigma')$.
Remark that by Proposition 126 only a gluing of type LL is possible such that $dim(E) = dim(E') = 4$. Proposition 138 shows that $QU(E,q,k,\sigma))$ and $QU(E', q',k', \sigma'$ are both dually isomorphic to orthogonal quadrangles. For the integrability of the Moufang foundation we therefore refer to case I.5.

$Z(k) \neq k$

<u>Case I.11</u> $\Delta_{12} = QO(E, q, k)$ and $\Delta_{23} = QU(E', q', k', \sigma')$ Remark that Proposition 125 implies that only a gluing of type LP is possble. For the integrability of F we refer to the results proved in [23].

<u>Case I.12</u> $\Delta_{12} = QU(E, q, k, \sigma)$ and $\Delta_{23} = QU(E', q', k', \sigma')$.

The gluing is of type PP.
Remark that Theorem 151 implies that if k is not a generalized quaternion algebra with standard involution β_{123} induces a (anti)-isomorphism from k to k'. The integrability of the Moufang foundation follows from Proposition 153.

The gluing is of type PL.
Proposition 130 shows that this case cannot occur unless $Z(k) = k$ a contradiction.

The gluing is of type LL.
Theorem 151 yields that the foundation can only be integrated if $dim(E)$ $= dim(E') = 4$, k is a generalized quaternion algebra with standard involution and σ' is a generalized quaternion algebra with standard involution σ'. But then we find that $QU(E, q, k, \sigma)$ is dually isomorphic to an orthogonal quadrangle $QO(\bar{E}, \bar{q}, Z(k))$ and similarly $QU(E', q', k', \sigma')$ is dually isomorphic to an orthogonal quadrangle $QO(\bar{E}',\bar{q}', Z(k'))$ such that $dim(\bar{E}) = dim(E') = 8$. Thus F is isomorphic to a foundation $\bar{F} = (QO(\bar{E}, \bar{q}, \bar{k}),QO(\bar{E}',\bar{q}', \bar{k}'),\bar{\beta}_{123}))$ of type PP. The integrability of F follows therefore from Proposition 153.

4.4.2 Case II : $\mathcal{M}_{R_2(c_{12})}(\Delta_{12}) \cong \mathcal{M}O(\bar{V}, \bar{q}, \bar{k})$

To avoid unneccesary work we will avoid to rephrase the cases where we a priori know that $\mathcal{M}_{R_2(c_{12})}(\Delta_{12})$ is a projective Moufang set.
Case II.1 $\Delta_{12} = QO(E, q, k)$ and $\Delta_{23} = QO(E', q', k')$.

The gluing is of type PP, PL or LP.
As in this case $\mathcal{M}_{R_2(c_{12})}(\Delta_{12})$ is isomorphic to projective Moufang set we refer to case I.5.

The gluing is of type LL and $\mathcal{M}_{R_2(c_{12})}(\Delta_{12})$ is not commutative. Proposition 147 implies that $QO(E, q, k)$ is isomorphic to $QO(E', q', k')$ in this case. The integrability of the foundation follows from Proposition 155.

The gluing is of type LL and $\mathcal{M}_{R_2(c_{12})}(\Delta_{12})$ is commutative.
Remark that in this case Lemma 120 implies that $codim(Rad(f)) = 2$ and $codim(Rad(f')) = 2$. Hence Δ_{12} and Δ_{23} can be seen as indifferent

quadrangles by Proposition 136. This means that the Moufang foundation F is isomorphic to a Moufang foundation of type LL involving two mixed quadrangles. The integrability of F follows from Proposition 156.

Case II.2 $\Delta_{12} = QO(E,q,k)$ and $\Delta_{23}= QU(E',q',k',\sigma')$.

The gluing is of type PP, PL or LP we refer to case I.7

The gluing is of type LL and $char(k) \neq 2$.
Proposition 127 implies that $dim(E') = 4$.
We distinguish between two subcases :
σ' is the standard involution. Proposition 138 shows that $QU(E',q',k',\sigma')$ is dually isomorphic to an orthogonal quadrangle $QO(\bar{E}',\bar{q}',Z(k'))$. This means that F is isomorphic to a Moufang foundation $\bar{F}$ $= (QO(E,q,k), Q(\bar{E}',\bar{q}',Z(k')), \bar{\beta}_{123})$ of type LP. For a discussion on the integrability of F we can therefore refer to case I.5.
σ' is not the standard involution. Lemma 115 implies that $\mathcal{M}_{R_2(c_{23})}(\Delta_{23})$ is isomorphic to a non-commutative orthogonal Moufang set $\mathcal{M}O(\bar{E}',\bar{q}',Z(k'))$ with $dim(\bar{E}') = 7$. By Proposition 127 we find that $dim(E) = 7$ and that q and q' are proportional up to an isomorphism α from k to $Z(k')$. The integrability of F follows from the results in [23].

The gluing is of type LL and $char(k) = 2$.
By Proposition 127 we find that in this case $codim(Rad(f')) = 2$, and k' is a generalized quaternion algebra. This means that $\mathcal{M}_{R_2(c_{12})}(\Delta_{12})$ is an extended polar line. If $dim(E') = 4$ the integrability of F can be proved as above in the characteristic non 2 case. If $dim(E') > 4$ we the integrability of F follows from the results in [23].

Case II.3 $\Delta_{12} = QO(E,q,k)$ and $\Delta_{23} = Q(k',k'';l',l'')$.

The gluing is of type PL or PP.
We refer to case I.

The gluing is of type LP.
Proposition 131 implies that $codim(Rad(f)) = 2$. Hence $QO(E,q,k)$

is an indifferent quadrangle by Proposition 136. The integrability of F therefore follows from Proposition 156.

CaseII.4 $\Delta_{12} = Q(k, \tilde{k}; l, \tilde{l})$ and $\Delta_{23} = Q(k', \tilde{k}'; l', \tilde{l}')$.
The integrability of F follows in this case from Proposition 156.

4.4.3 Case III : $\mathcal{M}_{R_2(c_{12})}(\Delta_{12}) \cong \mathcal{M}H(\bar{V}, \bar{q}, \bar{k}, \bar{\sigma})$.

To avoid unnecessary work we will avoid to rephrase cases where we know a priori that $\mathcal{M}_{R_2(c_{12})}(\Delta_{12})$ is a projective Moufang set.

Case III.1 : $\Delta_{12} = QH(E, q, k, \sigma)$ and $\Delta_{23} = QH(E', q', k', \sigma')$.

The gluing is of type PP, PL or LP.
We refer to case I.8.

The gluing is of type LL.
If $dim(E) = 4$, Proposition 128 shows that $dim(E') = 4$. By Proposition 135 we have that $QH(E, q, k, \sigma)$ is dually isomorphic to an orthogonal quadrangle $QO(\bar{E}, \bar{q}, Fix(\sigma))$ and $QH(E', q', k', \sigma')$ is dually isomorphic to an orthogonal quadrangle $QO(\bar{E}', \bar{q}', Fix(\sigma'))$. This means that F is isomorphic to a foundation $\bar{F} = (QO\ (\bar{E},\ \bar{q},\ Fix(\sigma)),\ QO\ (\bar{E}',\ \bar{q}',\ Fix(\sigma')),\ \bar{\beta}_{123}))$ of type PP. The integrability of F follows then from Proposition 153.
If $dim(E) > 5$ Proposition 128 implies that $dim(E') > 5$ and both $\mathcal{M}_{R_2(c_{12})}(\Delta_{12})$ and $\mathcal{M}_{R_{23}(c_{23})}(\Delta_{23})$ have non commutative root groups. Remark that in this case $QH(E, q, k, \sigma)$ is isomorphic to $QH(E', q', k', \sigma')$ by Proposition 147. The integrability of F follows from Proposition 155.
Remark that the case where $dim(V) = dim(V') = 5$ is still left open.

Case III.2 : $\Delta_{12} = QH(E, q, k, \sigma)$ and $\Delta_{23} = QU(E', q', k', \sigma')$.

The gluing is of type PP, PL or LP.
We refer to I.9.

The gluing is of type LL.
Theorem 151 implies that in this case $dim(E) = dim(E') = 4$, k' is a generalized quaternion algebra with standard involution σ' such that

$Z(k') \cong Fix(\sigma)$. But then Proposition 135 implies that $QH(E,q,k,\sigma)$ is dually isomorphic to an orthogonal quadrangle $QO(\bar{E},\bar{q},Fix(\sigma))$ and by Proposition 138 we find that $QU(E',q',k',\sigma')$ is dually isomorphic to an orthogonal quadrangle $QO(\bar{E}',\bar{q}',\ Z(k'))$. This means that the foundation F is isomorphic to a foundation $\bar{F}=\ (QO(\bar{E},\bar{q},Fix(\sigma)),$ $QO(\bar{E}',\bar{q}',Z(k'),\sigma')$ of type PP. The integrability of F then follows from Proposition 153

4.4.4 Case IV : $\mathcal{M}_{R_2(c_{12})}(\Delta_{12}) \cong \mathcal{M}U(\bar{V},\bar{q},\bar{k},\bar{\sigma})$

By Theorem 151 we can refer to case I for a discussion on the integrability of F in this case.

4.4.5 Case V : $\mathcal{M}_{R_2(c_{12})}(\Delta_{12}) \cong \mathcal{P}(\bar{l};\bar{k})$

As earlier mentioned we will not consider the cases where we a priori know that $\mathcal{M}_{R_2(c_{12})}(\Delta_{12})$ is isomorphic to a projective Moufang set.

Case V.1 $\Delta_{12} = QO(E,q,k)$ and $\Delta_{23} = QO(E',q',k')$.

The gluing is of type PP, PL or LP. We refer to case I.5

The gluing is of type LL.
Propositions 131 and 136 show that $QO(E,q,k)$ and $QO(E',q',k')$ are isomorphic to indifferent quadrangles. Thus F is isomorphic to a foundation of type LL involving two indifferent quadrangles. The integrability of F then follows from Proposition 156.

Case V.2 $\Delta_{12} = QO(E,q,k)$ and $\Delta_{23} = Q(k',k'';l',l'')$.

The gluing is of type PP, PL. We refer to case I.5.

The gluing is of type LP.
Proposition 131 and 136 yield that $QO(E,q,k)$ is isomorphic to an indifferent quadrangle. Hence F is isomorhpic to a Moufang foundation of type LL involving two indifferent quadrangles. For the integrability of F we refer to Proposition 156.

The gluing is of type LL.
By Proposition 3.4.4 in [?] we know that $Q(k', k''; l', l'')$ is dually isomorphic to the indifferent quadrangle $Q(k'', (k')^2; l'', (l')^2)$. Hence F is isomorphic to a Moufang foundation $\bar{F}$ $(QO(E, q, k), Q(k'', (k')^2; l'', (l')^2)$ of type LP. The integrability of F follows then as above.

Case V.3 $\Delta_{12} = Q(k, \bar{k}; l, \bar{l})$ and $\Delta_{23} = Q(k', \bar{k}'; l', \bar{l}')$. The integrability of F follows from Proposition 156.

4.5 Non-existence in 443 case

Definition 157 Let M_{443} be the Coxeter matrix defined over the set $I = \{1, 2, 3\}$ with $m_{12} = m_{23} = 4$, $m_{13} = 3$. A Coxeter matrix M isomorphic to M_{443} is said to be of *type* 443. A *root system of type* 443 is defined as a root system of type M_{443}, a *building of type* 443 is a building of type M_{443} and a *Moufang foundation of type* 443 is defined as a Moufang foundation of type M_{443}.

In this section we will assume that for the (σ, ϵ)-quadratic forms with $\sigma \neq 1$ involved $\epsilon = -1$. Using Lemma 92 and section 3.12.3 we see that this does not put any restrictions on the forms.
Using similar reasonings as for the $\tilde{B}_2$ case we will show the non-existence of certain Moufang foundations of type 443.

Theorem 158 *Let $M = (m_{ij})_{i,j \in I}$ with $I = \{1, 2, 3\}$ a Coxeter matrix of type 443, Φ a root system of type 443 with root base $\Lambda = \{\alpha_i \mid i \in I\}$ such that $\bar{m}_{\alpha_1, \alpha_2} = \bar{m}_{\alpha_2, \alpha_3} = 4$ and $\bar{m}_{\alpha_1, \alpha_3} = 3$. Suppose that $((\Delta_{ij})_{\{i,j\} \in E(M)}, (c_{ij})_{\{i,j\} \in E(M)}, (\beta_{ijk})_{\{i,j\}\{j,k\} \in E(M)})$ is a Moufang foundation of type 443 where the for every $\{i, j\} \in E(M)$ the system $(U_{\alpha_{ij}^k})_{\alpha_{ij}^k \in \Phi_{\alpha_i \alpha_j}}$ forms a root groups system for Δ_{ij}. Suppose Δ_{12} is a unitary quadrangle QU (E, q, k, σ), Δ_{23} is a unitary quadrangle the form QU (E', q', k', σ') and Δ_{13} is a Desarguesion projective plane defines over a division ring $\bar{k}$, $\mathcal{M}_{R_2(c_{12})}(Q(E, q, k, \sigma)$ and $\mathcal{M}_{R_2(c_{23})}(Q(E', q', k', \sigma')$ are both line pencils, $\mathcal{M}_{R_1(c_{13})}(\Pi)$ is a line pencil and $\mathcal{M}_{R_3(c_{13})}(\Pi)$ is a point row. Assume moreover that $\mathcal{M}_l(Q(E, q, k, \sigma))$ and $\mathcal{M}_l(Q(E', q', k', \sigma'))$ are Moufang sets with non-commutative root groups such that if k or k' is a generalized quaternion algebra with standard involution, $Rad(f) = 0$ or $Rad(f') = 0$.*
If then $((\Delta_{ij})_{\{i,j\} \in E(M)}, (c_{ij})_{\{i,j\} \in E(M)}, (\beta_{ijk})_{\{i,j\}\{j,k\} \in E(M)})$ is integrable one of the following possibilities occurs :

(i) β_{312} *induces an anti-isomorhpism vrom* k *to* k', k *is a generalized quaternion algebra with standard involution* σ *and* k' *is a generalized quaternion algebra with standard involution* σ',

(ii) β_{312} *induces an isomorphism from* k *to* k' *and both* k *and* k' *are generalized quaternion algebras.*

proof :

Choose a coordinatization of QU (E, q, k, σ) associated to the decomposition $e_{-2}k \oplus e_1 k \oplus E_0 \oplus e_1 k \oplus e_2$. Similarly we choose a coordinatization of QU (E', q', k', σ') associated to the decomposition $e'_{-2}k' \oplus e'_{-1}k' \oplus E'_0 \oplus e'_1 k' \oplus e'_2 k'$. The assumptions of the theorem imply that there exists a 3-dimensional right $\bar{k}$-vector space U such that $\Pi \cong PG(U)$. Denote the dual space of U by U^*, then U^* is a right $\bar{k}^{opp}$-vector space. Choose a base fixed base $\{u_1,\ u_2,\ u_3\}$ of U with dual base $\{u_1^*,\ u_2^*,\ u_3^*\}$. For the rest of the proof elements of U of the form $u_1 w_1 + u_2 w_3 + u_3 w_3$ will be written as the row (w_1, w_2, w_3) and similarly every element of U^* of the form $u_1^* w_1^* + u_2^* w_2^* + u_3^* w_3^*$ will be denoted by $(w_1^*, w_2^*, w_3^*)^*$. Call $\beta_{312} = \alpha_1$, $\beta_{132} = \alpha_3$ and $\beta_{123} = \alpha_2$. Then we can assume without loss of generality that : α_1 defines a Moufang set isomorphism from $\mathcal{M}_{\Gamma((1,0,0))}$ $(PG(E))$ to $\mathcal{M}_{\Gamma([(0,0)])}(Q(E, q, k, \sigma))$, α_2 defines a Moufang set isomorphism from $\mathcal{M}_{\Gamma((0))}$ $(Q(E, q, k, \sigma))$ to $\mathcal{M}_{\Gamma((0))}$ $(Q(E', q', k', \sigma'))$, α_3 defines a Moufang set isomorphism from $\mathcal{M}_{\Gamma(\langle(0,1,0)(1,0,0)\rangle)}$ $(PG(E))$ to $\mathcal{M}_{\Gamma([(0,0)])}(Q(E', q', k', \sigma'))$.
Let $\mathcal{P}(\bar{k})$ and $\mathcal{P}(\bar{k}^{opp})$ be projective lines defined over $\bar{k}$ and $\bar{k}^{opp}$. Choose canonical coordinatizations of both projective Moufang sets. The calculations in section 3.4 show that the bijection γ from $\mathcal{P}(k)$ to $\mathcal{M}_{\Gamma(\langle(1,0,0),(0,1,0)\rangle)}$ defined by :

$$\begin{aligned}\gamma((v)) &= \langle(v,1,0)\rangle,\ \forall v \in \bar{k}\\ \gamma((\infty)) &= \langle(1,0,0)\rangle\end{aligned}$$

determines a Moufang set isomorphism and similarly that the bijection γ^* from $\mathcal{P}(\bar{k}^{opp})$ to $\mathcal{M}_{\Gamma((1,0,0))}$ defined by :

$$\begin{aligned}\gamma^*((v^*)) &= \langle(0,v^*,1)^*\rangle\\ \gamma^*((\infty)) &= \langle(0,1,0)^*\rangle\end{aligned}$$

determines a Moufang set isomorphism.
To simplify notations we will identify in the sequel the point set of $\mathcal{P}(k)$ with

the point set of $\mathcal{M}_{\Gamma(\langle(1,0,0),(0,0,1)\rangle)}(\Pi)$ via γ and similarly identify the point set of $\mathcal{P}(\bar{k}^{opp})$ with the point set of $\mathcal{M}_{\Gamma((1,0,0))}(\Pi)$ via γ^*. Without loss of generality we can assume that $\alpha_1((1)) = ((0,0),1)$ and $\alpha_3((1)) = ((0,0),1)$. Using Theorem 124 we see that α_1 defines a field (anti)-isomorphism from $\bar{k}^{opp}$ to k which we also denote as α_1 and is defined by :

$$\alpha_1((\bar{v}^*)) = ((\bar{v}^*)^{\alpha_1}),\ \forall \bar{v}^* \in \bar{k}.$$

In a similar way α_3 defines a field (anti)-isomorphism from $\bar{k}$ to k' also denoted by α_3 and define by :

$$\alpha_3((\bar{v})) = (\bar{v}^{\alpha_3}),\ \forall \bar{v} \in \bar{k}.$$

Without loss of generality we can assume that $\alpha_2\ ([0,(0,1)]) = [0,(0,1)]$. In this case Theorem 129 implies that the Moufang set isomorphism α_2 from $\mathcal{M}_{\Gamma((0))}(QU(E,q,k,\sigma)$ to $\mathcal{M}_{\Gamma((0))}(QU(E',q',k',\sigma')$ sends $\{[0,(0,\theta)] | \theta \in Tr(\sigma)\}$ to $\{[0,(0,\theta')] | \theta' \in Tr(\sigma')\}$ such that :

$$\alpha_2([0,(0,\theta)]) = [0,(0,\theta^{\gamma}],$$

with γ a field isomorphism from k to k' satisfying :

$$\lambda^{\sigma\gamma} = \lambda^{\gamma\sigma'},\ \forall \lambda \in k.$$

This implies that if the Moufang foundation $((\Pi, QU(E,q,k,\sigma),$ $QU(E',q',k',\sigma'),(c_{ij})_{\{i,j\}\in E(M)},\ (\alpha_i)_{1\leq i\leq 3})$ is integrable also the Moufang foundation $((\Pi, QU(W,q,k,\sigma), QU(W',q',k',\sigma'), (c_{ij})_{\{i,j\}\in E(M)}, (\alpha_i)_{1\leq i\leq 3})$ will be integrable where $W = \langle e_{-2},\ e_{-1},\ e_1,\ e_2\rangle$ and $W' = \langle e'_{-2},\ e'_{-1},\ e'_1,\ e'_2\ \rangle$. Thus there exists a Moufang building (Δ, W, S, d) with root groups $(U_\alpha)_{\alpha\in\Phi}$ of type 443 such that $((\Pi, QU(W,q,k,\sigma), QU(W',q',k',\sigma'), (\alpha_i, 1 \leq i \leq 3))$ $\cong (R_{ij}(c_+),(c_{ij})_{\{i,j\}\in E(M)},\ (\beta_{ijk})_{\{i,j\},\{j,k\}\in E(M)})$ with $c_+ \in \Delta$, $c_{ij} = c_+$, β_{ijk} $= Id$, $\forall \{i,j\}$, $\{j,k\} \in E(M)$. Therefore we can reduce the situation to the case where $E_0 = 0$, $E'_0 = 0$ and α_2 induces a field isomorphism also denote by α_2 from k to k' such that :

$$\alpha_2([0,(0,\theta)]) = [0,(0,\theta^{\alpha_2})]\ \ \forall [0,(0,\theta)] \in \mathcal{M}_{\Gamma((0))}(QU(V,q,k,\sigma)) \qquad (4.9)$$

and :

$$\lambda^{\sigma\alpha_2} = \lambda^{\alpha_2\sigma'},\ \forall \lambda \in k. \qquad (4.10)$$

Let $(b) \in \mathcal{M}_{\Gamma(\langle(010)(100)\rangle)}(\Pi)$. Consider the automorphism $s_{(b)}s_{(1)}$. Clearly this defines an automorphism of Π with matrix representation with respect to the base $\{u_1, u_2, u_3\}$:

$$\begin{pmatrix} -b & 0 & 0 \\ 0 & -b^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We have that for $(x^*) \in \mathcal{M}_{\Gamma(100)}(PG(E))$, $s_{(b)}\ s_{(1)}\ ((x^*)) = (-b^{-1}x^*)$ and $s_{(b)}\ s_{(1)}\ (\infty) = (\infty)$. Clearly $s_{(b)}s_{(1)}$ defines an automorphism $h_{(b)}$ fixing an aparment in the Moufang building in which the Moufang foundation is integrated. Without loss of generality we can thus assume that h_b defines an automorphism of the Moufang foundation $((\Pi, QU(E,q,k,\sigma), QU(E',q',k',\sigma'),$ $(c_{ij})_{\{i,j\}\in E(M)}, (\alpha_i, 1 \leq i \leq))$.
This means in particular that h_b defines automorphisms $g_{\alpha_1(b)}$ of $QU(E,q,k,\sigma)$ and $g_{\alpha_3(b)}$ of $QU(E',q',k',\sigma')$. In particular $g_{\alpha_1(b)}$ will define an automorphism of $\mathcal{M}_{\Gamma((0))}$ $(QU(W,q,k,\sigma))$ and $g_{\alpha_3(b)}$ and automorphism of $\mathcal{M}_{\Gamma((0))}$ $(QU\ (W',\ q',\ k',\sigma'))$. Without loss of generality we can assume that the apartment given by $\{[\infty], (\infty), [(0,0)], ((0,0),0), [(0,0),0,(0,0)], (0,(0,0),0),$ $[0,(0,0)], (0)\}$ in $QU(E,q,k,\sigma)$ is fixed under $g_{\alpha_1(b)}$ and that the same apartment given in $QU(E',q',k',\sigma')$ is fixed by $g_{\alpha_3(b)}$.
By construction the automorphisms $g_{\alpha_1(b)}$ and $g_{\alpha_3(b)}$ are representations of the action of h_b on the whole building. If the Moufang foundation is integrable this implies that the actions of $g_{\alpha_1(b)}$ on $\mathcal{M}_{\Gamma((0))}(QU(W,q,k,\sigma))$ and of $g_{\alpha_3(b)}$ on $\mathcal{M}_{\Gamma((0))}(QU(V',q',k',\sigma'))$ should coïncide after identification via α_2. In other words :

$$g_{\alpha_1(b)}([0,(0,\theta)]) = \alpha_2^{-1} g_{\alpha_3(b)}\alpha_2([0,(0,\theta)])),\ \forall\theta \in Tr(\sigma).$$

Using formula (4.9) this gives :

$$g_{\alpha_1(b)}([0,(0,\theta)]) = \alpha_2^{-1} g_{\alpha_3(b)}([0,(0,\theta^{\alpha_2})]),\ \forall\theta \in Tr(\sigma). \tag{4.11}$$

We calculate the action of $g_{\alpha_3(b)}$ on $QU(E',q',k',\sigma')$. As h_b is the global action of $s_{(b)}s_{(1)}$ on the whole building and α_3 defines a Moufang set isomorphism we have that $g_{\alpha_3(b)} = s_{\alpha_3(b)}s_{\alpha_3(b)}$ where a matrix representation of $s_{\alpha_3(b)}$ with respect to the base $\{e_{-2}, e_{-1}, e_1, e_2\}$ is given by :

$$\begin{pmatrix} 0 & b^{\alpha_3} & 0 & 0 \\ -b^{\alpha_3-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & b^{\alpha_3\sigma'} \\ 0 & 0 & -b^{\alpha_3\sigma'-1} & \end{pmatrix}.$$

Thus $g_{\alpha_3(b)}$ has matrix representation :

$$\begin{pmatrix} b^{\alpha_3} & 0 & 0 & 0 \\ 0 & b^{\alpha_3 - 1} & 0 & 0 \\ 0 & 0 & b^{\alpha_3\sigma'} & 0 \\ 0 & 0 & 0 & b^{\alpha_3\sigma' - 1} \end{pmatrix}.$$

And thus for $[0,(0,\theta')] \in \mathcal{M}_{\Gamma([(0)])}\ QU(V',q',k',\sigma')$ we have :

$$g_{\alpha_3(b)}([0,(0,\theta')]) = [0,(0,b^{\alpha_3}\theta' b^{\alpha_3\sigma'})].$$

Remains to calculate the action of $g_{\alpha_1(b)}$ and translate formula (4.9). By Theorem 133 we know that $g_{\alpha_1(b)}$ is induced by a semi-linear transformation φ with associated field automorphism γ such that :

$$\begin{aligned} g_{\alpha_1(b)}\langle x\rangle &= \langle\varphi(x)\rangle,\ \forall\langle x\rangle \in QU(W,q,k,\sigma) \\ c(f(x,y))^{\alpha} &= f(\varphi(x),\varphi(y)),\ \forall x,y \in W \\ c(q(x))^{\alpha} &= q(\varphi(x)),\ \forall x \in W \end{aligned}$$

where $c \in k$ is a constant which satisfies

$$c\lambda^{\sigma\alpha}c^{-1} = \lambda^{\alpha\sigma},\ \forall\lambda \in k.$$

This means that with respect to the ordered base $\{e_{-2},\ e_{-1},\ e_1,\ e_2\}$, $g_{\alpha_1(b)}$ has matrix representation :

$$\begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{pmatrix}$$

with $\lambda_i \in k$ satisfying :

$$\begin{aligned} \lambda_1^{\sigma}\lambda_4 &= -c \\ \lambda_2^{\sigma}\lambda_3 &= c \end{aligned}$$

By construction we know that for $((0,0),x) \in \mathcal{M}_{\Gamma[(0,0)]}(QU(V,q,k,\sigma)$:

$$g_{\alpha_1(b)}(((0,0),x)) = ((0,0),(-b^{-1}x^{\alpha_1^{-1}})^{\alpha_1}),\ \forall x \in k.$$

Two cases occur :
First case : α_1 defines a field anti-isomorphism.

Then we have :

$$g_{\alpha_1(b)}(((0,0),x)) = ((0,0),(-xb^{-1\alpha_1}), \forall x \in k.$$

This means that $g_{\alpha_1(b)}$ defines a linear transformation with a matrix representation of the form :

$$\begin{pmatrix} z & 0 & 0 & 0 \\ 0 & zb^{\alpha_1} & 0 & 0 \\ 0 & 0 & ((zb^{\alpha_1})^\sigma)^{-1}c & 0 \\ 0 & 0 & 0 & -(z^{-1})^\sigma c \end{pmatrix}$$

with $z \in Z(k)$ and $c \in k$. As c satisfies $c\lambda^\sigma c^{-1} = \lambda^\sigma$, $\forall \lambda \in k$ we find that $c \in Z(k)$. Consequently $g_{\alpha_1(b)}$ acts on the Moufang set $\mathcal{M}_{\Gamma((0))}$ $(QU(W,q,k,\sigma))$ by :

$$g_{\alpha_1(b)}([0,(0,\theta)]) = [0,(0,-zz^\sigma c^{-1}\theta)], \forall [0,(0,\theta)] \in \mathcal{M}_{\Gamma((0))}(QU(V,q,k,\sigma)).$$

Using property (4.10) condition (4.11) thus becomes :

$$-zz^\sigma c^{-1}\theta = b^{\alpha_3\alpha_2^{-1}}\theta(b^{\alpha_3\alpha_2^{-1}})^\sigma, \forall \theta \in Tr(\sigma).$$

As $\alpha_3\alpha_2^{-1}$ defines a bijection from $\bar{k}$ to k this yields :

$$\theta^{-1}\lambda\theta\lambda^\sigma \in Z(k), \forall \lambda \in k, \forall \theta \in Tr(\sigma).$$

If we put $\theta = 1$ in this equation we get :

$$\lambda\lambda^\sigma \in Z(k), \forall \lambda \in k.$$

But then we have also that $(1+\lambda)(1+\lambda)^\sigma = 1 + \lambda\lambda^\sigma + (\lambda + \lambda^\sigma) \in Z(k)$. Therefore we find that $Tr(\sigma) \subset Z(k)$. Lemma 8.13 in [29] implies that k is a generalized quaternion algebra and σ its standard involution. By symmetric arguments one finds that k' is a generalized quaternion algebra with standard involution σ'. As $\bar{k}$ is isomorphic to k and k' it is also a generalized quaternion algebra.

Second case : α_1 is an induces an isomorphism from $\bar{k}$ to k.

By similar arguments as for the first case one deduces that $g_{\alpha_1(b)}$ has a matrix representation with respect to the ordered base $\{e_{-2}, e_{-1}, e_1, e_2\}$ of the form :

$$\begin{pmatrix} zb^{-1\alpha_1} & 0 & 0 & 0 \\ 0 & z & 0 & 0 \\ 0 & 0 & (z^{-1})^\sigma c & 0 \\ 0 & 0 & 0 & -(z^{-1}b^{\alpha_1})^\sigma c^\sigma \end{pmatrix},$$

with $z \in Z(k)$, and $c \in Z(k)$ as c satisfies $c\lambda^\sigma c^{-1} = \lambda^\sigma$. For $[0,(0,\theta)] \in \mathcal{M}_{\Gamma((0))}(QU(V,q,k,\sigma))$ we thus find :

$$g_{\alpha_1(b)}([0,(0,\theta)]) = [0,(0,-zz^\sigma(c^{-1})^\sigma(b^{-1})^{\alpha_1}\theta(b^{-1})^{\alpha_1\sigma})],\ \forall\theta \in Tr(\sigma).$$

In this case condition (4.11) thus becomes :

$$-zz^\sigma(c^{-1})^\sigma b^{-1\alpha_1}\theta b^{-1\alpha_1\sigma} = b^{\alpha_3\alpha_2^{-1}}\theta(b^{\alpha_3\alpha_2^{-1}})^\sigma,\ \forall\theta \in Tr(\sigma).$$

Thus we find for every $b \in \bar{k}$ a $z_b \in Z(k)$ such that :

$$z_b b^{-1\alpha_1}\theta b^{-1\alpha_1\sigma} = b^{\alpha_3\alpha_2^{-1}}\theta(b^{\alpha_3\alpha_2^{-1}})^\sigma,\ \forall\theta \in Tr(\sigma), \tag{4.12}$$

Inserting $\theta = 1$ in this equation gives

$$b^{\alpha_3\alpha_2^{-1}\sigma}b^{\alpha_1\sigma} = (b^{\alpha_1}b^{\alpha_3\alpha_2^{-1}})^{-1}z_b$$

and (4.12) becomes :

$$\theta = b^{\alpha_1}b^{\alpha_3\alpha_2^{-1}}\theta(b^{\alpha_1}b^{\alpha_3\alpha_2^{-1}}),\ \forall b \in \bar{k}.,\quad (6)$$

Suppose that if k is a generalized quaternion algebra, σ is not its standard involution. By Lemma 47 we know that in this case $Tr(\sigma)$ generates k as a ring. But then (6) yields :

$$(b^{\alpha_1})(b^{\alpha_1})^{\alpha_1^{-1}\alpha_3\alpha_2^{-1}} \in Z(k),\ \forall b \in \bar{k}.$$

By assumption α_1 defines an isomorphism from $\bar{k}$ to k and $\alpha_1^{-1}\ \alpha_3\ \alpha_2^{-1}$ an (anti)-automorphism of k. This means that if we put $\delta = \alpha_1^{-1}\alpha_3\ \alpha_2^{-1}$ then :

$$\lambda\lambda^\delta \in Z(k),\ \forall\lambda \in k.$$

In particular

$$(1+\lambda)(1+\lambda)^\delta \in Z(k)$$

leads to $(\lambda + \lambda^\delta) \in Z(k)$. By then every $\lambda \in k$ is solution of a quadratic polynomial $P_\lambda(X)$ with coëfficïents in $Z(k)$, namely $P_\lambda(X) = X^2 - (\lambda + \lambda^\delta)X + \lambda\lambda^\delta$. Lemma 51 implies that this is only possible if k is a generalized quaternion algebra.

In any case we thus find that k is a generalized quaternion algebra, hence the same is valid for $\bar{k}$ and k'. This completes the proof. □

Appendix A

Nederlandstalige samenvatting

A.1 Inleiding en situering

Gebouwen verschenen impliciet het eerst in 1959 toen J. Tits een meetkundige interpretatie gaf aan een bepaalde veel bestudeerde algebraïsche groep (cfr. [28]). In die periode was het woord gebouw echter nog niet officieel geïntroduceerd in de abstracte en algebraïsche meetkunde. Het zou een 15 tal jaar duren vooraleer gebouwentheorie door het standaard werk van J. Tits [29] een feit werd. Dit werk heeft als voornaamste doel een volledige classificatie te geven van sferische gebouwen (gebouwen met eindige diameter) en rang groter dan 3.

Tweeling gebouwen, het vakgebied van deze thesis, dateren van een hele tijd later. Eind jaren 80 traden nieuwe algebraïsche structuren op de voorgrond als gevolg van de ontwikkelingen in de theoretische natuurkunde. Dit waren de zogenaamde Kac-Moody algebra's (cfr. [16]). Deze algebra's kunnen het best gezien worden als veralgemeningen van algebraïsche groepen. In algebraïsche groepentheorie gaat men er namelijk vanuit dat de dimensie van de vectorruimten waarin de groepen gedefiniëerd worden eindig dimensionaal is. Indien men deze veronderstelling laat vallen en oneidig dimensionale vectorruimten toelaat verkrijgt men onder een bepaalde voorwaarde Kac-Moody algebra's. Gezien J. Tits er destijds in geslaagd was een succesvolle theorie te ontwikkelen die algebraïsche groepen in een meetkundig perspectief stelde, was men ervan overtuigd dat een gelijkaardige theorie voor Kac-Moody algebra's diende opgesteld te worden.

Met dit als doeleinde stelden M. Ronan en J. Tits in 1990 het begrip tweeling gebouw voor. Gezien tweeling gebouwen een veralgemening zijn van sferische gebouwen gaf J. Tits in de standaard referentie [32], een ruw plan van hoe een classificatie van tweeling gebouwen eruit zou moeten zien. Deze bescrhijving zette B. Mühlherr er toe aan te beginnen werken aan een classificatie van 2 sferische tweeling gebouwen (2 sferische tweeling gebouwen zijn tweeling gebouwen met lokaal eindige diameter).

Tijdens het schrijven van zijn proefschrift was B. Mühlherr onrechstreeks in contact gekomen met technieken die nuttig zouden blijken voor een classificatie. Het eerste resultaat dat een oplossing gaf voor een probleem dat cruciaal was om deze classificatie te kunnen aanvatten, werd opgelost door B. Mühlherr en M. Ronan in [18]. Een ander belangrijke techniek, die volgens J. Tits zou moeten gebruikt worden, was de techniek van Galois cohomologie. B. Mühlherr slaagde erin in [21] om deze techniek uit te breiden naar het veld van tweeling gebouwen. Hierna bleek de theorie sterk van B. Mühlherr sterk genoeg ome de klassicatie van 2-sferische tweeling gebouwen te reduceren tot een classificatie van 3 types tweeling gebouwen : tweeling gebouwen van type $\tilde{A}_2$, $\tilde{B}_2$ end type 443.

Gezien gebouwen van type $\tilde{A}_2$ reeds goed gekend zijn door o.a. het werk op affiene gebouwen van H. Van Maldeghem en K. Van Steen bleven enkel de types $\tilde{B}_2$ and 443 over als onopgelost. Het voornaamste doel van dit proefschrift was dan ook te werken aan deze beide types meetkunden. Naarmate de theorie vorderde dienden een aantal verwante, vaak algebraïsch gerichte vragen opgelost te worden. Zo was een klassifikatie van klassieke en gemengde Moufang verzamelingen noodzakelijk een probleem dat nauw verwant leek met Borel-Tits theorie (cfr. [2]) en de theorie van orthogonale, hermitische en unitaire groepen en algebraïsche krommen (cfr. [6, 7]). In een aantal gevallen leidde dit zelf tot een aan aantal karakterisatiestellingen (cfr. Stellingen 74, 101, 132).
Om de uiteindelijk klassifikatie van $\tilde{B}_2$ te bekomen met bepaalde residue's,

werd gekozen om tweeling gebouwen te zien als Moufang gebouwen. (Moufang gebouwen zijn gebouwen waarvoor aan hoge symmetrie eisen voldaan is.) Dit had als voordeel dat stellingen en de presentatie konden worden vereenvoudigt. Gezien door het werk van B. Mühlherr (cfr. [18]) en het tweede deel van dit proefschrift 2-sferische Moufang gebouwen en tweeling gebouwen bijna altijd dezelfde zijn, legde dit geen extra beperkingen op.

Dit proefschrift werd verdeeld in vier delen (Chapters).
In het eerste deel worden een aantal definities en notaties gegeven.
Het tweede deel behandelt de oplossing van twee problemen betreffende tweeling gebouwen.
Het derde deel beschrijft een classificatie van gemengde en klassieke Moufang verzamelingen.
In vierde deel wordt een partiële classificatie gegeven van tweeling gebouwen van type $\tilde{B}_2$. Tevens wordt hier een eerste stelling bewezen die het niet bestaan van een aantal meetkunden van type 443 aantoont.

Definities

We vermelden in deze paragraaf de voornaamste definities en notaties. Als belangrijkste verwijzingen in deze context vermelden we [1], [20], [29],[32], [25] en [37].

A.1.1 Coxeter matrices, Coxeter systemen en wortelsystemen

Definitie 1 Zij I een eindige verzameling. Een Coxeter matrix over I is een symmetrische matrix $M = (m_{ij})_{i,j \in I}$ zodat $m_{ij} \in \mathbb{N} \cup \{\infty\}$, $m_{kl} \geq 2$, $\forall k, l \in I$ zodat $k \neq l$ en $m_{ii} = 1$, $\forall i \in I$.

Definitie 2 Een Coxeter matrix $M = (m_{ij})_{i,j \in I}$ noemt men *2 sferisch* indien $m_{ij} < \infty$, $\forall i, j \in I$. Indien $M = (m_{ij})_{i,j \in I}$ een Coxeter matrix is noteert met $E(M) = \{i, j\} \subset I$ waarvoor $m_{ij} \geq 3$.

Definitie 3 Zij $M = (m_{ij})_{i,j}$ een Coxeter matrix over een eindige verzameling I. Een *Coxeter systeem van type* M is een paar $(W, (s_i)_{i \in I})$, waarbij W een groep is met presentatie $W = \langle s_i | (s_i s_j)^{m_{ij}} \rangle$.

Stel dat $(W,(s_i)_{i\in I}$ een Coxeter systeem is. Voor $x \in W$ definiëren we $l(x)$ dan als $min\{m|x = s_{i(1)}s_{i(2)}\dots s_{i(m)}\ |i(j) \in I,\ 1 \leq j \leq m\}$ Bovendien noemt men elk element van de vorm ws_iw^{-1} *een spiegeling.* Elke spiegeling induceert een permutatie van W als men stelt :

$$ws_iw^{-1}(x) = ws_iw^{-1}x,\ \forall x \in W.$$

Men kan nu makkelijk bewijzen dat elke spiegeling ws_iw^{-1}, een partitie van W in twee helften invariant laat. Deze helften noemt men *wortels in W (behorend bij ws_iw^{-1}).* Deze wortels noteert men met $\alpha_{ws_iw^{-1}}$ en $-\alpha_{ws_iw^{-1}}$, waarbij $1 \in \alpha_{ws_iw^{-1}}$. Indien α en $-\alpha$ twee wortels zijn noteert men

$$\partial\alpha = \{\{x,y\}|x,y \in W \text{ en } s_\alpha(x) = y\}.$$

Als $\{x,y\} \in \partial\alpha$, dan noemt men $\{x,y\}$ ook *een paneel dat ligt op $\partial\alpha$.*

Definitie 4 Zij $M = (m_{ij})_{i,j}$ een Coxeter matrix over I en $(W,(s_i)_{i\in I})$ een Coxeter systeem van type M. Dan noemt men de verzameling van alle wortels in W een *wortelsysteem van type M.*

Definitie 5 Zij $M = (m_{ij})_{i,j\in I}$ een Coxeter matrix over I en $(W,(s_i)_{i\in I}$ een Coxeter systeem van type M met wortelsysteem Φ. Twee wortels α en β van W worden *prenilpotent* genoemd indien $\alpha \cap \beta \neq \emptyset$ en $-\alpha \cap -\beta \neq \emptyset$.
Indien α en β prenilpotent zijn noteert men :

$$[\alpha,\beta] = \{\gamma \in \Phi|\gamma \subseteq \alpha \cap \beta \text{ and } -\gamma \subseteq -\alpha \cap -\beta\}.$$

en :

$$(\alpha,\beta) = [\alpha,\beta] \setminus \{\alpha,\beta\}.$$

A.1.2 Gebouwen en Moufang gebouwen

Definitie 6 Zij $M = (m_{ij})_{i,j}$ een Coxeter matrix over een verzameling I, $(W,(s_i)_{i\in I}$ een Coxeter systeem van type M. Een *gebouw (van type M)* is een quadruple $(\Delta, W, (s_i)_{i\in I}, s)$ waarbij Δ een verzameling is, wiens elementen kamers worden genoemd, en d een functie is van $\Delta \times \Delta$ naar W zodat :

Bu1 $d(x,y) = 1$, als en slechts als $x = y$, $\forall x,y \in \Delta$

Bu2 Stel dat voor x, $y \in \Delta$, $d(x,y) = w$ en z een kamer zodat $d(y,z) = s$ met $s \in S$ dan geldt $d(x,z) \in \{w, ws\}$. Als in het bijzonder $l(ws) > l(w)$, dan vinden we $d(x,z) = ws$.

Bu3 Stel x, $y \in \Delta$ met $d(x,y) = w$. Dan bestaat er voor elke s_i minstens één kamer $z \in \Delta$ zodat $d(x,z) = ws$

Definitie 7 Als $(\Delta, W, (s_i)_{i\in I}, d)$ een gebouw is en $c \in \Delta$ dan noemt men een verzameling $R_{s_i}(c)$ $\{x \in \Delta | d(x,c) \in \{1, s_i\}$ een s_i-*paneel* of ook wel kortweg een *paneel* in Δ.

Het eenvoudigste voorbeeld van een gebouw van type $M = (m_{ij})_{i,j\in I}$ wordt gegeven door het quadrupel $(W, W, (s_i)_{s\in I}, d_W)$ waarbij $(W, (s_i)_{i\in I})$ een Coxeter systeem is van type I en d_W wordt gegeven door :

$$d_W(x,y) = x^{-1}y, \; \forall x, y \in W.$$

Noteer dit gebouw als Δ_W.
Als $(\Delta, W, (s_i)_{i\in I}, d)$ een gebouw is van type M, dan kan men bewijzen dat er deelverzamelingen Σ in Δ zijn die als gebouw isomorf zijn met Δ_W. Zulke verzameling noemt met een *appartement van* (Δ). Aangezien elk appartment Σ isomorf is met Δ_W kan men tevens spreken over *spiegelingen, wortel, wortelsysteem prenilpotente wortels in* Σ.

Definitie 8 Zij $M = (m_{ij})_{i,j\in I}$ een Coxeter matrix over I, $(W, (s_i)_{i\in})$ een Coxeter systeem van type M en $(\Delta, W, (s_i)_{i\in}, d)$ een gebouw van type M. Stel dat Σ_0 een vast apartement in Δ is. Noteer all wortels in Σ_0 door Φ_0. Dan noemen we $(\Delta, W, (s_i)_{i\in I}, d)$ een *Moufang gebouw* als er een family $(U_\alpha)_{\alpha\in\Phi_0}$ automorfisme groepen van $(\Delta, W, (s_i)_{i\in I}, d)$ bestaan (*wortelgroepen* genoemd) zodat :

Mo1 Elk element $u_\alpha \in U_\alpha$ fixeert alle kamers van α. Stel dat π een paneel is gelegen op $\partial\alpha$, en c een kamer van π die in α gelegen is. Dan werkt U_α regulier op alle kamers van $\pi \setminus \{c\}$.

Mo2 Als $\{\alpha, \beta\}$ een paar prenilpotente wortels is, geldt :

$$[U_\alpha, U_\beta] \subset U_{(\alpha,\beta)}$$

waarbij $U_{(\alpha,\beta)}$ de groep is voortgebracht door U_γ met $\gamma \in (\alpha, \beta)$.

Mo3 Voor elk element $u_\alpha \in U_\alpha \setminus\{1\}$ bestaat er een element $m(u_\alpha) \in U_{-\alpha}$ u_α $U_{-\alpha}$ waarvoor gelt $m(u_\alpha)$ (Σ_0) Σ_0.

Mo4 Stel voor $u_\alpha \in U_\alpha$, $n = m(u_\alpha)$ dan geldt voor elke wortel $\beta \in \Sigma_0$

$$nU_\beta n^{-1} = U_{n(\beta)}.$$

A.1.3 Tweeling gebouwen

Definitie 9 Zij $M = (m_{ij})_{i,j\in I}$ een Coxeter matrix over I, en $(W, (s_i)_{i\in I}$ een Coxeter systeem van type M. Een *tweeling gebouw (van type M)* is een paar gebouwen $(\Delta_+, W, (s_i)_{i\in I}, d_+)$, $(\Delta_-, W, (s_i)_{i\in I}, d_-)$ voorzien van een *complementaire aftandsfunctie* d^*, gaande van $\Delta_+ \sqcup \Delta_- \cup \Delta_- \sqcup \Delta_+$ naar W zodat ($\epsilon \in \{-1, 1\}$, $x \in \Delta_\epsilon$, $y \in \Delta_{-\epsilon}$ en $d^*(x, y) = w$) :

Tw1 $d^*(y, x) = w^{-1}$.

Tw2 Als z een kamer is in $\Delta_{-\epsilon}$ met $d_{-\epsilon}(y, z) = s_i$ em $l(ws_i) < l(w)$ dan geldt dat $d^*(x, z) = ws_i$.

Tw3 Voor elke $s_i \in S$ bestaat er ten minste één kamer $z \in \Delta_{-\epsilon}$ zodanig dat $d^*(x, z) = ws_i$.

A.1.4 Moufang verzamelingen

Moufang verzamelingen werden het eerst formeel geïntroduceerd in [32]. Een aantal gekende Moufang verzamelingen werden reeds voorheen bestudeerd onder een andere naam en met andere notatie. Ze kunnen het best gezien worden als de kleinst mogelijke tweeling gebouwen. Bovendien is elk Moufang gebouw samengesteld uit een groot aantal Moufang verzamelingen.

Definitie 10 Een *Moufang verzameling* is een verzameling X met ten minste 3 elementen, en een familie groepen $(U_x)_{x\in X}$ (wortelgroepen genaamd) zodat :

MoS1 Elke groep U_x werkt regulier op $X \setminus \{x\}$.

MoS2 Elke groep U_x stabiliseert de verzameling groepen $\{U_y | y \in X\}$ door conjungatie

Definitie 11 Een *isomorfisme* tussen twee Moufang verzamelingen $(X, (U_x)_{x\in X})$ en $(Y, (U_y)_{y\in Y})$ is een bijectie β van X naar Y zodanig dat voor elke $x \in X$ en $u_x \in U_x$ geldt dat :

$$\beta \circ u_x \circ \beta^{-1} \in U_{\beta(x)}$$

A.2 Algemene resultaten

A.2.1 Tweeling gebouwen en Moufang gebouwen.

Door het werk van B. Mühlherr en M. Ronan (cfr. [18]) was reeds gekend dat onder bepaalde lokale voorwaarden elk tweeling gebouwen kan gezien worden als een Moufang gebouw. In het artikel [32] haalt J. Tits aan dat men tevens het omgekeerde kan bewijzen, en geeft hij een aantal hints. Hij vermeldt er echter bij dat het geen triviaal resultaat is dat enig werk vereist. Dit probleem was dan ook een uitdaging om me vertrouwd te maken met de theorie van tweeling gebouwen. De stelling luidt als volgt :

Stelling 12 *(Theorem 74) Zij $(\Delta, W, (s_i)_{i\in I}, d)$ een Moufang gebouw van type M dan kan $(\Delta, W, (s_i)_{i\in I}, d)$ gezien worden als de helft van een tweeling gebouw, i.e. er bestaat een gebouw $(\Delta_-, W, (s_i)_{i\in I}, d_-)$ en een complementaire afstandsfunctie d^* zodat $((\Delta, W, (s_i)_{i\in}, d)$, $(\Delta_-, W, (s_i)_{i\in}, d_-)$,$d^*)$ een tweeling gebouw is.*

A.2.2 Lokale karakterisatie van tweeling gebouwen

De volgende stelling is het resultaat van het onderzoek verricht naar abstrakte voorstellingen van tweeling gebouwen. Dit resultaat werd tevens onafhankelijk gevonden door P. Abramenko en H. Van Maldeghem.
(Doorheen deze paragraaf stelt ϵ telkens een element uit de verzameling $\{-1, 1\}$ voor.)
De volgende definitie kan tevens teruggevonden worden in [19].

Definitie 13 Een *1-koppeling* tussen een paar gebouwen $(\Delta_+, W, (s_i)_{i\in I}, d_+)$ en $(\Delta_-, W, (s_i)_{i\in I}, d_-)$ van hetzelfde type is een symmetrische binaire relatie $\mathcal{O} \subset \Delta_+ \times \Delta_- \sqcup \Delta_- \times \Delta_+$ zodat als voor $c_\epsilon \in \Delta_\epsilon$ en $c_{-\epsilon} \in \Delta_{-\epsilon}$, geldt dat $(c_\epsilon, c_{-\epsilon}) \in \mathcal{O}$, dan bevat elk paneel van Δ_ϵ waarop c_ϵ ligt juist één kamer z zodanig dat $(z, c_{-\epsilon}) \notin \mathcal{O}$. Als $\mathcal{O}$ een 1-koppeling definieert tussen $(\Delta_+, W, (s_i)_{i\in I}, d_+)$ en $(\Delta_-, W, (s_i)_{i\in I}, d_-)$ en $c_\epsilon \in \Delta_\epsilon$, dan noteren we :

$$c_\epsilon^o = \{y \in \Delta_{-\epsilon} | (c_\epsilon, y) \in \mathcal{O}\}.$$

Stel dat $((\Delta_+, W, (s_i)_{i\in I}, d_+)$, $(\Delta_-, W, (s_i)_{i\in I}, d_-)$,$d^*)$ een tweeling gebouw is. Dan definieert de relatie Opp met :

$$(x, y) \in Opp \Leftrightarrow d^*(x, y) = 1$$

een 1-koppeling tussen $(\Delta_+, W, (s_i)_{i\in I}, d_+)$ en $(\Delta_-, W, (s_i)_{i\in I}, d_-)$. Men noemt deze 1-koppeling ook de *oppositie relatie tussen* Δ_+ *en* Δ_- *bepaald door* d^*. Men kan aantonen dat tweeling gebouwen tevens kunnen gedefinieerd worden in termen van de oppositie relatie. Met andere woorden, het is voldoende de relatie *Opp* te kennen teneinde d^* te reconstrueren.

Definitie 14 Stel dat $\mathcal{O}$ een 1-koppeling definieert tussen $(\Delta_+, W, (s_i)_{i\in I}, d_+)$ en $(\Delta_-, W, (s_i)_{i\in I}, d_-)$. Dan zeggen we dat $\mathcal{O}$ voldoet aan de voorwaarde *Ptw* voor een kamer $c \in \epsilon$ als :
$\forall\ y \in \Delta_{-\epsilon}$, $\forall\ c_y$, $\bar{c}_y \in c^o$ zodanig dat $l(d_{-\epsilon}(c_y, y)) = l(d_{-\epsilon}(\bar{c}_y, y))$ $= min\{l(d_{-\epsilon}(z, y))|z \in c^o\}$, $\forall\ y_c \in y^o$ met $l(d_\epsilon(c, y_c)) = min\{l(d_\epsilon(v, c))$ $|v \in c^o\}$ geldt :

$$d_{-\epsilon}(c_y, y) = d_{-\epsilon}(\bar{c}_y, y) = d_\epsilon(c, y_c).$$

Het belang van voorwaarde *Ptw* en tweeling gebouwen wordt gegeven in volgende stelling :

Stelling 15 *(Theorem 101) Stel dat* $\mathcal{O}$ *een dikke* 1*-koppeling definieert tussen twee gebouwen* $(\Delta_+, W, (s_i)_{i\in I}, d_+)$ *en* $(\Delta_-, W, (s_i)_{i\in I}, d_-)$. *Dan definieert* $\mathcal{O}$ *een oppositie relatie tussen* Δ_+ *en* Δ_- *(i.e. er bestaat een tweeling gebouw* $((\Delta_+, W, (s_i)_{i\in I}, d_+)$, $(\Delta_-, W, (s_i)_{i\in I}, d_-)$,$d^*$ *zodat* $Opp = \mathcal{O}$*) als en slechts als voorwaarde Ptw voldaan is voor ten minste één kamer uit* Δ_+ *of* Δ_-.

A.3 Resultaten over Moufang verzamelingen

Vooraleer het resultaat neer te schrijven, geven we waar mogelijk een korte beschrijving van de Moufang sets welke in dit proefschrift beschouwd werden. Als referentie geven we in dit kader Hoofdstuk 8 op van [29].

Definitie 16 Stel k een lichaam met involutie σ, $\epsilon \in k$ en V een rechtse k-vectorruimte. Een (σ, ϵ)-*hermitische vorm* is een afbeelding f van $V \times V$ naar k zodat :

$$\begin{aligned} f(x\lambda, y\mu) &= \lambda^\sigma f(x, y)\mu,\ \forall\lambda, \mu \in k, x, y \in V \\ f(x + y, z) &= f(x, y) + f(x, z),\ \forall x, y, z \in V \\ f(x, y) &= f(y, x)^\sigma\epsilon,\ \forall x, y \in V \end{aligned}$$

Voorts noteert men $k^{(\sigma,\epsilon)}$ $k/k_{\sigma,\epsilon}$ waarbij $k_{\sigma,\epsilon} = \{t - t^\sigma\epsilon | t \in k\}$.

Definitie 17 Stel k een lichaam met involutie σ, $\epsilon \in k$ en V een rechtse k-vectorruimte. Een functie q gaande van V naar $k^{(\sigma,\epsilon)}$ noemt men een (σ,ϵ)-*quadratisch vorm* als $q(x\lambda) = \lambda^\sigma q(x)\lambda + k_{\sigma,\epsilon}$, $\forall\ \lambda \in k$, $x \in V$ en indien er een (σ,ϵ)-hermitische vorm f op $V \times V$ bestaan zodanig dat :

$$q(x+y) = q(x) + q(y) + (f(x,y) + k_{\sigma,\epsilon}.$$

Indien q een (σ,ϵ)-quadratische vorm is op een rechte k-vectorruimte V, kan met bewijzen dat $q^{-1}(0)$ unie is van deel vectorruimten van V, welke *totaal isotrope deelruimten van V worden genoemd.* Door het gebruik van Zorns lemma volgt bovendien dat alle maximale deelruimten in $q^{-1}(0)$ dezelfde dimensie hebben, welke de *Witt index van q* wordt genoemd.

> Indien q een (σ,ϵ)-quadratische vorm is van Witt index 2 vormen totale isotrope deelruimten een meetkundige structuur welke bekend staat als een *veralgemeende vierhoek*, genoteerd als $Q(V,q,k,\sigma)$. We zullen hier niet nader ingaan op de theorie van veralgemeende vierhoeken. Voor meer informatie verwijzen we naar het standaard werk [37].

> Indien q een (σ,ϵ)-quadratische vorm is van Witt index 1, vormen de totale isotrope deelruimten een Moufang verzameling, genoteerd $\mathcal{M}(V,q,k,\sigma)$.

De Moufang verzamelingen van de vorm $\mathcal{M}(V,q,k,\sigma)$ vormen een grote klasse van diegene die in dit werk bestudeerd worden. Ze worden in 3 klassen onderverdeeld

> *Orthogonale Moufang verzamelingen*,
> Moufang verzamelingen van de vorm $\mathcal{M}(V,q,k,\sigma)$ met $\sigma = 1$.
>
> *Hermitische Moufang verzamelingen*,
> Moufang verzamelingen van de vorm $\mathcal{M}(V,q,k,\sigma)$ met $Z(k) = k$ en $\sigma \neq 1$.
>
> *Unitaire Moufang verzamelingen* Moufang verzamelingen van de vorm $\mathcal{M}(V,q,k,\sigma)$ met $Z(k) \neq k$.

Naast Moufang verzamelingen geassocieerd met (σ, ϵ)-quadratische vormen vermelden we nog de twee andere klassen welke in classificatie opgenomen werden.

Projective Moufang verzamelingen,
genoteerd als $\mathcal{P}(k)$. Deze Moufang verzamelingen vertalen in feite alle bepalende eigenschappen van de projective rechte over een lichaam k.

Gemengde Moufang verzamelingen,
genoteerd als $\mathcal{P}(k, k'; l, l')$, waarbij k en k' twee velden in karakteristiek 2 voorstellen met deelverzamelingen l en l' die aan bepaalde voorwaarden voldoen.

Definitie 18 Een Moufang verzameling $(X, (U_x)_{x\in X})$ noemt men *klassiek* indien $(X, (U_x)_{x\in X})$ isomorf is met een projectieve Moufang verzameling of een Moufang verzameling van de vorm $\mathcal{M}(V, q, k, \sigma)$

In Hoofdstuk 3 wordt in het kader van de classificatie van $\tilde{B}_2$ Moufang gebouwen, een classificatie gegeven van klassieke en gemengde Moufang verzamelingen. Als belangrijk resultaat dat volgt uit deze classificatie geven we :

Stelling 19 *(cfr. Theorems 124, 125,126,127, 128, 129,130 en 131) Indien de dimensie van V en V' groter is dan 5, en indien de wortelgroepen van beide Moufang verzamelingen niet commutatief zijn indien $Z(k) \neq k$, dan bestaat er voor elk isomorfisme β van $\mathcal{M}(V, q, k, \sigma)$ naar $\mathcal{M}(V', q', k', \sigma')$ een semi-lineare afbeelding φ met α een isomorfisme van k naar k' en een constante $c' \in k'$ zodat :*

$$\begin{aligned} \beta(\langle x\rangle) &= \langle\varphi(x)\rangle, \forall\langle x\rangle \in \mathcal{M}(V, q, k, \sigma) \\ (f(x, y))^\alpha &= c' f'(\varphi(x), \varphi(y)),\ \forall x, y \in V \\ (q(x))^\alpha &= c' q'(\varphi(x)),\ \forall x \in V \end{aligned}$$

Deze stelling vormt een uitbreiding van Borel-Tits theorie in het geval van een algebraïsche groep van rang 1.
Als gevolg van de classificatie en gemengde Moufang verzamelingen kon een lokale karakterisatiestelling voor klassieke Moufang verzamelingen opgesteld worden.

Stelling 20 *(cfr. Theorem 132) Een Moufang verzameling* $(X,(U_x)_{x\in X})$ *is van de vorm* $\mathcal{M}(V,q,k,\sigma)$, *met* $dim(V) \geq 5$ *en als* k *een veralgemeende quaterniaanse algebra is, is* σ *niet de standaard involutie, als en slechts als er twee punten* y_1, $y_2 \in X$ *en een familie* $(Y_i)_{i\in I}$ *Moufang deelverzamelingen van* $(X,(U_x)_{x\in X})$ *zodat :*

(i) Elke Y_i *is isomorf onder* φ_i *met een Moufang set* $\mathcal{M}(V_i,q_i,k_i,\sigma_i)$, *waarbij* $dim(V_i) \geq 4$ *als* $Z(k_i) = k_i$ *en* $\sigma_i \neq 1$ *en* σ_i *niet gelijk is aan de standaard involutie als* k_i *een veralgemeende quaterniaanse algebra is. Alle* Y_i *zijn van hetzelfde type.*

(ii) Elke Moufang deelverzameling Y_i *bevat* y_1 *en* y_2 *alle drie de punten* x_1, x_2 *en* x_3.

(iii) Als de Y_i *orthogonale Moufang verzamelingen zijn, geldt :*
voor elk paar i, $j \in I$ *is de Moufang verzameling* $Y_i \cap Y_j$ *niet kommutatief en :*

$$Z(Fix_{TY_i}\{y_1,y_2\}) = Z(Fix_{TY_j}\{y_1,y_2\}.$$

Als $char(k_i) = 2$, *is elke Moufang verzameling* $\varphi_i^{-1}(Y_j)$ *van de vorm* $\mathcal{M}(V_{ij},q_{ij},k_{ij},\sigma_j)$ *waarbij* V_{ij} *een deelruimte van* V_j *voorstelt en* $q_{ij} = q_i|_{V_{ij}}$.

(iv) Als de Y_i *niet orthogonaal zijn, geldt :*

$$Z(Stab_{U_{y_1}}(Y_i)) = Z(Stab_{U_{y_1}}(Y_j)), \forall i,j \in I.$$

(v) Als de Y_i *hermitische Moufang verzamelingen zijn, bestaat er een* Y_0 *behorende tot de family* $(Y_i)_{i\in I}$, *zodat voor elk paar* i, $j \in I$, $Y_0 \cap Y_i \cap Y_j$ *een Moufang verzameling is met niet-kommutatieve wortelgroepen.*

A.4 classificatie van $\tilde{B}_2$ Moufang gebouwen en het niet bestaan van Moufang gebouwen van type 443

Stel dat $(\Delta, W, (s_i)_{i\in I}, d)$ een Moufang gebouw is van type $M = (m_{ij})_{i,j\in I}$. Zij $R_i(c)$ het s_i-paneel in Δ dat c bevat. Dan volgt uit de standaard theorie dat de Moufang structuur op Δ een structuur induceert zodat met $R_i(c)$ kan zien als een Moufang verzameling. Noteer dergelijke Moufang verzameling met $\mathcal{M}_{R_i(c)}$. We kunnen dan volgend begrip invoeren, waarover meer informatie kan teruggevonden worden in [32], [20] en [21].

Definitie 21 Zij $M = (m_{ij})_{i,j\in I}$ een Coxeter matrix over een verzameling I. Een *Moufang fundering (van type M)* is een tripel $((\Delta_{ij})_{i,j\in E(M)}$, $(c_{ij})_{\{i,j\}\in E(M)}$,$(\beta_{ijk})_{\{i,j\},\{j,k\}\in E(M)})$ met :

MoFo1 Voor elk paar $\{i,j\} \in E(M)$ is Δ_{ij} een Moufang gebouw van type $(m_{k,l})_{k,l\in\{i,j\}}$.

MoFo2 Voor elk paar $\{i,j\} \in E(M)$, stelt c_{ij} een kamer van Δ_{ij} voor en $c_{ij} = c_{ji}$, $\forall i,j \in I$

MoFo3 Voor elk koppel $\{i,j\}$ $\{j,k\} \in E(M)$, definieert β_{ijk} een isomorfisme tussen $\mathcal{M}_{R_j(c_{ij})}$ en $\mathcal{M}_{R_j(c_{jk})}$.

Men kan aantonen dat indien $(\Delta, W, (s_i)_i, d)$ een Moufang gebouw is van type M, hiermee een isomorfie klasse van Moufang funderingen van type M correspondeert die we noteren als $MoFo(\Delta)$. Moufang funderingen bleken door [32], [20] en [21] essentieel teneinde een klassifikatie van Moufang gebouwen te kunnen opstellen. Een rechtstreeks gevolg van [18] was dat een Moufang gebouw volledig bepaald wordt door $MoFo(\Delta)$. Er volgde namelijk uit dat in de meeste gevallen twee Moufang gebouwen Δ en Δ' isomorf zijn als en slechts als $MoFo(\Delta) = MoFo(\Delta')$. Een klassifikatie van Moufang gebouwen diende dus te beginnen met een klassifikatie van Moufang funderingen.

Definitie 22 Een Moufang fundering $F = ((\Delta_{ij})_{\{i,j\}\in E(M)}, (c_{ij})_{\{i,j\}\in E(M)}, (\beta_{ijk})_{\{i,j\},\{j,k\}\in E(M)})$ van type M noemt men *integreerbaar* indien $F \in MoFo(\Delta)$ waarbij Δ een Moufang gebouw van type M voorstelt.

Definitie 23 Stel $M_{\tilde{B}_2}$ de Coxeter matrix over $\{1,2,3\}$ met $m_{12} = m_{23} = 4$ en $m_{13} = 2$ en M_{443} de Coxeter matrix over $\{1,2,3\}$ met $m_{12} = m_{23} = 4$ en $m_{13} = 3$.

Definitie 24 Idien voor een Moufang fundering $F = ((\Delta_{ij})_{\{i,j\}\in E(M)}, (c_{ij})_{\{i,j\}\in E(M)}, (\beta_{ijk})_{\{i,j\},\{j,k\}\in E(M)})$, van type $M_{\tilde{B}_2}$, $\mathcal{M}_{R_2(c_{12})}$ en $\mathcal{M}_{R_2(c_{23})}$ beide puntenrijen zijn, noemt men F van *type PP*, $\mathcal{M}_{R_2(c_{12})}$ en $\mathcal{M}_{R_2(c_{23})}$ beide lijnen penselen, noemt met F van *type LL*, $\mathcal{M}_{R_2(c_{12})}$ een puntenrij en $\mathcal{M}_{R_2(c_{23})}$ een lijnenpenseel is, noemt met F van type PL, $\mathcal{M}_{R_2(c_{12})}$ een lijnenpenseel en $\mathcal{M}_{R_2(c_{23})}$ een puntenrij is, noemt men F van *type LP*.

A.4.1 Niet integreerbare Moufang funderingen

Volgende twee stellingen zijn cruciaal voor een classificatie van Moufang gebouwen van types $M_{\tilde{B}_2}$ en M_{443}.

Stelling 25 *(cfr. Theorem 151) Zij $F = ((\Delta_{ij})_{\{i,j\}\in E(M)}, (c_{ij})_{\{i,j\}\in E(M)}, (\beta_{ijk})_{\{i,j\},\{j,k\}\in E(M)})$ een Moufang fundering van type $M_{\tilde{B}_2}$.*
Stel $\Delta_{12} = Q(E, q, k, \sigma)$ en $\Delta_{23} = Q(E', q', k', \sigma')$, met $Z(k) \neq k$ en $Z(k') \neq k'$. Als F integreerbaar is, geldt :

(i) F is van type PP en er geldt:
$(i.a)$ β_{123} induceert een anti-isomorfisme van k naar k'
$(i.b)$ k en k' zijn veralgemeende quaterniaanse algebra's met standaard involutie σ en σ' en β_{123} definieert een isomorfisme van k naar k'

(ii) F is van type LL, k en k' zijn veralgemeende quaterniaanse algebra's en $dim(E) = dim(E') = 4$.

Stelling 26 *(cfr. Theorem 158) Zij $F = ((\Delta_{ij})_{\{i,j\}\in E(M)}, (c_{ij})_{\{i,j\}\in E(M)}, (\beta_{ijk})_{\{i,j\},\{j,k\}\in E(M)})$ een Moufang fundering van type M_{443} met $m_{12} = m_{23} = 4$ en $m_{13} = 3$. Stel dat $\Delta_{12} = Q(E, q, k, \sigma)$, $\Delta_{23} = Q(E', q', k', \sigma')$, met $Z(k) \neq k$, $Z(k') \neq k'$ en $Rad(f) = Rad(f') = 0$ (waarbij f de vorm is geassocieerd aan q en f' de vorm geassocieerd met q') als k of k' een veralgemeende quaterniaanse algebra's is met standaard involutie σ of σ'. Als F integreerbaar is, treedt één van volgende gevallen op :*

(i) β_{123} induceert een anti-isomorfisme tussen k en k', k is een veralgemeende quaterniaanse algebra met standaard involutie σ, k' is een veralgemeende quaterniaanse algebra met standaard involutie σ'

(ii) β_{123} induceert een isomorfisme tussen k en k', k is een veralgemeende quaterniaanse algebra en k' is een veralgemeende quaterniaanse algebra

A.4.2 Integreerbare Moufang funderingen van type $M_{\tilde{B}_2}$

Gebruik makend van voorgaande stellingen kunnen de problematische gevallen voor de classificatie van $\tilde{B}_2$ gebouwen volledig geëlimineerd worden. Resteerd ons dus nog over een lijst te geven van de integreerbare Moufang funderingen van type $\tilde{B}_2$. Bij het opstellen van deze lijst werd veelvuldig gebruik gemaakt van de resultaten van [23]. Gezien de lengte en het technisch karakter de lijst verwijzen we voor een expliciete opsomming naar paragraaf 4.4. We vermelden echter wel de voornaamste stellingen.

Stelling 27 *(cfr. Theorem 153) Elke Moufang fundering* $F = ((Q(E, q, k, \sigma), Q(E', q', k', \sigma'), c_{12}, c_{23}, \beta_{123}))$ *van type PP met* $Z(k) \neq k$ *zodanig dat* β_{123} *een anti-isomorfisme definieert van* k *naar* k' *is integreerbaar.*

Stelling 28 *(cfr. Theorem 155) Stel* k *en* k' *beide velden. Dan is elke Moufang fundering* $F = ((Q(E, q, k, \sigma), Q(E', q', k', \sigma'), c_{12}, c_{23}, \beta_{123}))$ *van type LL, zodat* $\mathcal{M}_l(Q(E, q, k, \sigma))$ *niet commutatief is, integreerbaar.*

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