MAF300 - Written Exam

Date: March 4, 2016 **Time**: 09:00 - 13:00 (4 hours)
Allowed aids: All written and printed matters; All approved calculators.
The exam contains 4 tasks, each of which is given approximately equal weight. Some sub-tasks may be continued, even if intermediate results have not been obtained. **Good Luck**!

1)

Consider the following initial value problem:

$$2\,x\,(t) + x'\,(t) + 3\,y'\,(t) = -2\,, \quad y\,(t) - x'\,(t) - 2\,x\,(t) = 0$$

with initial conditions $x\,(0) = 4$, $y\,(0) = 5\,$.
Work with 3 significant digits when performing numerical computations.
a) A higher-order initial value problem may always be written as a system of first-order differential equations with an appropriate set of initial conditions. Find the second-order differential equation for $x\,(t)$ and the complete set of initial conditions corresponding to the initial value problem given above.
b) Determine the type of the differential equation found in a) and give the general steps necessary to obtain an analytic solution.
c) Following the steps you outlined in b), solve the initial value problem for $x\,(t)$ analytically. Compute also the result for $y\,(t)$.
d) Solve the initial value problem (in one of its equivalent forms) numerically. Use Euler's method and perform 3 steps of size 0.1.

2)

Do not try this one. We only do linear regression

Consider the following data pairs: $(x_i, y_i) = (0.1, -0.2), (0.2, -0.1), (0.3, -0.06), (0.4, 0.1)\,$.
Assume that the values x_i are exact and that each of the ordinates y_i has a 'standard-deviation' of $\sigma_i = 0.02$. A theoretical model predicts that the data points approximately obey the relationship

$$g\,(x) = ax^3 + bx + c\,.$$

Note that the highest power is 3. The model parameters a, b and c are to be determined below. Work with 3 significant digits.
a) The model parameters shall be determined such that the value of

$$\chi^2 = \sum_{i=1}^{4} \frac{(y_i - g\,(x_i))^2}{{\sigma_i}^2}$$

is minimal. What linear system determines these optimal values of a, b and c?
b) Give the linear system of equations for the parameters in terms of its numerical coefficient matrix and right-hand side vector.
c) Solve the linear system obtained in b) using a numerically stable method and compute the optimal model parameters.
d) Write down the model function $g\,(x)$ and compute the corresponding value of χ^2. Can you trust the obtained least-squares fit?

Flip the page.

3)

Given be the expression

$$f(x) = \frac{e^x - 1 - x}{x^2}.$$

a) Compute the limits of $f(x)$ at $x = 0$, $x \to -\infty$ and $x \to +\infty$.

b) Why is the expression for $f(x)$ unstable around $x = 0$? Find an approximate expression which is stable in this region. [**Hint**: Use a series expansion of the numerator and truncate at appropriate order.]

c) Use the Lagrange form of the error term (remainder) of the series expansion to obtain an error estimate for the approximation. Assume that $|x| \leq x_0$ (x_0 is positive) and find an upper bound for the error (uncertainty).

d) What is the minimum order of the series expansion required to ensure that the absolute truncation error stays below $\Delta f = 0.04$ if the approximate expression of b) was to be used on the interval given by $|x| \leq 1$? Is your result obtained in b) sufficient to guarantee this additional requirement?

4)

Consider the real function f of two variables, given by

$$f(x, y) = \ln\left(\frac{x}{3} + \sqrt{1 - y}\right).$$

It shall be evaluated at $x^* = 1.12$, $y^* = 0.9998$. These values were obtained by optimal rounding to the given number of digits.

a) What are the absolute and relative uncertainties of the values x^* and y^*?

b) Evaluate $f(x^*, y^*)$ and compute also the values of the first partial derivatives of f with respect to x and y in this point.

c) Use error propagation formulas to compute the expected absolute and relative uncertainties of $f(x^*, y^*)$ due to the rounding error in the input data. Was it justified to round y^* to 4 significant digits, while x^* has only 3? Give the total result with absolute uncertainty (round the result appropriately).