STA500 Introduction to Probability and Statistics 2, autumn 2018.

Solution exercise set 3

Exercises from the book:

6.40

X=consumption of water is having a gamma distribution with $\alpha = 2$ and $\beta = 3$.

$$\begin{aligned} P(X > 9) &= \int_9^\infty \frac{1}{3^2\Gamma(2)} x^{2-1} e^{-x/3} dx = \frac{1}{9}\int_9^\infty x e^{-x/3} dx \\ &= \frac{1}{9}[x(-3)e^{-x/3}]_9^\infty - \frac{1}{9}\int_9^\infty -3e^{-x/3} dx \\ &= \frac{1}{9} 27e^{-3} + \frac{1}{9}[-9e^{-x/3}]_9^\infty = 3e^{-3} + e^{-3} = 4e^{-3} = \underline{\underline{0.199}} \end{aligned}$$

Here integration by parts is used to solve the integral. Alternatively the integration formula found on the last page of the formula sheets could have been used.

6.43

a) By the general formulas for expectation and variance in the gamma distribution we get: $\mathrm{E}(X) = \alpha\beta = 2 \cdot 3 = \underline{\underline{6}}$ and $\mathrm{Var}(X) = \alpha\beta^2 = 2 \cdot 3^2 = \underline{\underline{18}}$.

6.47

a) For the Weibull distribution with parameters α and β we have that (see the paragraph about the gamma function on the formula sheets)
$\mathrm{E}(X) = \alpha^{-1/\beta}\Gamma(1+\frac{1}{\beta})$. I.e.

$$\mathrm{E}(X) = 0.5^{-1/2}\Gamma(1+\frac{1}{2}) = \sqrt{2}\cdot\Gamma(\frac{3}{2}) = \sqrt{2}(\frac{3}{2}-1)\Gamma(\frac{1}{2}) = \sqrt{\frac{\pi}{2}} = \underline{\underline{1.25}}$$

b)

$$f(x) = \alpha\beta x^{\beta-1}e^{-\alpha x^\beta} = xe^{-\frac{1}{2}x^2} \quad , x \geq 0 \qquad \text{which gives that:}$$
$$P(X>2) = \int_2^\infty xe^{-\frac{1}{2}x^2}dx = [-e^{-\frac{1}{2}x^2}]_2^\infty = e^{-\frac{1}{2}2^2} = e^{-2} = \underline{\underline{0.135}}$$

6.56/6.54

We use the fact that when X is having a lognormal distribution then $\ln(X)$ is having a normal distribution and get:

$$\begin{aligned} P(X>270) &= 1-P(X\leq 270) = 1-P(\ln(X)\leq \ln(270)) \\ &= 1-P(\frac{\ln(X)-4}{2} \leq \frac{\ln(270)-4}{2}) \\ &= 1-P(Z\leq 0.80) = 1-0.7881 = \underline{\underline{0.2119}} \end{aligned}$$

6.57/6.55

For the lognormal distribution we have that $\mathrm{E}(X) = e^{\mu+\sigma^2/2}$ and $\mathrm{Var}(X) = e^{2(\mu+\sigma^2)} - e^{2\mu+\sigma^2}$ which here gives that:

$$\begin{aligned} \mathrm{E}(X) &= e^{4+2^2/2} = e^6 = \underline{\underline{403.4}} \\ \mathrm{Var}(X) &= e^{2(4+2^2)} - e^{2\cdot 4+2^2} = e^{16}-e^{12} = \underline{\underline{8723355.7}} \end{aligned}$$

6.58/6.56

a) Let X be the number of cars arriving at the intersection during one minute. From the information in the exercise text we get that X is having a Poisson distribution with expectation 5 such that:

$$P(X>10) = 1-P(X\leq 10) \overset{table}{=} 1-0.9863 = \underline{\underline{0.0137}}$$

b) Here we shall calculate the probability that it will take more than 2 minutes before 10 cars have appeared at the intersection. This can be calculated in two ways.

One (the simplest) way to calculate this is to define Y as the number of events in the interval $[0,2]$ and calculate $P(Y<10)$ (the event that it takes more than 2 minutes until 10 cars have appeared is the same as the event that less than 10 cars appear during the 2 minutes). Y is having a Poisson distribution with expectation $\lambda t = 5\cdot 2 = 10$ and we get:

$$P(Y<10) = P(Y\leq 9) \overset{table}{=} \underline{\underline{0.458}}$$

The other way is to define S_{10} as the time which elapses until car number 10 appears at the intersection and calculate $P(S_{10} > 2)$. We know that time until event number 10 in a Poisson process with $\lambda = 5$ is having a gamma distribution with parameters $\alpha = 10$ and $\beta = 1/5$ (being the sum of 10 exponentially distributed variables with expectation 1/5). We then get that:

$$\begin{aligned}
P(S_{10} > 2) &= 1 - P(S_{10} \le 2) = 1 - \int_0^2 \frac{1}{\beta^\alpha \Gamma(\alpha)} s^{\alpha-1} e^{-s/\beta} ds \\
&= 1 - \int_0^2 \frac{1}{(\frac{1}{5})^{10}\Gamma(10)} s^{10-1} e^{-5s} ds \overset{u=5s}{=} 1 - \int_0^{10} \frac{5^{10}}{\Gamma(10)} (\frac{u}{5})^{10-1} e^{-u} du/5 \\
&= 1 - \int_0^{10} \frac{u^{10-1}}{\Gamma(10)} e^{-u} du \overset{tableA.24}{=} 1 - 0.542 = \underline{\underline{0.458}}
\end{aligned}$$

6.59/6.57

The time between events in a Poisson process is having an exponential distribution with expectation $1/\lambda = 1/5$.

a)

$$P(X > 1) = 1 - \int_0^1 5e^{-5x} dx = 1 - [-e^{-5x}]_0^1 = 1 + e^{-5} - 1 = \underline{\underline{0.0067}}$$

b) $\mathrm{E}(X) = \beta = 1/5 = \underline{\underline{0.2}}$

Exercises from the note on extreme values:

Exercise 1:

We have a parallel system made up of two independent components, where the lifetime of each component is having an exponential distribution with parameter λ.

$$F_{X_i}(t) = 1 - \exp(-\lambda t)$$

Let the lifetime of the system be denoted V. We shall find the distribution of V. The system is functioning as long as at least one of the components are functioning.

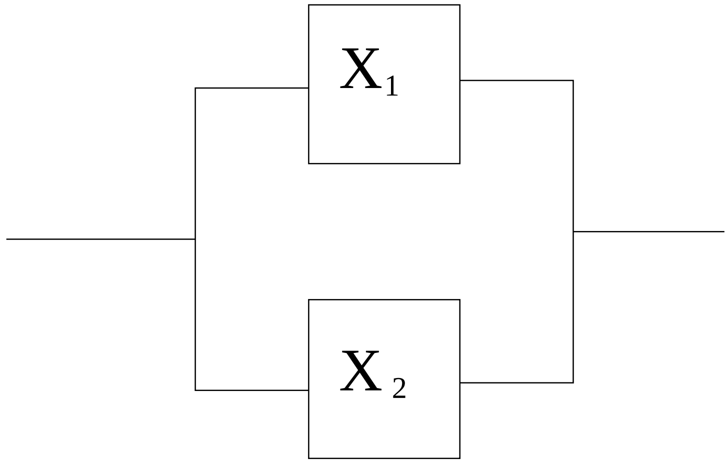

Figure 1: Parallel system with two components.

$$
\begin{aligned}
F_V(v) &= P(V < v) = P(\max(X_1, X_2) < v) = P(X_1 < v \cap X_2 < v) \\
&\overset{indep.}{=} P(X_1 < v) \cdot P(X_2 < v) = (1 - e^{-\lambda v})^2 \\
&= 1 - 2e^{-\lambda v} + e^{-2\lambda v} \\
f_V(v) &= F_V'(v) = 2\lambda e^{-\lambda v} - 2\lambda e^{-2\lambda v} = \underline{\underline{2\lambda(e^{-\lambda v} - e^{-2\lambda v})}} \quad , v \geq 0
\end{aligned}
$$

Expectation:

$$
\begin{aligned}
\mathrm{E}(V) &= \int_0^\infty v f_V(v) dv = \int_0^\infty v 2\lambda(e^{-\lambda v} - e^{-2\lambda v}) dv \\
&= 2\int_0^\infty v\lambda e^{-\lambda v} - \int_0^\infty v 2\lambda e^{-2\lambda v} = 2\frac{1}{\lambda} - \frac{1}{2\lambda} = \underline{\underline{\frac{3}{2\lambda}}}
\end{aligned}
$$

Notice that we recognize the two last integrals as the expression for the expectation for exponentially distributed variables with respectively parameter λ and parameter 2λ, and we thus know what these integrals are.

Exercise 2:

We have a series system made up of n independent components where the lifetime of component i is having a Weibull distribution with parameters α and β:

$$
P(X_i \leq x) = F_{X_i}(x) = 1 - e^{-\alpha x^\beta} \quad , x \geq 0
$$

Let the lifetime of the system be denoted U. We shall first find the distribution of U. The system is only functioning as long as all components are functioning.

$$
\begin{aligned}
F_U(u) &= P(U < u) = P(\min(X_1, X_2, ..., X_n) < u) = 1 - P(\min(X_1, X_2, ..., X_n) > u) \\
&\overset{indep.}{=} 1 - \prod_{i=1}^n P(X_i > u) = 1 - \prod_{i=1}^n e^{-\alpha u^\beta} = \underline{\underline{1 - e^{-n\alpha u^\beta}}}
\end{aligned}
$$

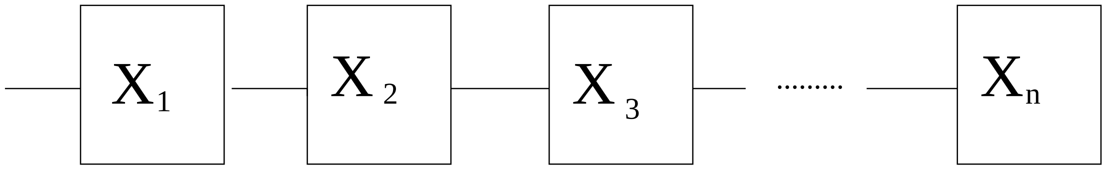

Figure 2: Series system with n components.

We see that this is a $\underline{\underline{\text{Weibull distribution with parameters } n\alpha \text{ and } \beta.}}$

For the Weibull distribution with parameters α and β we have that $\mathrm{E}(X) = \alpha^{-1/\beta}\Gamma(1+\frac{1}{\beta})$. I.e. for a single component we have that:

$$\mathrm{E}(X) = 0.1^{-1/0.5}\Gamma(1+\frac{1}{0.5}) = 100 \cdot \Gamma(3) = 100 \cdot 2! = \underline{\underline{200}}$$

For the whole system we get:

$$\mathrm{E}(U) = (n \cdot 0.1)^{-1/0.5}\Gamma(1+\frac{1}{0.5}) = \frac{1}{n^2}100 \cdot \Gamma(3) = \underline{\underline{\frac{200}{n^2}}}$$

Exercise 1:

a)

$$\begin{aligned}
\mathrm{E}(X) &= \int_{-\infty}^{\infty} xf(x)dx = \int_0^{\infty} x\frac{1}{\beta}e^{-x/\beta}dx = [-xe^{-x/\beta}]_0^{\infty} - \int_0^{\infty} -e^{-x/\beta}dx = 0 - [\beta e^{-y/\beta}]_0^{\infty} = \underline{\underline{\beta}} \\
\mathrm{E}(X^2) &= \int_0^{\infty} x^2\frac{1}{\beta}e^{-x/\beta}dx = [-x^2e^{-x/\beta}]_0^{\infty} - \int_0^{\infty} 2x(-e^{-x/\beta})dx \\
&= 0 + 2\beta\int_0^{\infty} x\frac{1}{\beta}e^{-x/\beta}dx = 2\beta^2 \\
\Rightarrow \mathrm{Var}(X) &= \mathrm{E}(X^2) - \mathrm{E}(X)^2 = 2\beta^2 - \beta^2 = \underline{\underline{\beta^2}}
\end{aligned}$$

Instead of integration by parts integration formulas found on the last page of the formula sheets could have been used to solve the integrals.

b) With $\beta = 1000$ we get

$$P(X > 1000) = \int_{1000}^{\infty} \frac{1}{1000}e^{-x/1000}dx = [-e^{-x/1000}]_{1000}^{\infty} = e^{-1} = \underline{\underline{0.368}}$$

Exercise 2:

a) The number of telephone calls per hour is having a Poisson distribution with expectation $\mu = \lambda t = 6 \cdot 1 = 6$, i.e.

$$P(X > 6) = 1 - P(X \leq 6) \overset{\text{table}}{=} 1 - 0.6063 = \underline{\underline{0.3937}}$$

b) The time until the first event in a Poisson process is having an exponential distribution with expectation $1/\lambda = 1/6$. Also note that 10 minutes is 1/6 hour. E.g.

$$P(T < 1/6) = \int_0^{1/6} 6e^{-6x}dx = [-e^{-6x}]_0^{1/6} = -e^{-1} + 1 = \underline{\underline{0.6321}}$$

(Alternatively the exercise can be solved by looking at the probability of having at least one event in a 10 minute interval)

c) The time until the second event in a Poisson process is having a gamma distribution with parameters $\alpha = 2$ and $\beta = 1/\lambda = 1/6$. Since 20 minutes is 1/3 hour we then get:

$$\begin{aligned}
P(S_2 < 1/3) &= \int_0^{1/3} \frac{1}{(1/6)^2\Gamma(2)}x^{2-1}e^{-x/(1/6)}dx = \int_0^{1/3} 36xe^{-6x}dx \\
&= [-6xe^{-6x}]_0^{1/3} - \int_0^{1/3}(-6e^{-6x})dx \\
&= -2e^{-2} - [e^{-6x}]_0^{1/3} = -2e^{-2} - e^{-2} + 1 = \underline{\underline{0.5940}}
\end{aligned}$$

Alternatively we can solve the problem by defining Y=the number of telephone calls in [0,1/3], Y is then having a Poisson distribution with expectation $\mu = \lambda t = 6 \cdot (1/3) = 2$. If it takes less than 20 minutes until the second telephone call, this means that there will be at least 2 calls during the 20 first minutes (1/3 hour) and we get:

$$P(S_2 < 1/3) = P(Y \geq 2) = 1 - P(Y \leq 1) \overset{\text{table}}{=} 1 - 0.4060 = \underline{\underline{0.5940}}$$

d) The number of telephone calls during 7.5 hours is having a Poisson distribution with expectation $\mu = \lambda t = 6 \cdot 7.5 = 45$. In principle this can be used to calculate $P(X > 50)$ exactly, however, in this case it is easier to use the approximation to the normal distribution. Since $\mu > 15$ the approximation to the normal distribution is good, and we get

$$\begin{aligned} P(X > 50) &= 1 - P(X \leq 50) = 1 - P(\frac{X-45}{\sqrt{45}} \leq \frac{50+0.5-45}{\sqrt{45}}) \\ &= 1 - P(Z \leq 0.82) = 1 - 0.7939 = \underline{\underline{0.2061}} \end{aligned}$$

e) The number of events in any interval of length t in a Poisson process is having a Poisson distribution with expectation λt, i.e. the number of telephone calls during 10 minutes = 1/6 hour is having a Poisson distribution with expectation $\mu = 6 \cdot (1/6) = 1$, i.e.

$$P(X = 0) = \frac{1^0}{0!} e^{-1} = \underline{\underline{0.3679}}$$

f) Since the number of events in non-overlapping intervals in a Poisson process are indendent (independent increments) what has happened in the previous 10 minutes has no influence on what will happen the next 10 minutes - i.e. the probability of no telephone calls the next 10 minutes is the same as the probability of no calls in any 10 minute interval, $\underline{\underline{0.3679}}$.

Exercise 3:

a) We first find:

$$F_X(x) = \int_0^x 0.02ue^{-0.01u^2} du = [-e^{-0.01u^2}] = 1 - e^{-0.01x^2}$$

Further we have:

$$\begin{aligned} F_U(u) &= P(U < u) = P(\min(X_1, X_2, X_3) < u) = 1 - P(\min(X_1, X_2, X_3) > u) \\ &\overset{indep.}{=} 1 - P(X_1 > u) \cdot P(X_2 > u) \cdot P(X_3 > u) = 1 - [1 - F_X(u)]^3 \\ &= 1 - [e^{-0.01u^2}]^3 = \underline{\underline{1 - e^{-0.03u^2}}} \end{aligned}$$

We could also calculate the probability density: $f_U(u) = F'_U(u) = \underline{\underline{0.06ue^{-0.03u^2}}} \quad , u > 0$

b)

$$\begin{aligned} P(X < 5) &= F_X(5) = 1 - e^{-0.01 \cdot 5^2} = \underline{\underline{0.221}} \\ P(U < 5) &= F_U(5) = 1 - e^{-0.03 \cdot 5^2} = \underline{\underline{0.528}} \end{aligned}$$

c) If we compare the probability density in a) with the Weibull distribution,

$$f(x) = \alpha\beta x^{\beta-1} e^{-\alpha x^\beta} \quad , x \geq 0,$$

we see that X is having a Weibull distribution with $\alpha = 0.01$ and $\beta = 2$, while U is having a Weibull distribution with $\alpha = 0.03$ and $\beta = 2$. From the expression for the expectation in the Weibull distribution, $\mathrm{E}(X) = \alpha^{-1/\beta}\Gamma(1 + \frac{1}{\beta})$ we then get

$$\begin{aligned} \mathrm{E}(X) &= 0.01^{-1/2}\Gamma(1 + \frac{1}{2}) = 0.01^{-1/2}\frac{1}{2}\Gamma(\frac{1}{2}) = 0.01^{-1/2}\frac{1}{2}\sqrt{\pi} = \underline{\underline{8.86}} \\ \mathrm{E}(U) &= 0.03^{-1/2}\Gamma(1 + \frac{1}{2}) = 0.03^{-1/2}\frac{1}{2}\Gamma(\frac{1}{2}) = 0.03^{-1/2}\frac{1}{2}\sqrt{\pi} = \underline{\underline{5.12}} \end{aligned}$$