STA500 Introduction to Probability and Statistics 2, autumn 2018.

Exercise set 9

Note on Markov Processes: Exercise 1, 2, 3, 4, 5.

Exercise 1:

Suppose that the chance of rain tomorrow only depends on whether it is raining today and not on past weather conditions. Let the states be labelled, 0: no rain, 1: rain. Suppose that:

$$\begin{aligned} P(\text{rain tomorrow}|\text{rain today}) &= 0.75 \\ P(\text{rain tomorrow}|\text{not rain today}) &= 0.35 \end{aligned}$$

a) Find the transition probability matrix. Show that:

$$P^2 = \begin{pmatrix} 0.51 & 0.49 \\ 0.35 & 0.65 \end{pmatrix}$$

b) If it is raining today, what is the probability that it is not raining tomorrow?
If it is not raining today, what is the probability that it will be raining the day after tomorrow?
If it was raining yesterday, what is the probability that it will not be raining tomorrow?

Suppose now that the chance of rain on a day does not only depend on the weather the previous day, but also on the weather two days ago. Then we can not use the simple Markov model above, but we can actually still model the situation as a Markov chain by augmenting the state space to contain the weather sequence the two last days (see also example 12.11 in Ghahramani). Let state 0: ss (sun-sun), state 1: rs (rain-sun), state 2: sr (sun-rain), state 3: rr (rain-rain). Let $\{X_n : n = 0, 1, 2, \ldots\}$ be a stochastic process with the four states above as state space.

Suppose the chance of rain is 0.3 if it was not raining the two previous days, 0.5 if it was not raining the day before but was raining two days before, 0.6 if it was raining the day before but not two days before and 0.8 if it was raining the two previous days.

c) Explain why $\{X_n : \ n = 0, 1, 2, \ldots\}$ is a Markov chain with transition probability matrix

$$P = \begin{pmatrix} 0.7 & 0 & 0.3 & 0 \\ 0.5 & 0 & 0.5 & 0 \\ 0 & 0.4 & 0 & 0.6 \\ 0 & 0.2 & 0 & 0.8 \end{pmatrix}$$

It is given that:

$$P^2 = \begin{pmatrix} 0.49 & 0.12 & 0.21 & 0.18 \\ 0.35 & 0.20 & 0.15 & 0.30 \\ 0.20 & 0.12 & 0.20 & 0.48 \\ 0.10 & 0.16 & 0.10 & 0.64 \end{pmatrix}$$

d) It is raining today, but was not raining yesterday, what is the probability that it is not raining tomorrow? If it was not raining today and yesterday, what is the probability it will be raining both the two next days? If it was raining yesterday, but not today, what is the probability it will be raining tomorrow but not the day after?

Exercise 2:

Using certain criteria the stock marked has what could be called a bad day (state 0), an average day (state 1) or a good day (state 2). Let X_n be the state on day n. The process $\{X_n : \ n = 0, 1, 2, 3, \ldots\}$ is assumed to be a Markov chain with the following transition matrix,

$$P = \begin{pmatrix} 0.7 & 0.2 & 0.1 \\ 0.3 & 0.5 & 0.2 \\ 0.1 & 0.4 & 0.5 \end{pmatrix}$$

and with

$$P^2 = \begin{pmatrix} 0.56 & 0.28 & 0.16 \\ 0.38 & 0.39 & 0.23 \\ 0.24 & 0.42 & 0.34 \end{pmatrix}.$$

a) Find $P(X_1 = 2|X_0 = 0)$, $P(X_4 = 1|X_3 = 0, X_2 = 1)$, $P(X_2 = 1|X_0 = 1)$ and $P(X_{21} = 1|X_{19} = 2)$.

(Wait with point b) below until you have learned about steady state probabilities)

b) Find the steady state (equilibrium) equations for this system and solve them. If the system starts in $X_0 = 0$, what is the probability that $P(X_n = 0|X_0 = 0)$ when n becomes large?

Some answers:

Note on Markov Processes: 1 Classes $\{0, 1\}$, $\{2, 3\}$. **3** Irreducible and aperiodic. **5** $\pi_0 = 1/(2 - p)$.

Exercise 1 b) 0.25, 0.49 and 0.35; d) 0.4, 0.18, and 0.20.

Exercise 2 a) 0.1, 0.2, 0.39 and 0.42; b) 17/40, 14/40 and 9/40, 17/40;