MAT331: Infinite-dimensional geometry

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Analysis and geometry on infinite-dimensonal spaces is an active research field with many applications in mathematics and physics. We would like to point out two examples:

- Spaces of differentiable functions can only be modelled using infinitely many parameters. (Riemannian) Geometry on these spaces is studied extensively in the field of shape analysis. This theory is motivated by a plethora of applications ranging from medical imaging to computer graphics.
- Groups arising from problems related to differential geometry, fluid dynamics and the symmetry of evolution equations are often naturally infinite dimensional manifolds with smooth group operations. Prime examples are the diffeomorphism groups $\mathrm{Diff}(K)$ for K a smooth and compact manifold. If K is a three dimensional torus, the group $\mathrm{Diff}(K)$ arises naturally in fluid mechanics. The motion of a particle in the fluid corresponds, under periodic boundary conditions, to a curve in $\mathrm{Diff}(K)$, see [2].

To deal with these examples, one frequently needs to leave the theory of differential geometry on Banach manifolds [4]: For example, due to Omori's theorem, it is impossible to modell $\mathrm{Diff}(K)$ on Banach spaces.

In the lecture we will give an introduction to infinite-dimensionale (differential) geometry. We shall explain the notions of calculus and manifolds in the general setting. Then we will investigate some of the easiest non-trivial (and often used examples) of infinite-dimensional manifolds: spaces of differentiable mappings and their submanifolds. This will allow us to highlight two basic concepts of infinite-dimensional geometry: Lie groups (such as the famous loop groups from physics) and Riemannian metrics (such as the L^2-metric from shape analysis). Note that we will only deal with the easiest examples of this complex theory in the lecture. In particular, we begin with the basics (What is...? infinite-dimensional calculus, a Lie group, a Riemannian metric etc.) and develop the theory from the basics.

Prerequesite knowledge MAT242 Topology, MAT243 Manifolds,
optional (would allow interesting connections if sufficiently many of the participants have the knowledge): MAT232 Functional analysis, MAT342 Differential geometry

Target audience Master and PhD students at the norwegian universities. The course will be held at the university of Bergen in fall 2020 with three meetings of 2-3 days between September and November 2020. As the course is supported by Trond Moen Stiftelse travel support is available to attend the meetings at the university of Bergen. In case travelling and meetings are not permitted due to the Corona virus pandemic in fall, the course will be given via digital meetings.

If you are interested in the lecture (especially if you are based at another norwegian university) please contact

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for signing up and for more information.

Topics to be discussed in the lecture

The following list of topics can still be ammended with other topics depending on the interest of the audience.

1. Infinite-dimensional calculus beyond Banach spaces, [3]
    - What is a locally convex space (and why should you care)?

- Bastiani calculus (vs. convenient calculus)
- infinite-dimensional manifolds modelled on locally convex spaces
- elements of differential geometry: submersions and immersions

2. Spaces and manifolds of differentiable mappings [1, Appendix A]
   - the locally convex spaces $C^\ell(K,E)$ for K compact, E a locally convex space
   - the exponential law $C^\infty(K \times L, E) \cong C^\infty(K, C^\infty(L,E))$ and its consequences
   - the (topological) spaces $C^\ell(K,M)$ for a manifold M.

3. Lifting geometry to mapping spaces I: Lie groups
   - What is an (infinite-dimensional) Lie group, [5]?
   - the current groups $C^\ell(K,G)$ for a Lie group G; Application: loop groups $LG = C^\infty(\mathbb{S}^1, G)$ [6]

4. Lifting geometry to mapping spaces II: (weak) Riemannian metrics
   - What is a (weak) Riemannian metric, [4]?
   - the L^2-metric on $C^\infty(K,H)$ for a Hilbert space H; Application: Shape analysis

References

[1] H. Amiri, H. Glöckner, A. Schmeding *Lie groupoids of mappings taking values in a Lie groupoid*, arXiv1811.02888.

[2] D.G. Ebin, J. Marsden, *Groups of diffeomorphisms and the motion of an incompressible fluid.* Annals of Mathematics 92 (1970)(1):102–163

[3] A. Kriegl, P.W. Michor *The convenient setting of global analysis.* Mathematical Surveys and Monographs, Vol. 5 AMS 1997

[4] S. Lang *Fundamentals of differential geometry*, Springer 1999

[5] K.–H. Neeb *Towards a Lie theory for locally convex groups*, Japanese Journal of Math. 1 (2006), 291-468

[6] A. Pressley, G. Segal *Loop groups*, Oxford University Press 1986