Kinematical Klein Geometries

José Miguel Figueroa-O'Farrill (University of Edinburgh)

I report on my visit to UiT, the Arctic University of Norway (Tromsø), during the month of May 2019.

In the beginning there was Newton. His famous equations model the universe as a four-dimensional affine space $\mathbb{A}^4 \to \mathbb{A}^1$ fibered over an affine line, which is to be interpreted as **time**. The fibres are affine spaces consisting of **simultaneous events**. The invariant notions in the Newtonian universe are the time intervals and the euclidean distance between simultaneous events and the **relativity group** which preserves them is the **galilean group**, a ten-dimensional Lie subgroup of the affine group $\mathrm{Aff}(\mathbb{A}^4)$ consisting of spatial rotations, translations in both space and time and (galilean) boosts.

The unified description of electromagnetic phenomena was based on the **Maxwell's equations**, which unlike Newton's equations, are not galilean invariant. Departing from this fact, Einstein arrived at his **special theory of relativity**, which Minkowski famously geometrised, proposing his eponymous spacetime: a four-dimensional affine space $\mathbb{A}^4$ with an invariant notion of **proper distance** between **spacetime events**. The Lie subgroup of $\mathrm{Aff}(\mathbb{A}^4)$ preserving the proper distance is the **Poincaré group** and it too is a ten-dimensional Lie group consisting of rotations, spatio-temporal translations and (lorentzian) boosts. Special relativity famously posits that there is a maximum speed (that of light), denoted c. The Newtonian universe can be understood as the $c \to \infty$ limit of Minkowski spacetime. In this limit, the Poincaré group **contracts** to the galilean group.

The galilean and Poincaré Lie groups are examples of **kinematical Lie groups** and they act transitively on Minkowski spacetime and the newtonian universe, respectively. In other words, the spacetimes of both Newton and Minkowski are homogeneous spaces of kinematical groups. Homogeneous spaces are the main subjects of Felix Klein's **Erlangen programme**, which proposes to study a geometry via its Lie group of automorphisms.

Half a century ago, Bacry and Lévy-Leblond formulated the classification problem of homogeneous spacetimes of kinematical Lie groups, launching what could be termed a **kinematical Erlangen programme**. They introduced the notion of a kinematical Lie algebra and gave a classification in (spacetime) dimension 4 subject to some "by no means compelling" assumptions which were lifted twenty years later by Bacry and Nuyts, completing the classification in that dimension. They observed that for every kinematical Lie algebra $\mathfrak{k}$ there is a **Klein pair** $(\mathfrak{k}, \mathfrak{h})$ suggesting the existence of an associated homogeneous spacetime. Nevertheless they stopped short of showing that every such kinematical Klein pair had a geometric realisation.

Using deformation theory, I recently rederived the Bacry–Nuyts classification of kinematical Lie algebras and extended it to arbitrary (spacetime) dimension — the three-dimensional case in collaboration with Tomasz Andrzejewski. These classifications formed the basis of recent work with Stefan Prohazka where we classified simply-connected kinematical Klein geometries in arbitrary dimension, refining and completing the programme started by Bacry and Lévy-Leblond more than 50 years ago. Except for some *exotic* two-dimensional spacetimes, all others fall into one of several classes, depending on which invariant structure they possess: **lorentzian**, **riemannian**, **galilean**, **carrollian** or **aristotelian**. In our most recent work, arXived shortly before arriving to Tromsø, Stefan Prohazka, Ross Grassie and I have continued the study of these homogeneous geometries. We determined their invariant connections and their infinitesimal (conformal) automorphisms, which in the galilean and carrollian cases are typically infinite-dimensional and reminiscent of the famous BMS (Bondi–Metzner–Sachs) Lie algebra of asymptotic symmetries of asymptotically flat spacetimes. In fact, I believe that this is not an accident and hope to explain this in a more general setting.

Kinematical Klein geometries include and generalise the maximally symmetric lorentzian manifolds, and they are expected to play a similar starring rôle in our further understanding of holography and quantum theories of gravity, to that which Minkowski and the de Sitter spacetimes have played in quantum field theory and the gauge/gravity correspondence.