Report on the visit of Erwan Brugallé to UiO in Spring 2019

My collaboration with Kristin Shaw is currently oriented towards applications of tropical homology, recently introduced by Itenberg, Katzarkov, Mikhalkin and Zharkov. This theory, initially developed to study non-singular tropical varieties, has now been extended to the more general framework of polyhedral complexes (with no balancing condition nor notion of non-singularity). This generalization provides, in particular, an algebro-geometric intuition for more combinatorial objects. Surprisingly, this intuition turned out to be fruitful to establish combinatorial analogs of theorems from algebraic geometry even for polyhedral complexes that are far from algebraic geometry. For example, there are polyhedral versions of the Lefschetz $(1,1)$ Theorem and also Poincaré duality for matroidal spaces.

During my visit to Oslo I continued a research project with Kristin Shaw (UiO), Benoît Bertrand (Toulouse, France), and Arthur Renaudineau (Lille, France) on applications of a recent work by Shaw and Renaudineau to real algebraic geometry. The study of the topology of real algebraic varieties is a classical subject, that goes back to Hilbert's 16th problem asking for the classification of isotopy types realized by real algebraic curves of a fixed degree in the real projective plane $\mathbb{R}P^2$. It has enjoyed spectacular progress since 1970, when it has been given a new birth essentially under the influence of Arnol'd, Rokhlin, Viro and Kharlamov. Since then, the area has known several tremendous developments and breakthroughs, in particular through the works of people like Itenberg, Mikhalkin, or Welschinger. For a real algebraic variety X of any dimension, the Smith-Thom inequality provides an upper bound for the total Betti number of the real part of the real part $\mathbb{R}X$ of X. The problem of determining when this bound is sharp remains widely open. Itenberg and Viro claimed to have establish sharpness for projective hypersurfaces 25 years ago, nevertheless the proof has never been published.

In his thesis, Haas provided necessary and sufficient conditions for curves in $\mathbb{R}P^2$ obtained from a procedure called patchworking to be maximal in the sense of the Smith-Thom inequality. In a joint work with Bertrand, Renaudineau and Shaw, we are working on generalizations of Haas Theorem for curves to real varieties of all dimensions arising from patchworking. The initial Haas Theorem for curves was proven prior to tropical homology but turns out to have a natural interpretation in the homological realm. Furthermore, this point of view promises to be fruitful for generalizations to higher dimensions. As a main application, we are working on a new proof of the aforementioned Itenberg-Viro's claim about the existence of maximal projective hypersurfaces of any degree and dimension.

During my visit, I also gave a series of two talks in the algebra seminar and the ASGARD conference. The topic of these lectures were enumerative geometry, from the three different but intertwining viewpoints of real, complex and tropical geometry.

Finally, I have been a member of the jury for masters student Edvard Aksnes' thesis defence. The topic of Aksnes' thesis was to give necessary and sufficient conditions for polyhedral fans to exhibit Poincaré duality for tropical homology, extending the current results known in the case of matroids to a larger combinatorial realm.