Sup Norm Estimates for the d-bar Equation

The Cauchy-Riemann equation is central in the study of functions of complex variables.

A function is holomorphic on a domain if and only if it satisfies the Cauchy-Riemann equation in each variable. This can be formulated as follows

$$\sum_{j=1}^{n} \frac{\partial f}{\partial \overline{z}_j} d\overline{z}_j \equiv 0$$

in Ω.

A central subject in the field is to decide if a holomorphic function can have certain properties. An example is: Given a domain Ω and a boundary point p, is it possible to find a function f which is continuous on the closure of Ω and holomorphic on Ω such that

- $f(p) = 1$
- $|f(z)| < 1$ on $\overline{\Omega} \setminus \{p\}$.

In dimension 1 this is easy if we have smooth boundary, but in higher dimension it is much more complicated.

A natural approach to such a question is to find a smooth function with the wanted properties and then modify it to get a holomorphic function with the same properties.

The procedure goes as follows: We start with our smooth function g, let $\omega := \sum_{j=1}^{n} \frac{\partial g}{\partial \overline{z}_j} = \overline{\partial} g$ which is a so called $(0,1)$-form. Now $\overline{\partial}\overline{\partial} = 0$ so this particular $(0,1)$-form is $\overline{\partial}$ closed.

Next we look for a "minimal" u such that $\overline{\partial} u = \omega$. Let $f = g - u$, then f is holomorphic.

We end up with a question of the following type:

Given a domain $\Omega \subset\subset \mathbb{C}^n$, a closed $(0,1)$-form ω, when can we find a "minimal" function u such that $\overline{\partial} u = \omega$.

In our case we want to know if ω bounded would imply that we can choose u, such that u also is bounded.

From early on one have known that this depends a lot on the geometry of Ω. For example it is easy to see that if Ω is not a domain of holomorphy (pseudoconvex), then there are bounded closed $(0,1)-$forms which do not admit a bounded solution to the equation $\omega = \overline{\partial} u$.

But it is not enough to require that Ω is pseudoconvex or a domain of holomorphy. Sibony found an example of a smoothly bounded pseudoconvex domain and a bounded $(0,1)$-form on the domain that did not admit a bounded solution.

On the other hand, in the 1970's Henkin was able to show that if Ω is strictly pseudoconvex (type 2), then there are integral kernels that give bounded solutions if ω is bounded.

In between just smooth pseudoconvex and strictly pseudoconvex are domains of finite type.

The method used by Henkin relied on the existence of holomorphic support functions. An example by Kohn and Nirenberg shows that this is not always the case when Ω is pseudoconvex and of finite type.

In the 1980's Fornaess and Range were able to work around this in the case where Ω is in $\mathbb{C}^2$.

When Ω is in $\mathbb{C}^n, n \geq 3$ there are a lot of added difficulties, like that type do not only change from point to point, but also in different directions at a given boundary point. Moreover one also has to deal with singular complex curves that meet the boundary to high order.

Together with Dusty Grundmeier and Lars Simon I have been able to handle most of the difficulties in complex dimension 3.

The Month in Oslo was spent on giving a series of lectures on the solution to this problem together with working out several of the estimates needed.

The paper has now been posted on arXiv.