Motivic homotopy, arithmetic invariants and absolute Galois groups

One of the most spectacular works in Mathematics in recent times is Voevodsky's proof of the Milnor and Bloch-Kato conjectures, for which Voevodsky was awarded the Fields medal. In order to do so Voevodsky, along with Morel, also introduced the area of motivic homotopy which is now a major research area on its own with very large impact. I am particularly interested in reconnecting this research direction with its roots, which lie in the study of absolute Galois groups, and attack with its help the Massey vanishing conjecture, which is likely to have a similar importance in the development of the field as the Milnor and Bloch-Kato conjectures were.

The Massey vanishing conjecture claims that certain higher homotopical products vanish in the Galois cohomology of fields. It gives a very strong restriction on the possible structure of absolute Galois groups, for example it implies that a large class of embedding problems over any field are always solvable. It generated a lot of interest in the last few years; it was proved for triple Massey products of degree one elements in general, and more spectacularly it was proved for number fields by Harpaz–Wittenberg, relying on their earlier very deep Diophantine result on the fibration method, which in turn relies on the similarly very deep Green–Tao theorem.

My activities during my one month stay in Norway included a lecture series, which covered my joint work with Schlank on the computation of Massey products, which uses the Beilinson-Lichtenbaum conjectures to transport the computation of Massey products from Galois theory to motivic cohomology, which is much more geometric in nature. Another part of my lecture series were on my joint work with Krishnamoorty on an analogue of Simpson's conjecture over finite fields. We expect that an l-adic Galois representation of the étale fundamental group of a smooth quasi-projective variety should be motivic in origin under mild assumptions. In this joint work with Krishnamoorthy I recently proved a special case of this conjecture, which characterises all l-adic representations which are coming from elliptic curves. The proof relies on the Langlands correspondence over global function fields and the theory of p-adic coefficients.

I also talked about my joint work with Frank Neumann on an arithmetic refinement of the Yau–Zaslow formula by replacing the classical Euler characteristic in Beauville's argument by a variant of Levine's motivic Euler characteristic. The result is strong enough to derive several similar formulas for other related invariants, including Saito's determinant of cohomology, and a generalisation of a formula of Kharlamov and Rasdeaconu on counting real rational curves on real K3 surfaces. As a part of my activities in Norway I also visited Trondheim for a week, where I gave a lecture on the aforementioned arithmetic refinement of the Yau–Zaslow formula, and worked on a joint project with Gereon Quick on a question of Hopkins and Wickelgren. In their seminal paper on the vanishing of triple Massey products Hopkins and Wickelgren asked if a stronger property is true, namely the differential graded algebra of continuous cochains of the absolute Galois group of a field is formal. Positselsky provided counterexamples, but we can prove that the question has a positive answer in a large class of non-trivial cases, including function fields of curves over real fields. Our method uses Schneiderer's calculation of the Galois cohomology of these fields, Kadeishvili's work on A_∞-algebras and computations in Hochschild cohomology.