THE HARMONIC COMPONENT OF A HOMOGENEOUS POLYNOMIAL

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Abstract. We consider geometric constructions related to the projection of the space of homogeneous polynomials of degree n on $\mathbb{R}^3$ onto its subspace of harmonic polynomials.

1. Introduction

The spherical harmonics were introduced by Legendre in 1782. The name "spherical harmonics", that is common now, at first was used by Thomson (Lord Kelvin) and Tait in the early (1867) edition of their *Treatise on Natural Philosophy* (see [14]). The word "harmonic" relates to harmonic functions. Recall that they are defined as the solutions to the equation $\Delta f = 0$, where $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is the Laplace operator.[1] The spherical harmonics of degree n are the harmonic homogeneous polynomials on $\mathbb{R}^3$ of this degree as well as their traces on the unit sphere S^2. We denote by $\mathcal{H}_n$ both spaces. The traces on S^2 are eigenfunctions for the eigenvalue $-n(n+1)$ of the angular part Δ_{S^2} of Δ, which is the Laplace–Beltrami operator on S^2.

1.1. **Brief hystory.**

1.1.1. *Celestial mechanics.* The basic contribution to the theory of spherical harmonic was made by Legendre and Laplace in the middle 1780th. They were motivated by problems of the Celestial mechanics, in particular, by the problem of the stability of the Solar system. Due to Newton's gravitation law, many problems of astronomy can be modeled by a perturbation of the motion of a point mass in the central gravitational field. Its potential, up to a shift, is proportional to $\frac{1}{r}$, where

$$r = \sqrt{x^2 + y^2 + z^2}.$$

Note that $\frac{1}{r}$ is SO(3)-invariant and harmonic. The perturbation is a sum of spherical harmonics. Using approximation of this kind, Laplace solved many problems of the Celestial mechanics. For example, he explained the acceleration of the Moon motion by the long period oscillation of the eccentricity of the Earth orbit. Thus, a time will come for the deceleration.

[1] Basic definitions have obvious extensions onto higher dimensions but we consider only the dimension 3. It has a rich history and admits some special constructions.

It is widely believed that the harmonics are the spherical analogue of a Fourier series. Fourier attended École Royale Militaire of Auxerre at the age 14 when Laplace introduced the functions $Y_{n,m}$ that form an orthonormal basis in $\mathcal{H}_n$. They are eigenfunctions of the circle group of the rotations about some axis. Thus the consruction of spherical harmonics includes the idea of the Fourier series.

1.1.2. *Electricity and Magnetism.* In 19th century, the theory of electromagnetic fields had similar problems. Maxwell in his *Treatise on electricity and magnetism* suggested a method for constructing the spherical harmonics by differentiation of the function $\frac{1}{r}$ in suitable directions several times. He treated them as a scaling limits of the fields of several points charges with zero total charge as the distance between points tends to zero (see [10, Ch. 9]) for details). Maxwell did not give a rigorous mathematical proof.

The common points of S^2 and the rays in the mentioned directions are called poles. Sylvester noticed that a real harmonic uniquely defines the set of poles up to their signs and gave a sketch of the proof at the end of his note [13]. The book *Methoden der Mathematischen Physik* by Courant and Hilbert contains the complete proof ([5, Ch. 7]). Vladimir Arnold clarified the topological meaning of this construction (see [2, Appendix A]).

Maxwell applied his theory of poles to computation of the integrals of the products of two harmonics from $\mathcal{H}_n$ by differentiation of one of them in directions of the poles of the other (see [10, Ch. 9, (31)]). This contains implicitly the ideas of the scalar product of subsection 2.2 as well as the projection of subsection 3.1 that seems that this was not noticed yet. His ideas in this area were rediscovered several times (for instance, [3, Ch. 5], [4]).

1.1.3. *Hydrogen atom.* The 20th century clarified the significance of the notion of symmetry in both the mathematics and the physics. An outstanding example is the hydrogen atom. The Schrödinger equation for the wavefunction ψ of the electron in it may be written as

$$\left(\Delta + \frac{K}{r}\right)\psi = -LE\psi,$$

where K, L are some constants depending on the mass, the charge of the electron, and the Planck constant. It is evidently SO(3)-invariant. The separation of variables leads to the solutions of the type $f(r)h$, where h is a spherical harmonic. Together with rules of quantum mechanics this makes it possible to build a very satisfactory model of the hydrogen atom, in particular, to find all possible energy levels and the radii of the electron orbits.

1.1.4. *CMB.* In 21th century we have a new challenge which necessarily involves the spherical harmonics. As in 18th century, it comes from the sky. This is *the Cosmic Microwave Background* (CMB) previously known as *the* *relic radiation*. It fills the universe since an early stage of its evolution and contains the valuable records about it. The radiation is almost isotropic but non-isotropic. The desired information is hidden in the perturbations, which can be approximated by spherical harmonics. It is important to know if the statistic of CMB is Gaussian or not because the predictions of cosmological parameters depend of this property. There is the well known decomposition

$$\mathcal{P}_n = \sum\nolimits_{j \in J_n} r^{n-j} \mathcal{H}_j, \tag{1.1}$$

where $J_n = \{j \in \mathbb{Z} : 0 \le j \le n,\ n-j \text{ even}\}$ and $\mathcal{P}_n$ is the space of the homogeneous polynomials of degree n. For any Gaussian SO(3)-invariant distribution its components are pairwise independent but their statistic shows a strong correlation between the harmonics of some different degrees (see [11], [4]). To explain this is one of the problems of the modern cosmology.

1.2. **About this note.** The spherical harmonics is a useful tool in mathematics and its applications. It is also a very interesting object in its own. The aim of this note is to show links of spherical harmonics to some mathematical tools and objects. It contains no new result (maybe, except for Theorem 3.4).

Unless the contrary is explicitly stated we assume that the scalar product is that of $L^2(S^2)$ and that the functions are real valued. Complex function spaces are equipped with the upper index $\mathbb{C}$. Everywhere "invariant" means "SO(3)-invariant".

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2. The projection onto $\mathcal{H}_n$

The summands of (1.1) are irreducible and pairwise non-equivalent. Hence every commuting with SO(3) linear operator in $\mathcal{P}_n$ preserves all spaces $\mathcal{H}_j$ and is either trivial or invertible in each of them. Let

$$\pi_n : \mathcal{P}_n \to \mathcal{H}_n$$

be the projection that agrees with (1.1). It follows that π_n is independent of the choice of the invariant inner product in $\mathcal{P}_n$. Also, the invariant inner products in $\mathcal{H}_j$ may differ only by a scaling factor.

In this note we describe several geometric constructions related to the following problem: given $p \in \mathcal{P}_n$, find $h_n = \pi_n p$. If we know how to find h_n, then we may obtain the decomposition $p = h_n + r^2 h_{n-2} + \dots$ replacing p with $\frac{p-h_n}{r^2}$ repeatedly.

2.1. **The reproducing kernel.** For the completeness, we mention the standard integral representation

$$\pi_n p(u) = \int_{S^2} \Phi_n(u,v) p(v)\, d\sigma(v),$$

where $\Phi_n(u,v) = \sum_{k=1}^{2n+1} f_k(u) f_k(v)$ for every orthonormal bases $f_1, \dots, f_{2n+1}$ in $\mathcal{H}_n$. Also,

$$\Phi_n(u,v) = \langle \phi_u, \phi_v \rangle\,,$$

where $u \to \phi_u$ is the evaluation mapping defined by $\langle f, \phi_u \rangle = f(u)$ for all f in $\mathcal{H}_n$. The function $\phi_u(v) = \Phi_n(u,v)$ is the zonal harmonic $c_n P_n(\langle u,v\rangle)$, where P_n the nth Legendre polynomial, c_n is a normalizing constant, $\langle\ ,\ \rangle$ stands for the scalar product of $L^2(S^2,\sigma)$, where σ is the invariant probability measure on S^2, and $u, v \in S^2$.

2.2. **Another scalar product in $\mathcal{P}_n$.** The formula

$$\langle p, q \rangle = p\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) q.$$

defines the well known and a very useful inner product in $\mathcal{P}_n$. It is invariant and has the remarkable property

$$\langle pq, f \rangle = \left\langle q, p\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) f \right\rangle$$

which immediately follows from the definition. In particular, the above equality implies that $h \in \mathcal{P}_n$ is orthogonal to $r^2\mathcal{P}_{n-2}$ if and only if $\Delta h = 0$, where Δ is the Euclidean laplacian. Thus,

$$\mathcal{P}_n = r^2\mathcal{P}_{n-2} \oplus \mathcal{H}_n,$$

where the sum is orthogonal. Notice that distinct monomials are orthogonal. This property together with the equality

$$|x^j y^k z^l|^2 = j!k!l!$$

provides another definition which shows that the inner product is positive definite.

This scalar product must be proportional on $\mathcal{H}_n$ to that of $L^2(S^2)$ since $\mathcal{H}_n$ is irreducible and they are both invariant. Thus we can compute integrals of a product of two harmonics by differentiation. Actually, this method is due to Maxwell ([10, Ch. 9, (31)]). It had been rediscovered several times ([3],[12]).

2.3. **A restatement of the problem.** For any $p \in \mathcal{P}_n$ we have the unique decomposition

$$p = r^2 q + h, \tag{2.1}$$

where $q \in \mathcal{P}_{n-2}$ and $h \in \mathcal{H}_n$ and the problem "given p, find $\pi_n p$" may be stated as follows:

given $p \in \mathcal{P}_n$, find $h \in \mathcal{H}_n$ such that $p - h$ is divisible by r^2.

2.4. **Spherical harmonics on $\mathbb{C}^3$ and polynomials on $\mathbb{C}^2$.** To avoid awkward notation, we write $\mathcal{P}_{2n}^{\mathbb{C}}$ for complex homogeneous polynomials on $\mathbb{C}^2$ and $\mathcal{P}_n^{\mathbb{C}}$ for that on $\mathbb{C}^3$ of degrees $2n$ and n, respectively (i.e., we drop $\mathbb{C}^2$ and $\mathbb{C}^3$). We have

$$\dim \mathcal{H}_n^{\mathbb{C}} = \dim \mathcal{P}_{2n}^{\mathbb{C}} = 2n+1.$$

The representations of the groups SO(3) and SU(2) in these spaces are equivalent.[2] This can be verified by a simple geometric construction. Let $\mathcal{Q}_0$ be the cone $r^2 = 0$. The mapping

$$\kappa(\zeta) = \left(\frac{\zeta_1^2 + \zeta_2^2}{2}, i\frac{\zeta_2^2 - \zeta_1^2}{2}, i\zeta_1\zeta_2 \right)$$

covers $\mathcal{Q}_0$ twice by $\mathbb{C}^2$ identifying $\pm\zeta$. We omit the computation which shows that κ intertwines the actions of SO(3) on $\mathcal{Q}_0$ and SU(2) on $\mathbb{C}^2$. Set

$$Rp = p \circ \kappa.$$

Since $r^2 \circ \kappa = 0$, Rp is independent of the first summand of the decomposition (2.1). Also, R agrees with the actions of SU(2) in $\mathcal{P}_{2n}$ and SO(3) in $\mathcal{H}_n$. The representations in these spaces are irreducible, hence either $R = 0$ or R is invertible on $\mathcal{H}_n$. The first is obviously false whence the second is true. Thus, the mapping $R : \mathcal{H}_n \to \mathcal{P}_{2n}$ is a bijection which intertwines the group actions.

2.5. **How to find the harmonic component of p knowing Rp.** It is sufficient to find the harmonics in $\mathcal{H}_n^{\mathbb{C}}$ relating to the monomials in $\mathcal{P}_{2n}^{\mathbb{C}}$. Put

$$\begin{aligned} \zeta &= x + iy, \\ \bar\zeta &= x - iy, \\ \omega &= -iz. \end{aligned}$$

The bar does not mean the complex conjugation: we consider $\zeta, \bar\zeta, \omega$ as functions on $\mathbb{C}^3$. Let $c = (a, b) \in \mathbb{C}^2$. An easy calculation shows that

$$(\zeta, \bar\zeta, \omega) \circ \kappa(c) = (a^2, b^2, ab).$$

Hence

$$R(\zeta^j \bar\zeta^k \omega^l) = a^{2j+l} b^{2k+l}.$$

Let us fix $\alpha = 2j + l$ and $\beta = 2k + l$. If $Rp_1 = Rp_2 = a^\alpha b^\beta$ for some $p_1, p_2 \in \mathcal{P}_n$, then $R(p_1 - p_2) = 0$ and consequently r^2 divides $p_1 - p_2$. Therefore, there is at most one harmonic polynomial h in the linear span of the products $p = \zeta^j \bar\zeta^k \omega^l$ such that $Rp = a^\alpha b^\beta$. Such a polynomial exists since $Rp \neq 0$ for any p with this property. In fact,

$$h = \begin{cases} \zeta^m p_{n,m}, & \alpha \geq \beta, \\ \bar\zeta^m p_{n,m}, & \alpha \leq \beta, \end{cases}$$

[2]Actually, we have the action of $\mathrm{SO}(3) = \mathrm{SU}(2)/\mathbb{Z}_2$ in $\mathcal{P}_{2n}$ since SU(2) has nontrivial kernel $\pm I$.

where $n = \frac{\alpha+\beta}{2}$, $m = \frac{|\alpha-\beta|}{2}$, and the polynomial $p_{n,m}$ depends only on r^2 and z. The common case $\alpha = \beta$ relates to the Legendre polynomial P_n. Then h is the zonal harmonic. If $m > 0$, then $\zeta^m p_{n,m}$ corresponds to the classical $Y_{n,m}$.

2.6. **A determinant formula.** Let $\langle\ ,\ \rangle$ be the bilinear extension of the standard inner product in $\mathbb{R}^3$ onto $\mathbb{C}^3$. Set

$$\ell_u(v) = \langle v, u\rangle\,.$$

If $u \in \mathcal{Q}_0$, then ℓ_u^n is harmonic for all $n \in \mathbb{N}$. An easy calculation verifies this.

The mapping κ defines one-to-one correspondence between lines in $\mathbb{C}^2$ (i.e. $\mathbb{C}P^1$) and lines in the cone $\mathcal{Q}_0 \subset \mathbb{C}^3$ since κ glues points which lies in the same line. A polynomial $q \in \mathcal{P}_{2n}$ of two complex variables a, b is a product of $2n$ linear functions due to the fundamental theorem of algebra. For a generic q the set of its zeros is the union of $2n$ lines. The same is true for the restriction of $p \in \mathcal{P}_n$ to $\mathcal{Q}_0$ if

$$p \circ \kappa = q.$$

Let h be the harmonic component of p and let $c_1, \dots, c_{2n} \in \mathcal{Q}_0$ be generating vectors for these lines. Then the determinant

$$H_u(v) = \det\begin{pmatrix} \langle c_1, c_1\rangle^n & \dots & \langle c_1, c_{2n}\rangle^n & \langle c_1, u\rangle^n \\ \vdots & \ddots & \vdots & \vdots \\ \langle c_{2n}, c_1\rangle^n & \dots & \langle c_{2n}, c_{2n}\rangle^n & \langle c_{2n}, u\rangle^n \\ \langle v, c_1\rangle^n & \dots & \langle v, c_{2n}\rangle^n & \langle v, u\rangle^n \end{pmatrix}$$

vanishes on the same lines in $\mathcal{Q}_0$ as p. Hence $H_u \circ \kappa$ is proportional to q and, consequently, the harmonic component of H_u is equal to th for some $t \in \mathbb{C}$. A computation with determinants shows that $t \neq 0$ for a generic $u \in \mathcal{Q}_0$. Furthermore, the functions $\ell_{c_k}^n$ are harmonic since $c_k \in \mathcal{Q}_0$. Hence H_u is harmonic if $u \in \mathcal{Q}_0$. Then it is proportional to h since $H_u(c_k) = 0$. Replacing $\langle v, u\rangle^n$ in the right lower corner with the holomorphic extension of the zonal harmonic $\phi_u(v)$ on u, we get the equality

$$h(v)h(u) = \tau(u)H_u(v),$$

where τ is a nontrivial function on $\mathcal{Q}_0$ and v runs over $\mathbb{C}^3$.

There is a real version of this representation for harmonics on S^2 with $\Phi_n(c_k, c_j)$ instead of $\langle c_k, c_j\rangle^n$ which makes it possible to construct harmonics with prescribed zeros in S^2. See [7] for details.

Thus we obtain a circuitous path to compute $h = \pi_n p$. We'll keep trying to find others.

2.7. **The Casimir operator.** Let $D_{xy} = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}$ be the vector field corresponding to the group of the rotations about the z-axis and let D_{yz}, D_{zx} be defined similarly. Set

$$C = D_{xy}^2 + D_{yz}^2 + D_{zx}^2.$$

The operator C commutes with the group action and annihilates radial functions. Hence it commutes with the multiplication on them. The spaces $\mathcal{H}_n$ are its eigenspaces with the eigenvalues $\lambda_n = -n(n+1)$. Due to the decomposition (1.1), for any $p \in \mathcal{P}_n$ we have $p = \sum_{j\in J_n} r^{n-j} h_j$, where $h_j \in \mathcal{H}_j$. Then

$$Cp = \sum_{j\in J_n} \lambda_n r^{n-j} h_j.$$

It follows that

$$\pi_n = \prod\nolimits_{j\in J_{n-2}} (\lambda_n - \lambda_j)^{-1}(C - \lambda_j I).$$

The operator C is the Casimir operator for the group SO(3) acting on $\mathbb{R}^3$ or $\mathbb{C}^3$. It can be defined in more general setting as follows. Let ρ denote a representation of a Lie algebra $\mathfrak{g}$ and its extension onto the universal enveloping algebra $\mathfrak{U}_\mathfrak{g}$. If z is a central element of $\mathfrak{U}_\mathfrak{g}$, then $\rho(z)$ is called the Casimir operator. If $E_1, \dots, E_m$ is an orthonormal basis in $\mathfrak{g}$ for an $\mathrm{Ad}_\mathfrak{g}$-invariant scalar product, then $\rho(E_1)^2 + \cdots + \rho(E_m)^2$ is a Casimir operator.

The definition of C allows to treat it as an operator acting in spaces of smooth functions on S^2. Then it coincides with the Laplace–Beltrami operator on S^2 defined by the standard Riemannian metric on it. Sometimes it is more convenient to work with C than with Δ_{S^2} in local coordinates.

3. Maxwell's multipoles

3.1. **Maxwell's projection.** The above mentioned Maxwell's method for constructing spherical harmonics in fact defines the projection onto $\mathcal{H}_n^\mathbb{C}$.

Theorem 3.1. *The operator* $\mu_n : p \to \frac{(-1)^n r^{2n+1}}{(2n-1)!!} p\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \frac{1}{r}$ *coincides with the projection* $\pi_n : \mathcal{P}_n^\mathbb{C} \to \mathcal{H}_n^\mathbb{C}$.

Proof. It is an easy calculation that $\mu_n \mathcal{P}_n^\mathbb{C} \subseteq \mathcal{P}_n^\mathbb{C}$. It follows from the definition that μ_n annihilates the polynomials $r^2 q$ because $\frac{1}{r}$ is harmonic on $\mathbb{R}^3 \setminus \{0\}$. Due to (2.1), it is sufficient to prove that μ_n is identical on $\mathcal{H}_n^\mathbb{C}$.

If $r^2(a) = 0$, then ℓ_a^n is harmonic for all $n \in \mathbb{N}$ and, moreover,

$$\begin{aligned} \ell_a\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \frac{1}{r^s} &= -\frac{s\ell_a}{r^{s+2}}, \\ \ell_a\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \ell_a^j &= 0,\ j \in \mathbb{N}. \end{aligned}$$

Therefore, $\mu_n \ell_a^n = c_n \ell_a^n$, where c_n depends only on n. A computation shows that $c_n = 1$. Hence μ_n is identical on the complex linear span of the family of the functions $\{\ell_a^n : a \in \mathcal{Q}_0\}$ that equals $\mathcal{H}_n^\mathbb{C}$ since it is SO(3)-invariant and $\mathcal{H}_n^\mathbb{C}$ is irreducible. This completes the proof of the theorem. $\square$

3.2. **Multipoles of real harmonics.** Set

$$D_v f(x) = \frac{d}{dt} f(x + tv)|_{t=0} = \ell_v \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) f$$

for $v \in \mathbb{R}^3$. Maxwell suggested to obtain spherical harmonics in $\mathcal{H}_n$ applying $D_{v_1}, \dots, D_{\bar{v}_n}$ to the function $\frac{1}{r}$. Then we get $\mu_n p$, where $p = \ell_{v_1} \cdots \ell_{v_n}$.

Let us call a polynomial *decomposable* if it is the product of linear functions.

Theorem 3.2. *For any $h \in \mathcal{H}_n$ there is the unique real decomposable polynomial $p \in \mathcal{P}_n$ such that $\mu_n p = h$.*

It is assumed in the theorem that the factors are real linear functions. We wait with the proof until it became evident (see subsection 3.5). Notice that the product decompositions of p may differ by the ordering of the factors and by their scalings. The common points of S^2 and the lines $\mathbb{R}v_1, \dots, \mathbb{R}v_n$ are called *Maxwell's poles* or *multipoles.*

3.3. **Real decomposable harmonic polynomials.** The followings description of real harmonic polynomials which admit factorization to real linear functions was obtained by Agranovsky and Quinto in [1] in connection with the problem of a characterization of stationary points for solutions to the wave equation.

Theorem 3.3. *A real decomposable polynomial is harmonic if and only if its zero hyperplanes form a Coxeter system.*

The latter means that the reflections in the zero hyperplanes generate a finite linear group. The proof is based on two observations: first, a harmonic function is odd with respect to the reflection in any hyperplane that is contained in its nodal set and second, the euclidean Laplace operator commutes with the reflections and decreases the degree of the polynomial.

3.4. **Complex multipoles.** We assume the polynomials complex in the sequel. Since $\mu_n = \pi_n$, we have $Rh = Rp$ for $h = \pi_n p$. Suppose p decomposable and set $q = Rp$. Clearly,

$$p = \ell_1 \cdots \ell_n \implies q = (\ell_1 \circ \kappa) \cdots (\ell_n \circ \kappa).$$

Every factor $\ell_j \circ \kappa$ is quadratic. Recall that any homogeneous polynomial of two complex variables is decomposable. Hence $\ell_j \circ \kappa$ is the product of a couple of linear divisors of the decomposable polynomial $q \in \mathcal{P}^{\mathbb{C}}_{2n}$. Any partition by pairs (coupling for short) of its linear factors defines the product decomposition of q by quadratic ones. The mapping κ relates to a quadratic polynomial on $\mathbb{C}^2$ a linear function on $\mathbb{C}^3$. The correspondence is one-to-one. Thus a coupling defines a decomposable polynomial $p \in \mathcal{P}^{\mathbb{C}}_n$. Any such p corresponds to $2n$ points of $\mathbb{C}P^1$ (complex multipoles; every complex hyperplane in $\mathbb{C}^3$ either intersects the cone $\mathcal{Q}_0$ in two lines or touchs it) which in turn define it up to a constant factor.

The varieties of the decomposable complex polynomials were investigated in the paper [8] by G. Katz.

3.5. **Decomposable and harmonic polynomials on $\mathbb{C}^3$.** If the linear factors of $q \in \mathcal{P}_{2n}$ are pairwise non-proportional, then there are $(2n-1)!!$ couplings and each of them defines a decomposable polynomial $p \in \mathcal{P}_n^{\mathbb{C}}$ such that $Rp = q$.

If $Rp = Rh$, then r^2 divides $p - h$. If additionally $h \in \mathcal{H}_n^{\mathbb{C}}$, then $\pi_n p = h$ (see subsection 2.3). Moreover, h is the unique harmonic polynomial in $\mathcal{P}_n^{\mathbb{C}}$ such that $Rh = q$. Let

$\mathcal{D}_n$ denote the variety of decomposable polynomials in $\mathcal{P}_n^{\mathbb{C}}$,
$\mathfrak{L}_q$ be the set of all linear factors of q,
Π_q be the family of all partitions of $\mathfrak{L}_q$ by pairs, and, for $\varpi \in \Pi_q$,
q_ϖ be the product of linear functions on $\mathbb{C}^3$ corresponding to pairs in ϖ.

Let h be harmonic and let $q = Rh$. The above arguments show that

$$(h + r^2\mathcal{P}_{n-2}) \cap \mathcal{D}_n = \{p \in \mathcal{D}_n : \pi_n p = h\} = \{q_\varpi : \varpi \in \Pi_q\}. \tag{3.1}$$

This in particular implies that the set on the left is finite and, moreover, consists of $(2n-1)!!$ points counted with multiplicities. To prove this, it is sufficient to note that this is true for the right-hand part. Also, this makes Theorem 3.2 evident since the complex conjugate to a linear factor of a real polynomial divides it. This uniquely defines for real h the coupling and the decomposable real polynomial p such that $\pi_n p = h$.

3.6. **Example.** If $n = 2$, then $\mathcal{D}_n$ coincides with the family of degenerate quadratic forms on $\mathbb{C}^3$. Indeed, if a nonzero quadratic form q is degenerate, then it can be reduced either to x^2 or to x^2+y^2 which are both decomposable. Conversely, the product of two linear functions is a degenerate quadratic form. Therefore,

$$\mathcal{D}_2 \cap (p + \mathbb{C}r^2) = \{p - \lambda r^2 : \lambda \in \mathbb{C} \text{ and } \operatorname{rank}(p - \lambda r^2) < 3\}.$$

The set of the above λ is the spectrum of the quadratic form p. For any nonzero $p \in \mathcal{P}_2^{\mathbb{C}}$ we have three complex numbers $\lambda_1, \lambda_2, \lambda_3$ and six linear functions $\ell_1, \dots, \ell_6$ on $\mathbb{C}^3$ such that

$$p - \ell_{2j-1}\ell_{2j} = \lambda_j r^2,\ j = 1, 2, 3.$$

The products $\ell_{2j-1}\ell_{2j}$ are independent of the choice of a non-degenerate polynomial in the line $p + \mathbb{C}r^2$ and

$$Rp = R(\ell_{2j-1}\ell_{2j}),\ j = 1, 2, 3.$$

We may assume without loss of generality that $p = \lambda_1 x^2 + \lambda_2 y^2 + \lambda_3 z^2$. Then p is harmonic if and only if $\lambda_1 + \lambda_2 + \lambda_3 = 0$. An easy computation shows that $p = \frac{1}{3}(\ell_1\ell_2 + \ell_3\ell_4 + \ell_5\ell_6)$ in this case. Therefore,

$$\pi_2 p = \frac{1}{3}(\ell_1\ell_2 + \ell_3\ell_4 + \ell_5\ell_6)$$

for all $p \in \mathcal{P}_2$. In other words, the harmonic polynomial in $p + r^2\mathcal{P}_{n-2}$ is the center of gravity of the set $(p + r^2\mathcal{P}_{n-2}) \cap \mathcal{D}_n$ for $n = 2$. This is true for $n > 2$ as well.

3.7. **Another construction for π_n.** Thus, we get a method to find $\pi_n p$ for any $p \in \mathcal{P}_n$. It probably will never be used on this purpose but the relation itself may be useful. Set $\Lambda(p) = \mathcal{D}_n \cap \big(p + r^2\mathcal{P}^{\mathbb{C}}_{n-2}\big)$.

Theorem 3.4. *Let $p \in \mathcal{P}^{\mathbb{C}}_n$. Then*

$$\pi_n p = \frac{1}{(2n-1)!!} \sum\nolimits_{\tilde{p} \in \Lambda(p)} \tilde{p}. \tag{3.2}$$

In other words, $h = \pi_n p$ is the center of gravity of the finite set $\Lambda(p)$, where points of $\Lambda(p)$ are counted with the multiplicities defined by (3.1).

If $n = 1$, then $\mathcal{D}_1 = \mathcal{P}_1$ and the theorem is true if we set $\mathcal{P}_{-1} = 0$. For $n = 2$ it is proved above. We give a sketch of the proof in the general case. Let us denote the mapping in the right-hand side of (3.2) by A and set $h = \pi_n p$. Since $\Lambda(p) = \Lambda(h)$, it is sufficient to prove that $Ah = h$.

Clearly, A commutes with $\mathrm{SO}(3)$. Suppose A linear. Since the components of the decomposition (1.1) are pairwise non-equivalent, this implies $A\mathcal{H}_n \subseteq \mathcal{H}_n$. Thus h and Ah are harmonic. For any $p \in \Lambda(h)$ we have $Rp = Rh$. Hence $RAh = Rh$ and $Ah = h$ according to subsection 2.5.

The mappings $q \to q_{\varpi}$ is not linear. Hence it is not quite trivial that A is linear. The proof of this fact needs some calculation but the idea is clear. The coefficients of Ah are symmetric polynomials of linear factors of $q = h \circ \kappa$ (i.e., all permutations of the set $\mathfrak{L}_q$ preserve them). Moreover, they have the same degree as the coefficients of q and are linear on the coefficients of any linear factor. It follows that they depend on q linearly. The same is true for h due to the construction of subsection 2.5.

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