Jet functors in noncommutative geometry

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Fundamental idea of noncommutative geometry

- Study a geometric object via an algebra of “regular” functions over it (e.g. $C^\infty(M)$, $\mathcal{O}(M)$, $k[x_1, \ldots, x_n]/I$).
- Generalize the commutative algebra $C^\infty(M)$ of smooth functions on a manifold M to an arbitrary (associative, unital) algebra A.
- If A happens to be commutative, then constructions should reproduce classical geometry.
- Results such as the Gelfand-Naimark theorem and the Serre-Swan theorem show that this has merit.

Our setting

- We need a way to encode the smooth structure on A. We borrow some notions from quantum groups,
  1. Exterior algebra
  2. First order differential calculi
- In particular, we can work with any framework that has a notion of exterior derivative.
  1. Differential geometry
  2. Supergeometry
  3. “Quantum geometry”
  4. Countless toy examples

Definition

A first order calculus (Ω^1_d, d) *for a unital associative algebra* A *is*

1. *An* A*-bimodule* Ω^1_d*,*
2. *a map* $d\colon A \to \Omega^1_d$ *satisfying* $d(ab) = adb + (da)b$*,*
3. *such that* $AdA = \Omega^1_d$*.*

Notation and lexicon

Geometry	Algebra	NCG analogue	Structure
point	$\mathbb{R}$ (or $\mathbb{C}$)	k	commutative ring
M	$C^\infty(M)$	A	unital k-algebra
vector bundle E	$\Gamma(E)$	E	F.G.P. left A-module
$E \to F$	$\Gamma(E) \to \Gamma(F)$	$E \to F$	left A-linear map
T^*M	$\Omega^1(M)$	Ω^1_d	first order calculus

We will denote the categories of left and right A-modules by ${}_A\mathrm{Mod}$ and Mod_A, and A-bimodules by ${}_A\mathrm{Mod}_A$.

- The full subcategory of flat left A-modules is denoted ${}_A\mathrm{Flat}$.
- The full subcategory of projective left A-modules is denoted ${}_A\mathrm{Proj}$.
- The full subcategory of finitely generated projective left A-modules is denoted ${}_A\mathrm{FGP}$.

Universal calculus and 1-jets

Definition

The universal calculus Ω^1_u *for* A *is given by the kernel of the multiplication map,*

$$\Omega^1_u = \ker(\cdot) \subset A \otimes A.$$

The corresponding universal differential is

$$d_u \colon a \mapsto 1 \otimes a - a \otimes 1$$

Proposition

We have

$$0 \to \Omega^1_u \to A \otimes A \to A \to 0$$

Moreover, the universal prolongation $j^1_u \colon a \mapsto 1 \otimes a$ *splits the sequence in* Mod_A.

For any first order calculus Ω^1_d, there is a surjective map $p_d : \Omega^1_u \to \Omega^1_d$ which takes d_u to d, and we have

$$0 \to N_d \to \Omega^1_u \to \Omega^1_d \to 0$$

Analogously, for any left A-module E, we define

$$N_d(E) = \ker(p_{d,E}) = \Big\{ \sum_i a_i \otimes e_i \Big| \sum_i a_i e_i = 0, \sum_i da_i \otimes_A e_i = 0 \Big\} \subset A \otimes E$$

Definition

The 1-jet module for a left A-module E is

$$J^1_d E := J^1_u E / N_d(E) = A \otimes E / N_d(E).$$

The prolongation operator is $j^1_{d,E} \colon E \to J^1_d E$, *given by* $j^1_{d,E}(e) = [1 \otimes e]$.
The projection map is $\pi^{1,0}_{d,E} \colon [\sum_i a_i \otimes e_i] \mapsto \sum_i a_i e_i$.

Functoriality

We may also lift morphisms. Let $\varphi : E \to F$ be an A-linear map. Then

$$J^1_d \varphi := \mathrm{id}_{J^1_d A} \otimes_A \varphi \colon J^1_d E \to J^1_d F$$

Theorem

The construction $J^1_d : E \mapsto J^1_d E$ is an additive endofunctor on ${}_A\mathrm{Mod}$. This realizes j^1_d and $\pi^{1,0}_d$ as natural transformations from, and to, the identity functor on ${}_A\mathrm{Mod}$, respectively. In the category of additive endofunctors on ${}_A\mathrm{Mod}$, we have the exact sequence

$$0 \longrightarrow \Omega^1_d \otimes - \xhookrightarrow{\iota^1_d} J^1_d \xrightarrow{\pi^{1,0}_d} id_{({}_A\mathrm{Mod})} \longrightarrow 0.$$

and j^1_d splits this sequence. Moreover,

- *If Ω^1_d is in ${}_A\mathrm{Flat}$ (resp. ${}_A\mathrm{Proj}$, ${}_A\mathrm{FGP}$), then J^1_d* **preserves ${}_A\mathrm{Flat}$ (resp. ${}_A\mathrm{Proj}$, ${}_A\mathrm{FGP}$)**

The latter is geometrically comparable to the fact that the jets of a vector bundle form a vector bundle.

We construct the following three families of functors:

- The nonholonomic jet functors $J_d^{(n)} \colon {}_A\mathsf{Mod} \to {}_A\mathsf{Mod}$
- The semiholonomic jet functors $J_d^{[n]} \colon {}_A\mathsf{Mod} \to {}_A\mathsf{Mod}$
- The holonomic jet functors $J_d^{n} \colon {}_A\mathsf{Mod} \to {}_A\mathsf{Mod}$

In particular we have $J_d^{(0)} = J_d^{[0]} = J_d^0 = \mathrm{id}_{{}_A\mathsf{Mod}}$, and $J_d^{(1)} = J_d^{[1]} = J_d^1$. These functors come equipped with natural transformations

$$j_d^{(n)} \colon \mathrm{id}_{{}_A\mathsf{Mod}} \longrightarrow J_d^{(n)} \qquad j_d^{[n]} \colon \mathrm{id}_{{}_A\mathsf{Mod}} \longrightarrow J_d^{[n]} \qquad j_d^{n} \colon \mathrm{id}_{{}_A\mathsf{Mod}} \longrightarrow J_d^{n},$$

which are respectively called the nonholonomic, semiholonomic, and holonomic jet prolongation maps. We also have the natural transformations,

$$\pi_d^{(n,n-1;m)} \colon J_d^{(n)} \longrightarrow J_d^{(n-1)}, \quad \pi_d^{[n,n-1]} \colon J_d^{[n]} \longrightarrow J_d^{[n-1]}, \quad \pi_d^{n,n-1} \colon J_d^{n} \longrightarrow J_d^{n-1},$$

respectively called the nonholonomic, semiholonomic, and holonomic jet projections.

Definition

An exterior algebra $\Omega_d^\bullet$ *over a* k*-algebra* A*, is an associative graded algebra* $(\Omega_d^\bullet = \bigoplus_{n\geq 0} \Omega_d^n, \wedge)$ *equipped with a map* d *such that*

1. $\Omega_d^0 = A$;
2. d *is a differential, that is a* k*-linear map* $d\colon \Omega_d^\bullet \to \Omega_d^\bullet$ *such that* $d(\Omega_d^n) \subseteq \Omega_d^{n+1}$ *for all* $n \geq 0$*, which satisfies* $d^2 = 0$ *and*
$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^n \alpha \wedge d\beta, \qquad \forall \alpha \in \Omega_d^n, \beta \in \Omega_d^h.$$
3. A, dA *generate* $\Omega_d^\bullet$ *via* $\wedge$.

Remark

Given an exterior algebra $\Omega_d^\bullet$*, the first grade and* $d\colon \Omega_d^0 = A \to \Omega_d^1$ *form a first order calculus for* A*. Vice versa, given a first order differential calculus,* Ω_d^1*, a maximal exterior algebra,* $\Omega_{d,\max}^\bullet$*, is given by quotienting the tensor algebra by the minimal relations for* $d^2 = 0$ *to hold.*

Quantum symmetric forms

In the classical case, the $C^\infty(M)$-module of differential forms with values in a bundle E can be seen as $\Omega^\bullet(M) \otimes_{C^\infty(M)} \Gamma(E)$.
So, given an exterior algebra $\Omega^\bullet_d$ over A, we can define the functors

$$\begin{aligned} \Omega^\bullet_d \colon {}_A\mathrm{Mod} &\longrightarrow {}_A\mathrm{Mod} & E &\longmapsto \Omega^\bullet_d \otimes_A E; \\ \Omega^n_d \colon {}_A\mathrm{Mod} &\longrightarrow {}_A\mathrm{Mod} & E &\longmapsto \Omega^n_d \otimes_A E. \end{aligned}$$

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We define the functors

$$S^0_d = \Omega^0_d = \mathrm{id}_{{}_A\mathsf{Mod}}, \qquad S^1_d = \Omega^1_d := \Omega^1_d \otimes_A -.$$

For $n \geq 0$, the functor of quantum symmetric forms S^n_d is defined by induction as the kernel of the following composition

$$\Omega^1_d \circ S^{n-1}_d \xrightarrow{\Omega^1_d(\iota^{n-1}_\wedge)} \Omega^1_d \circ \Omega^1_d \circ S^{n-2}_d \xrightarrow{\wedge_{S^{n-2}_d}} \Omega^2_d \circ S^{n-2}_d$$

and $\iota^n_\wedge \colon S^n_d \longrightarrow \Omega^1_d \circ S^{n-1}_d$ is the inclusion.

The following lemma shows other equivalent descriptions of S^n_d.

Lemma

If Ω^1_d and Ω^2_d are flat in ${}_A\mathrm{Mod}$, for all $n \geq 0$, the following subfunctors of the tensor algebra $T^n_d := (\Omega^1_d)^{\otimes_A n}$ coincide

1. S^n_d;
2. $\bigcap_{k=0}^{n-2} \ker\left(T^k_d \wedge_{T^{n-k-2}_d}\right)$;
3. $\bigcap_{\substack{h\geq 2\\ 0\leq k\leq n-h}} \ker\left(T^k_d(\wedge_h)_{T^{n-k-h}_d}\right)$, *where* $\wedge_h\colon T^h_d \longrightarrow \Omega^h_d$;
4. $\left(S^h_d \circ T^{n-h}_d\right) \cap \left(T^{n-k}_d \circ S^k_d\right)$ *for* $0 \leq h, k \leq n$ *such that* $h + k > n$.

Spencer cohomology

For all $k, h \geq 0$, consider the functor $\Omega^k_d \circ S^h_d$, and define $\delta^{h,k}$ as the following composition

$$\begin{tikzcd}
\Omega^k_d \circ S^h_d \arrow[r, "\Omega^k_d(\iota^n_\wedge)"] \arrow[rr, bend right=15, "\delta^{h,k}"'] & \Omega^k_d \circ \Omega^1_d \circ S^{h-1}_d \arrow[r, "(-1)^k \wedge^{k,1}_{S^{h-1}_d}"] & \Omega^{k+1}_d \circ S^{h-1}_d
\end{tikzcd}$$

We thus obtain a complex in the category of functors of type ${}_A\mathrm{Mod} \to {}_A\mathrm{Mod}$.

$$0 \longrightarrow S^n_d \xrightarrow{\delta^{n,0}} \Omega^1_d \circ S^{n-1}_d \xrightarrow{\delta^{n-1,1}} \Omega^2_d \circ S^{n-2}_d \xrightarrow{\delta^{n-2,2}} \Omega^3_d \circ S^{n-3}_d \xrightarrow{\delta^{n-3,3}} \cdots$$

Definition (Spencer cohomology)

We call this the Spencer δ-complex, its cohomology the Spencer cohomology, and we denote the cohomology at $\Omega^k_d \circ S^h_d$ by $H^{h,k}$.

Nonholonomic jet functors

Definition

We term the functor

$$J_d^{(n)} := (J_d^1)^{\circ n} = J_d^1 \circ \cdots \circ J_d^1 = (J_d^1 A)^{\otimes_A n} \otimes_A - \colon {}_A\mathrm{Mod} \to {}_A\mathrm{Mod}$$

the nonholonomic n-jet functor. The following composition is called the nonholonomic n-jet prolongation.

$$j_d^{(n)} := j^1_{d, J_d^{(n)}} \circ j^1_{d, J_d^{(n-1)}} \circ \cdots \circ j^1_{d, J_d^1} \circ j_d^1 \colon \mathrm{id} \longrightarrow J_d^{(n)}.$$

For all $1 \leq m \leq n$, we have the natural epimorphisms

$$\pi_d^{(n,n-1;m)} = J_d^{(n-m)} \pi^{1,0}_{d, J_d^{(m-1)}} \colon J_d^{(n)} \longrightarrow\!\!\!\!\rightarrow J_d^{(n-1)},$$

which will be called the nonholonomic n-jet projections.

2-jet functor

We build the (holonomic) 2-jet module with the aim that the following sequence is exact

$$\begin{tikzcd} 0 \arrow[r] & S^2_d E \arrow[r, hook, "\iota^2_{d,E}"] & J^2_d E \arrow[r, two heads, "\pi^{2,1}_{d,E}"] & J^1_d E \arrow[r] & 0 \end{tikzcd}$$

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\end{tikzcd}$$

$$\begin{tikzcd}
0 \arrow[r] & \Omega^1_d(J^1_d E) \arrow[r, hook, "\iota^1_{d,J^1_d E}"'] & J^{(2)}_d E \arrow[r, two heads, "\pi^{1,0}_{d,J^1_d E}"'] & J^1_d E \arrow[r] & 0
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0 \arrow[r] & \Omega^1_d(J^1_d E) \arrow[r, hook, "\iota^1_{d,J^1_d E}"'] & J^{(2)}_d E \arrow[r, two heads, "\pi^{1,0}_{d,J^1_d E}"'] & J^1_d E \arrow[r] & 0
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We assume Ω^1_d flat in Mod_A.

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\end{tikzcd}$$

We assume Ω^1_d flat in Mod_A. As for the classical case, we want the jet prolongation to agree with the nonholonomic one, i.e.

$$l^2_{d,E}\circ j^2_{d,E} = j^{(2)}_{d,E} = j^1_{d,J^1_d E}\circ j^1_{d,E}.$$

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0 \arrow[r] & S^2_d E \arrow[r, hook, "\iota^2_{d,E}"] \arrow[d, hook, swap, "\Omega^1_d(\iota^1_{d,E}) \circ \iota^2_{\wedge,E}"] & J^2_d E \arrow[r, two heads, "\pi^{2,1}_{d,E}"] \arrow[d, hook, "l^2_{d,E}"] & J^1_d E \arrow[r] \arrow[d, equal] & 0 \\
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$$l^2_{d,E} \circ j^2_{d,E} = j^{(2)}_{d,E} = j^1_{d,J^1_d E} \circ j^1_{d,E}.$$

Under these conditions, $Aj^{(2)}_d(E) + S^2_d E \subseteq J^{(2)}_d E$ satisfies the 2-jet short exact sequence.

We can describe $J_d^2 E$ implicitly as the kernel of a left-linear (bilinear for $E = A$) map

$$\widetilde{Đ}_E \colon J_d^{(2)} E \longrightarrow (\Omega_d^1 \ltimes \Omega_d^2)(E),$$

where $(\Omega_d^1 \ltimes \Omega_d^2)(E) \cong (\Omega_d^1 \ltimes \Omega_d^2) \otimes_A E$.
As a right A-module, $\Omega_d^1 \ltimes \Omega_d^2 \cong \Omega_d^1 \oplus \Omega_d^2$, but as an A-bimodule, it comes equipped with a non-trivial left action

$$f \star (\alpha + \omega) = f\alpha + df \wedge \alpha + f\omega, \qquad \forall f \in A,\ \alpha \in \Omega_d^1,\ \omega \in \Omega_d^2.$$

Explicitly, we have

$$\begin{aligned} \widetilde{Đ}_E \colon J_d^{(2)} E &\xrightarrow{\hspace{10em}} (\Omega_d^1 \ltimes \Omega_d^2)(E) \\ [a \otimes b] \otimes_A [c \otimes e] &\longmapsto \left(ad(bc) \otimes_A e, da \wedge d(bc) \otimes_A e\right). \end{aligned}$$

Semiholonomic jets

Definition

We define the semiholonomic n-jet functor, denoted $J_d^{[n]}$, to be the equalizer

$$J_d^{[n]} := \mathrm{Eq}\left(\pi_d^{(n,n-1;m)} \middle| 1 \leq m \leq n\right).$$

We denote by $\iota_{J_d^{[n]}} \colon J_d^{[n]} \hookrightarrow J_d^{(n)}$ the corresponding inclusion.

All the projections coincide for semiholonomic jets, so we call the resulting unique natural transformation the semiholonomic projection:

$$\pi_d^{[n,n-1]} \colon J_d^{[n]} \longrightarrow J_d^{[n-1]}.$$

Proposition

The nonholonomic prolongation factors through the semiholonomic jets.

Proof.

We can decompose the prolongation as $j_d^{(n)} = j^{(n-m)}_{d,J_d^{(m)}} \circ j_d^{(m)}$. We construct the following:

$$\begin{tikzcd}
\mathrm{id}_{{}_A\mathsf{Mod}} \arrow[r, hook, "j_d^{(m)}"'] \arrow[rr, hook, bend left, "j_d^{(n)}"] \arrow[d, equal] & J_d^{(m)} \arrow[r, hook, "j^{(n-m)}_{d,J_d^{(m)}}"'] \arrow[d, two heads, "\pi_d^{(m,m-1)}"'] & J_d^{(n)} \arrow[d, two heads, "\pi_d^{(n,n-1)}"] \\
\mathrm{id}_{{}_A\mathsf{Mod}} \arrow[r, hook, "j_d^{(m-1)}"] \arrow[rr, hook, bend right, "j_d^{(n-1)}"'] & J_d^{(m-1)} \arrow[r, hook, "j^{(n-m)}_{d,J_d^{(m-1)}}"] & J_d^{(n-1)}
\end{tikzcd}$$

For all m, the diagram shows that $\pi_d^{(n,n-1;m)} \circ j_d^{(n)} = j_d^{(n-1)}$, whence we conclude that $j_d^{(n)}$ factors through the equalizer of all the $\pi_d^{(n,n-1;m)}$. □

Definition (Holonomic n-jet functor)

Let A be a k-algebra endowed with an exterior algebra $\Omega^\bullet_d$. We define J^n_d as the kernel of the natural transformation

$$\widetilde{\text{Đ}}_{J^{n-2}_d} \circ J^1_d(l^{n-1}_d) \colon J^1_d \circ J^{n-1}_d \longrightarrow (\Omega^1_d \ltimes \Omega^2_d) \circ J^{n-2}_d,$$

where we denote the natural inclusion by $l^n_d \colon J^n_d \longrightarrow J^1_d \circ J^{n-1}_d$. We call J^n_d the (holonomic) n-jet functor.

It is natural to consider the following composition

$$\iota_{J^n_d} := J^{(n-2)}_d(l^2_d) \circ J^{(n-3)}_d(l^3_d) \circ \cdots \circ J^{(1)}_d(l^{n-1}_d) \circ l^n_d \colon J^n_d \longrightarrow J^{(n)}_d.$$

Remark

The natural transformation $\iota_{J^n_d}$ provides a mapping from the holonomic to the semiholonomic jet, but in general it is not injective (as has been noted before in the setting of synthetic differential geometry.)

Holonomic projections

Definition

We define the (holonomic) n-jet projection as the natural transformation

$$\pi_d^{n,n-1} := \pi^{1,0}_{d,J_d^{n-1}} \circ l_d^n \colon J_d^n \longrightarrow J_d^{n-1}.$$

More generally, by composing them, we get, for all $0 \leq m \leq n$*,*

$$\pi_d^{n,m} := \pi_d^{n,n-1} \circ \pi_d^{n-1,n-2} \circ \cdots \circ \pi_d^{m+1,m} \colon J_d^n \longrightarrow J_d^m.$$

The natural map ι_d^n *is defined by induction, for* $n \geq 2$ *as the unique morphism that commutes in the following diagram*

$$\begin{tikzcd}
S_d^n \arrow[r, hook, "\iota_\wedge^n"] \arrow[d, dashed, "\iota_d^n"] & \Omega_d^1 \circ S_d^{n-1} \arrow[r, "\Omega_d^1(\iota_d^{n-1})"] & \Omega_d^1 \circ J_d^{n-1} \arrow[d, hook, "\iota^1_{d,J_d^{n-1}}"] \\
J_d^n \arrow[rr, hook, "l_d^n"] & & J_d^1 \circ J_d^{n-1}
\end{tikzcd}$$

Lemma

The prolongation $j_d^{(n)} \colon \mathrm{id}_{_A\mathsf{Mod}} \to J_d^{(n)}$ *factors uniquely through* $\iota_{J_d^n} \colon J_d^n \to J_d^{(n)}$ *via a natural monomorphism* $j_d^n \colon \mathrm{id}_{_A\mathsf{Mod}} \to J_d^n$ *in the category of functors of type* ${}_A\mathsf{Mod} \to \mathsf{Mod}$.

Proof.

We proceed by induction on n. For $n = 0$, there is nothing to prove. For $n > 0$, and assume that we have the factorization $j_d^{(n-1)} = \iota_{J_d^n} \circ j_d^{n-1}$, then we have

$$\widetilde{\text{Đ}}_{J_d^{n-2}} \circ J_d^1(l_d^{n-1}) \circ j_d^{(n)}, = \widetilde{\text{Đ}}_{J_d^{n-2}} \circ J_d^1(l_d^{n-1}) \circ \iota_{J_d^n} \circ j_d^{n-1}, = 0$$

implying the existence of j_d^n. $\square$

Theorem (Holonomic jet exact sequence)

Let A be a k-algebra endowed with an exterior algebra $\Omega^\bullet_d$ such that Ω^1_d, Ω^2_d, and Ω^3_d are flat in Mod_A. For $n \geq 1$, if the Spencer cohomology $H^{m,2}$ vanishes, for all $1 \leq m < n-2$, then the following sequence is exact,

$$0 \longrightarrow S^n_d \xhookrightarrow{\iota^n_d} J^n_d \xrightarrow{\pi^{n,n-1}_d} J^{n-1}_d \longrightarrow H^{n-2,2}.$$

Therefore, if $H^{n-2,2} = 0$ we obtain a short exact sequence

$$0 \longrightarrow S^n_d \xhookrightarrow{\iota^n_d} J^n_d \xrightarrow{\pi^{n,n-1}_d} J^{n-1}_d \longrightarrow 0.$$

We will now introduce another type of jet functor, which we will only use as a tool to address the exactness of the jet sequence, although its classical counterpart is known.

Definition

The sesquiholonomic n-jet functor, denoted by $J_d^{\{n\}}$*, is the subfunctor of* $J_d^1 \circ J_d^{n-1}$ *defined by the kernel of*

$$\widetilde{Đ}^I_{J_d^{n-2}} \circ J_d^1(I_d^{n-1})\colon J_d^1 \circ J_d^{n-1} \longrightarrow \Omega_d^1 \circ J_d^{n-2}.$$

We denote the inclusion of $J_d^{\{n\}}$ *into* $J_d^1 \circ J_d^{n-1}$ *by* $I_d^{\{n\}}$*.*

Proposition

Let A be a k-algebra equipped with an exterior algebra $\Omega^\bullet_d$ such that Ω^1_d is flat in ${}_A\mathrm{Mod}$. If the holonomic $(n-1)$-jet sequence is exact

$$\begin{tikzcd}
0 \arrow[r] & S^{n-1}_d \arrow[r, hook, "\iota^{n-1}_d"] & J^{n-1}_d \arrow[r, two heads, "\pi^{n-1,n-2}_d"] & J^{n-2}_d \arrow[r] & 0,
\end{tikzcd}$$

then the sesquiholonomic n-jet sequence is also exact

$$\begin{tikzcd}
0 \arrow[r] & \Omega^1_d \circ S^{n-1}_d \arrow[r, hook, "\iota^{\{n\}}_d"] & J^{\{n\}}_d \arrow[r, two heads, "\pi^{\{n,n-1\}}_d"] & J^{n-1}_d \arrow[r] & 0.
\end{tikzcd}$$

Moreover, $\widetilde{\mathrm{Đ}}^{I}_{J^{n-2}_d}$ remains surjective even when restricted to $J^1_d \circ J^n_d$.

Proof.

$$\begin{tikzcd}
0 \arrow[r] & \Omega^1_d \circ S^{n-1}_d \arrow[r, hook, "\iota_d^{\{n\}}"] \arrow[d, hook, "\Omega^1_d(\iota^{n-1}_d)"'] & J^{\{n\}}_d \arrow[r, two heads, "\pi_d^{\{n,n-1\}}"] \arrow[d, hook, "l_d^{\{n\}}"] & J^{n-1}_d \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \Omega^1_d \circ J^{n-1}_d \arrow[r, hook, "\iota^1_{d,J^{n-1}_d}"] \arrow[d, two heads, "\Omega^1_d(\pi_d^{n-1,n-2})"'] & J^1 \circ J^{n-1}_d \arrow[r, two heads, "\pi^{1,0}_{d,J^{n-1}_d}"] \arrow[d, "\widetilde{\text{Đ}}^{l}_{J^{n-2}_d} \circ J^1_d(l^{n-1}_d)"] & J^{n-1}_d \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Omega^1_d \circ J^{n-2}_d \arrow[r, equal] & \Omega^1_d \circ J^{n-2}_d \arrow[r] & 0 \arrow[r] & 0
\end{tikzcd}$$

□

Spencer and the jet exact sequence

$$\begin{tikzcd}
0 \arrow[r] & S^n_d \arrow[r, hook, "\iota^n_d"] \arrow[d, hook, "\iota^n_\wedge"] & J^n_d \arrow[r, "\pi^{n,n-1}_d"] \arrow[d] & J^{n-1}_d \arrow[d, equal] \arrow[dddll, rounded corners, to path={-- ([xshift=4ex]\tikztostart.east) |- ([yshift=-2ex]\tikztostart.south |- \tikztostart.south) -- ([xshift=-20ex, yshift=-9ex]\tikztostart.south) -| ([xshift=-4ex]\tikztotarget.west) -- (\tikztotarget)}] & \\
0 \arrow[r] & \Omega^1_d \circ S^{n-1}_d \arrow[r, hook, "\iota^{\{n\}}_d"] \arrow[d, "\wedge"] & J^{\{n\}}_d \arrow[r, two heads, "\pi^{\{n,n-1\}}_d"] \arrow[d, "\widetilde{Ð}_{J^{n-2}_d} \circ J^1_d(l^{n-1}_d) \circ l^{\{n\}}"] & J^{n-1}_d \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \ker(\delta^{n-2,2}) \arrow[r, equal] \arrow[d] & \ker(\delta^{n-2,2}) \arrow[r] \arrow[d] & 0 \arrow[d] & \\
 & H^{n-2,2} \arrow[r] & C \arrow[r] & 0 &
\end{tikzcd}$$

Theorem (Stability)

Let A be a k-algebra.

1. *Let Ω^1_d be a first order differential calculus for A. If Ω^1_d is in ${}_A\mathrm{Flat}$ (resp. ${}_A\mathrm{Proj}$, ${}_A\mathrm{FGP}$), then $J^{(n)}_d$ preserves ${}_A\mathrm{Flat}$ (resp. ${}_A\mathrm{Proj}$, ${}_A\mathrm{FGP}$);*
2. *Let Ω^1_d be a first order differential calculus for A which is flat in Mod_A. If Ω^1_d is in ${}_A\mathrm{Flat}$ (resp. ${}_A\mathrm{Proj}$, ${}_A\mathrm{FGP}$), then $J^{[n]}_d$ preserves ${}_A\mathrm{Flat}$ (resp. ${}_A\mathrm{Proj}$, ${}_A\mathrm{FGP}$);*
3. *Let $\Omega^\bullet_d$ be an exterior algebra for A such that Ω^1_d, Ω^2_d, and Ω^3_d are flat in Mod_A. If $H^{m,2}$ vanishes and S^m_d is in ${}_A\mathrm{Flat}$ (resp. ${}_A\mathrm{Proj}$, ${}_A\mathrm{FGP}$), for all $1 \leq m \leq n$, then J^n_d preserves ${}_A\mathrm{Flat}$ (resp. ${}_A\mathrm{Proj}$, ${}_A\mathrm{FGP}$).*

Definition

Let $E, F \in {}_A\mathrm{Mod}$*. A k-linear map* $\Delta\colon E \to F$ *is called a linear differential operator of order at most n with respect to the exterior algebra* $\Omega^\bullet_d$*, if it factors through the holonomic prolongation operator* j^n_d*, i.e. there exists an A-module map* $\widetilde{\Delta} \in {}_A\mathrm{Hom}(J^n_d E, F)$ *such that the following diagram commutes:*

$$\begin{tikzcd}
J^n_d E \arrow[dr, "\widetilde{\Delta}"] & \\
E \arrow[u, "j^n_d"] \arrow[r, "\Delta"] & F
\end{tikzcd}$$

If n is minimal, we say that Δ *is a linear differential operator of order n.*

Examples of differential operators

- Suppose Ω^1_d is free and finitely-generated as a left A-module, i.e. parallelizable. Given a basis $\theta_1, \dots, \theta_n$, we can define the partial derivative operator $\partial_i \colon A \to A$, by $da = \sum_i \partial_i(a)\theta_i$.
- A (left) connection with respect to the first order differential calculus Ω^1_d on a left A-module E is a k-linear map

  $$E \longrightarrow \Omega^1_d \otimes_A E,$$

  satisfying the identity

  $$\nabla(fe) = df \otimes_A e + f\nabla e.$$

- The operator $Ð_E \colon J^1_d(E) \to (\Omega^1_d \ltimes \Omega^2_d)(E)$, whose lift $\widetilde{Ð}_E$ defines second order jets.

Proposition

Let $n \leq m$, then a differential operator of order at most n is also a differential operator of order at most m.

Proof.

Given $\Delta \in \mathrm{Diff}^n(E, F)$, let $\widetilde{\Delta} \colon J^n_d E \to F$ be the corresponding A-linear lift of Δ. The map $\widetilde{\Delta} \circ \pi^{m,n}_{d,E} \colon J^n_d E \to F$ is also left A-linear, and

$$\widetilde{\Delta} \circ \pi^{m,n}_{d,E} \circ j^m_{d,E} = \widetilde{\Delta} \circ j^n_{d,E} = \Delta.$$

□

Proposition

Let $\Delta_1 \colon E \to F$ and $\Delta_2 \colon F \to G$ be differential operators of order at most n and m, respectively. Then the composition $\Delta_2 \circ \Delta_1 \colon E \to G$ is a differential operator of order at most $n + m$.

Proof.

$$\begin{tikzcd}
J_d^{m+n}E \arrow[r, "l_{d,E}^{m,n}"] & J_d^m(J_d^n E) \arrow[dr, "J_d^m(\widetilde{\Delta}_1)"] & & \\
& J_d^n E \arrow[u, "j_{d,J_d^n E}^m"] \arrow[dr, "\widetilde{\Delta}_1"] & J_d^m F \arrow[dr, "\widetilde{\Delta}_2"] & \\
& E \arrow[uul, "j_{d,E}^{m+n}"] \arrow[u, "j_{d,E}^n"'] \arrow[r, "\Delta_1"] & F \arrow[u, "j_{d,F}^m"] \arrow[r, "\Delta_2"] & G
\end{tikzcd}$$

□

Corollary

There is a category Diff *with the same objects as* ${}_A$Mod *and with morphisms given by the differential operators.*

The algebra $\mathcal{D}$

The previous result shows, in particular, that the differential operators from A to A form an algebra under composition. We will call this algebra $\mathcal{D}$, following the classical notation.

Proposition

The algebra A^{op} embeds into $\mathcal{D}$ as zero order differential operators, via the right action of A on itself.

Remark

On the other hand, the left multiplication operators L_a need not be differential operators when A is non-commutative.

Terminal calculi

Proposition

Let $f \in A$, then there is a minimal first order differential calculus for which left multiplication by f is a first order differential operator, with $N_f = \ker(n \otimes m \mapsto nfm) \subset \Omega^1_u$. Left multiplication by f is a first order differential operator with respect to d for any flat A-module E if $N_d \subset N_f$.

The map $f_f \colon m \otimes n \mapsto mfn$ is well-defined and A-bilinear, with kernel N_f, and this realizes

$$\Omega^1_f := p_f(\Omega_u) = A[f, A] \subseteq A,\ d_f(a) = [f, a]$$

Definition

We call Ω^1_f the terminal calculus for f. If $S \subseteq A$, let $N_S = \cap_{f \in S} N_f$, and $\Omega^1_S = \Omega^1_u / N_S$ is the terminal calculus for S.

Proposition

Morphisms in the category of first order calculi are epimorphisms when considered in ${}_A\mathrm{Mod}_A$. Moreover, when they exist, morphisms between any two calculi are unique.

Proposition

Suppose there exists a morphism of calculi $\Omega^1_d \to \Omega^1_{d'}$. Let $\mathrm{Diff}^1_d(E, F)$ and $\mathrm{Diff}^1_{d'}(E, F)$ be the differential operators of order at most 1 from E to F with respect to Ω^1_d and $\Omega^1_{d'}$, respectively. Then $\mathrm{Diff}^1_{d'}(E, F) \subset \mathrm{Diff}^1_d(E, F)$.

There is a full subcategory, consisting of all calculi for which S are of order at most 1. Ω^1_S is a terminal object in this subcategory.

Example: quaternions

Definition

Let $\mathbb{H}$ be the ***Quaternions****; the unital non-commutative algebra with basis $1, i, j, k$ subject to the relations*

$$i^2 = j^2 = k^2 = ijk = -1$$

The quaternions form a normed division algebra.

Remark

Left (right) $\mathbb{H}$-modules are called left (right) quaternionic vector spaces. These are characterized by quaternionic dimension up to isomorphism. Quaternionic bimodules are not necessarily free.

Universal and terminal calculi for $\mathbb{H}$

The universal calculus $\Omega^1_u(\mathbb{H})$ is finite dimensional over $\mathbb{R}$, and generated as an $\mathbb{H}$-module as follows:

$$\Omega^1_u(\mathbb{H}) = \langle i \otimes i + 1 \otimes 1,\, j \otimes j + 1 \otimes 1,\, k \otimes k + 1 \otimes 1 \rangle$$

Consider the terminal calculus for i. We have $N_i = \ker(a \otimes b \mapsto aib)$. A direct computation gives

$$N_i = \langle i \otimes i + 1 \otimes 1,\, j \otimes j - k \otimes k \rangle$$

And similarly for j, k. Since N_i and N_j each have quaternionic dimension 2, their intersection has dimension at least 1. Indeed, we have

$$N_{\{i,j\}} = N_i \cap N_j = \langle 1 \otimes 1 + i \otimes i + j \otimes j - k \otimes k \rangle$$

Structure of the terminal calculus $\Omega^1_{\{i,j\}}$

Now, we may compute the structure of the calculus.

$$\begin{aligned} dk &:= [1 \otimes k - k \otimes 1] \\ &= [-k(k \otimes k + 1 \otimes 1)] \\ &= [-k(2 \otimes 1 + i \otimes i + j \otimes j)] \\ &= -jdi + idj \end{aligned}$$

This is the structure equation for the calculus, and it is parallelizable and generated by di, dj. We may also compute the bimodule structure,

$$\begin{gathered} idi = -(di)i, idj = -(dj)i \\ jdi = -(di)j, jdj = -(dj)j \\ kdi = (di)k, kdj = (dj)k \end{gathered}$$

Differential operators on $\mathbb{H}$

The zero order operators are the right multiplications, $1, R_i, R_j, R_k$. At first order, we have the partial derivative operators,

$$\partial_i : \begin{bmatrix} 1 \\ i \\ j \\ k \end{bmatrix} \mapsto \begin{bmatrix} 0 \\ 1 \\ 0 \\ -j \end{bmatrix}, \partial_j : \begin{bmatrix} 1 \\ i \\ j \\ k \end{bmatrix} \mapsto \begin{bmatrix} 0 \\ 0 \\ 1 \\ i \end{bmatrix}.$$

Proposition

The algebra $\mathcal{D}$ *is generated by* $1, R_i, R_j, R_k, \partial_i, \partial_j$. *The relations are* $\partial_i^2 = \partial_j^2 = [\![\partial_i, \partial_j]\!] = 0$ *together with the* $\mathbb{H}^{op}$*-bimodule structure*

$$\begin{aligned} [\![\partial_i, R_j]\!] = [\![\partial_j, R_i]\!] &= 0, & [\partial_j, R_k] &= R_i, \\ [\![\partial_i, R_i]\!] = [\![\partial_j, R_j]\!] &= 1, & [\partial_i, R_k] &= -R_j. \end{aligned}$$

The algebra $\mathbb{H}$ embeds into $\mathcal{D}$; L_i and L_j are of order 1 by construction and L_k is of order 2.

Quantum symmetric forms for $\mathbb{H}$

Proposition

We have that

$$S^2_d = \langle di \otimes dj - dj \otimes di \rangle, \qquad S^3_d = 0.$$

So jet dimension stabilizes,

$$J^1_d \mathbb{H} \simeq \mathbb{H}^3, \text{ and } J^2_d \mathbb{H} \simeq \mathbb{H}^4,$$

as left quaternionic vector spaces. Moreover $J^n_d \mathbb{H} \simeq J^2_d \mathbb{H}$ for $n \geq 2$.

We have $S^2_d \cong \mathbb{H}$ as a bimodule, and so Ω^1_d admits a unique quantum metric (to scale). We may compute its Laplacian,

$$\Delta = 2\partial_j \circ \partial_i.$$

Let M be a smooth manifold and consider the case $k = \mathbb{R}$ and $A = C^\infty(M)$ with the exterior de Rham differential

$$d \colon C^\infty(M) \to \Omega^1(M),$$

regarded as a first order differential calculus. Let $N \to M$ be a vector bundle, and let $E := \Gamma(M, N)$ be its $C^\infty(M)$-module of sections.

$$\begin{tikzcd}
0 \arrow[r] & \Omega^1_u(E) \arrow[r, "\iota^1_{u,E}"] \arrow[d, two heads, "p_{d,E}"] & J^1_u E \arrow[r, "\pi^{1,0}_{u,E}"] \arrow[d, two heads, "\widehat{p}_{d,E}"] & E \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \Omega^1(M,N) \arrow[r] & \Gamma(M, J^1 N) \arrow[r, "\pi^{1,0}"] & \Gamma(M,N) \arrow[r] & 0
\end{tikzcd}$$

$$\widehat{p}_{d,E} = j^1 \circ \pi^{1,0}_u + \iota_d \circ p_d \circ \rho_u, \qquad f \otimes \sigma \longmapsto [f\sigma]^1 - (df) \otimes \sigma.$$

And thus an isomorphism is given by

$$[f \otimes \sigma] \in J^1_d E \longmapsto [f\sigma]^1 + df \otimes \sigma \in \Gamma(M, J^1_d N).$$

This isomorphism intertwines the prolongation operators.

$$j^1_d(\sigma) \mapsto [\sigma]^1.$$

Proposition (Goldschmidt '68)

Let N be a vector bundle. There exists a unique differential operator

$$\rho\colon J^1(J^nN) \longrightarrow C^1_nN$$

of order 1, such that the symbol (i.e. the restriction of the lift $\widetilde{\rho}$ to $T^*\otimes J^nN$) is the natural projection $T^*\otimes J^nN \to C^1_nN$ and the sequence

$$0 \longrightarrow J^{n+1}N \longrightarrow J^1(J^nN) \longrightarrow C^1_nN \longrightarrow 0$$

is exact.

Here C^1_nN is the space $(T^*\otimes J^nN)/\delta(\mathrm{Sym}^{n+1}(T^*)\otimes N)$, and δ is the Spencer differential.

The uniqueness result of Goldschmidt identifies the map ρ with our map $\widetilde{Đ}$ (easy to see for $n = 2$). Thus, the kernel of $\widetilde{Đ}$ is the classical 2-jet module, seen as a submodule of the nonholonomic jets. The prolongation maps coincide with the composition of the first jet prolongation with itself both for our construction and classically. Thus

Proposition

We have $J^2_d E \cong \Gamma(M, J^2 N)$ in ${}_A\mathrm{Mod}$, the classical module of 2-jets of sections of N, and the isomorphism takes our prolongation to the classical prolongation.

Lemma (Quillen '64, Spencer '69)

Let N be a vector bundle. Then the following holds

$$J^{n+m}N = J^n(J^m N) \cap J^{n-1}(J^{m+1}N) \subset J^{(n+m)}N,$$

where $J^{(n+m)}N$ *denotes the (classical) nonholonomic jet bundle of order* $n+m$*.*

Theorem (Classical correspondence)

Let $A = C^\infty(M)$ for a smooth manifold M, let $\Omega^\bullet_d = \Omega^\bullet(M)$ equipped with the de Rham differential d, and let E be the space of smooth sections of a vector bundle. Then the classical nonholonomic and holonomic n-jet bundles of E are isomorphic to $J^{(n)}_d E$ and $J^n_d E$ in ${}_A\mathrm{Mod}$, respectively, and the prolongation maps and jet projections are compatible with the isomorphisms.

Sketch of proof.

The smooth one-forms $\Omega^1(M)$ are regular enough, in particular flat as a right module, that we can show $J^n_d E = J^1_d(J^{n-1}_d E) \cap J^2_d(J^{n-2}_d E)$. Combined with the Quillen result, the base cases $n = 1$ and $n = 2$, and verifying that prolongations coincide, this yields an inductive proof of the result for the holonomic case. ☐