Curvature restrictions on a manifold with a flat Higgs bundle

Geometric structures and supersymmetry

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Definition of the Higgs bundle

The notion of Higgs bundle is introduced by Hitchin for the one dimensional case and by Simpson for the general case. The following definition is due to Simpson:

Definition (Higgs bundle)

A Higgs bundle is a pair (H, θ), where H is a holomorphic vector bundle over a complex manifold B and θ is an $\mathrm{End}(H)$-valued holomorphic one form such that $\theta^2 \equiv 0$ on B.

By using the Higgs field θ, one may define a holomorphic bundle map from the tangent bundle T_B of B to $\mathrm{End}(H)$ as follows

$$\eta : v \mapsto v \lrcorner\, \theta := \theta_v, \ \ v \in T_B. \tag{1}$$

Then $\theta^2 \equiv 0$ iff $[\theta_v, \theta_w] \equiv 0$.

Admissible Higgs bundle

We shall also use the following definition:

Definition (Admissible Higgs bundle)

A Higgs bundle (H, θ) is said to be **admissible** *if the associated bundle map η is an injection from T_B to $\mathrm{End}(H)$.*

Example: It is known that there is a natural Higgs bundle structure on the base manifold of a proper holomorphic fibration. More precisely, let π be a proper holomorphic submersion from a complex manifold $\mathcal{X}$ to a complex manifold B with connected fibres $X_t := \pi^{-1}(t)$. Let us denote by

$$\kappa : v \mapsto \kappa(v) \in H^{0,1}(T_{X_t}), \tag{2}$$

the Kodaira-Spencer map.

Higgs bundle structure on the base manifold

Let E be a holomorphic vector bundle over the total space $\mathcal{X}$. If the following direct image sheaves

$$\mathcal{H}_E^k := \oplus_{p+q=k} \mathcal{H}_E^{p,q}, \ \mathcal{H}_E^{p,q} := R^q \pi_* \mathcal{O}(E \otimes \wedge^p T^*_{\mathcal{X}/B}).$$

are locally free then the Kodaira-Spencer map κ defines a natural holomorphic bundle map, say η, from T_B to $\mathrm{End}(\mathcal{H}_E^k)$ such that $\eta(v)$ maps $\mathcal{H}_E^{p,q}$ to $\mathcal{H}_E^{p-1,q+1}$. Moreover, we have

$$[\eta(v), \eta(w)] \equiv 0, \tag{3}$$

which implies that the $\mathrm{End}(\mathcal{H}_E^k)$-valued holomorphic one form, say θ, associated to η satisfies that $\theta^2 \equiv 0$. Thus $(\mathcal{H}_E^k, \theta)$ is a Higgs bundle over the base manifold B.

Two remarks

Remark (Relation with the period map)

In case E is flat and the total space is Kähler, we know that η is just the **differential of the period map**. *Thus in that case, $(\mathcal{H}_E^k, \theta)$ is admissible iff the period map is an immersion.*

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Remark (Natural metric on the base)

If (H, θ) is an admissible Higgs bundle then T_B can be seen as a **holomorphic subbundle** *of $\mathrm{End}(H)$. Fix a smooth metric h on H. Then one may define a natural Hermitian structure on the subbundle T_B of $\mathrm{End}(H)$ as follows:*

$$(v, w)_H := (\theta_v, \theta_w)_{h\otimes h^*}, \quad \forall\ v, w \in T_B, \tag{4}$$

here we identify $\mathrm{End}(H)$ with $H \otimes H^*$.

Hodge metric on the base manifold

If B is the base manifold of a Calabi-Yau family then the above metric is just the Hodge metric introduced by Zhiqin Lu. In general, we have:

Definition (Hodge metric)

Let (H, θ) be an admissible Higgs bundle. Then we call the Hermitian metric defined above the **Hodge metric** *on B.*

Hodge metric on the base manifold

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Remark (Hodge semi-metric)

In general, a Higgs bundle may not be admissible. Then we call the Hermitian form $(v, w)_H$ the Hodge semi-metric on B.

Gauss-Manin connection and flat Higgs bundle

If the total space $\mathcal{X}$ is Kähler and (E, h) is flat then we know that there is a natural **flat** connection (Gauss-Manin connection), say D^H, on $(\mathcal{H}^k_E, \theta)$. Moreover, D^H can be written as

$$D^H = D^h + \theta + \theta^*,$$

where $D^h := \overline{\partial} + \partial^h$ denotes the Chern connection on (H, h) and θ^* denotes the adjoint of θ. In general, we shall define:

Definition (Flat Higgs bundle, by Simpson)

Let (H, θ) be a Higgs bundle with a smooth Hermitian metric h. Then we call $D^H := D^h + \theta + \theta^*$ the **Higgs connection** *on H. And a Higgs bundle (H, θ, h) is said to be* **flat** *if its* **Higgs curvature** *$\Theta^H := (D^H)^2 \equiv 0$ on B.*

Nilpotent Higgs bundle

Let (H, θ) be a Higgs bundle. By iteration, one may also define η^j as a bundle map from $\otimes^j T_B$ to $\mathrm{End}(H)$, i.e.

$$\eta^j(v_1 \otimes \cdots \otimes v_j) := \theta_{v_1} \circ \cdots \circ \theta_{v_j}.$$

If $H = \mathcal{H}_E^k$ then each η^j is just the iterated Kodaira-Spencer map and $\eta^{k+1} \equiv 0$. In general, we have the following definition:

Definition (Nilpotent Higgs bundle)

Let k be a natural number. A Higgs bundle (H, θ) is said to be k-nilpotent if the associated bundle map satisfies $\eta^{k+1} \equiv 0$.

Now let us state our main theorem:

Main theorem

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If there is a flat admissible Higgs bundle, say H, over a complex manifold, say B, then the Hodge metric on B is a Kähler metric with semi-negative holomorphic bisectional curvature. Assume further that H is k-nilpotent. Then the holomorphic sectional curvature of B is bounded above by $-(k^2 \mathrm{Rank}(H))^{-1}$.

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If there is a flat admissible Higgs bundle, say H, over a complex manifold, say B, then the Hodge metric on B is a Kähler metric with semi-negative holomorphic bisectional curvature. Assume further that H is k-nilpotent. Then the holomorphic sectional curvature of B is bounded above by $-(k^2 \mathrm{Rank}(H))^{-1}$.

Remark

Recently, a better bound is obtained by QiongLing Li. For the variation of Hodge structure case, the above theorem is due to Griffiths-Schmid and Deligne (the Kähler part is due to Zhiqin Lu). Later we shall show how to use **curvature formula for the subbundle** *to prove our main theorem.*

Proof of the Kähler part

By definition, we know that the fundamental form of the Hodge metric associated to (H, θ) can be written as

$$\omega = i\{\theta, \theta\},$$

where $\{\cdot, \cdot\}$ denotes the sesquilinear product on $\mathrm{End}(H)$. Since θ is holomorphic, we have

$$\partial\omega = i\{\partial^{\mathrm{End}(H)}\theta, \theta\},$$

where $\partial^{\mathrm{End}(H)}$ denotes the $(1,0)$-part of the Chern connection on $\mathrm{End}(H)$. Since $\mathrm{End}(H) \simeq H \otimes H^*$, by definition of the Chern connection on the dual bundle and the tensor product bundle, we know that

$$\partial^{\mathrm{End}(H)}\theta = [\partial^h, \theta].$$

Notice that $[\partial^h, \theta]$ is just the $(2,0)$-part of the Higgs curvature, which is zero if H is flat. Thus ω is Kähler.

Proof of the second part

Since (H, θ) is admissible, we know that the associated bundle map $\eta : T_B \to \mathrm{End}(H)$, $v \mapsto \theta_v$, defines a holomorphic subbundle structure on T_B and the Hodge metric is just the induced metric on T_B. Let us denote by D^{T_B} the Chern connection on T_B with respect to the Hodge metric. Locally one may write

$$D^{T_B} = \sum dt^j \otimes D_j^{T_B} + \sum d\bar{t}^j \otimes \bar{\partial}_{t^j}, \ \Theta_{j\bar{k}}^{T_B} := [D_j^{T_B}, \bar{\partial}_{t^k}].$$

By the curvature formula for the subbundle, we have

$$\sum (\Theta_{j\bar{k}}^{T_B} v, v)_H \xi^j \bar{\xi}^k \leq \sum (\Theta_{j\bar{k}}^{\mathrm{End}(H)} \theta_v, \theta_v) \xi^j \bar{\xi}^k, \ \forall \ \xi \in \mathbb{C}^{\dim B}$$

where $\Theta_{j\bar{k}}^{\mathrm{End}(H)}$ are the Chern-curvature operators on $\mathrm{End}(H)$.

Proof of the second part

Since $\mathrm{End}(H) \simeq H \otimes H^*$, by the curvature formula of the dual bundle and the tensor product bundle, we have

$$\Theta^{\mathrm{End}(H)}_{j\bar{k}} \theta_v = [\Theta^h_{j\bar{k}}, \theta_v], \tag{5}$$

where $\Theta^h = (D^h)^2$ denotes the Chern connection on (H, h). By our assumption, (H, θ) is flat, thus we have

$$\Theta^h_{j\bar{k}} = [\theta^*_k, \theta_j],$$

where $\theta_j := \partial/\partial t^j \lrcorner\, \theta$. Moreover, $\theta^2 \equiv 0$ imples that $[\theta_v, \theta_j] \equiv 0$. Thus we have

$$[\Theta^h_{j\bar{k}}, \theta_v] = [[\theta^*_k, \theta_j], \theta_v] = [[\theta^*_k, \theta_v], \theta_j]. \tag{6}$$

Notice that

$$\begin{aligned} ([[\theta^*_k, \theta_v], \theta_j], \theta_v) &= ([\theta^*_k, \theta_v]\theta_j - \theta_j[\theta^*_k, \theta_v], \theta_v) \\ &= -([\theta^*_k, \theta_v], [\theta^*_j, \theta_v]). \end{aligned}$$

Proof of the second part

Thus we have

$$\sum(\Theta_{j\bar{k}}^{\mathrm{End}(H)}\theta_v,\theta_v)\xi^j\bar{\xi}^k = -||\sum[\theta_j^*,\theta_v]\xi^j||^2 \leq 0,$$

which implies the second part of our main theorem. In order to prove the last part, it suffices to show the following inequality:

$$||[\theta_j^*,\theta_j]||^2 \geq (k^2\mathrm{Rank}(H))^{-1}||\partial/\partial t^j||_H^4, \ \forall\ j. \tag{7}$$

Now let us prove this inequality:

Let us fix $t \in B$. Then $A := \theta_j(t)$ is a $\mathbb{C}$-linear transform from the fibre $V := H_t$ to itself. Since (H,θ) is k-nilpotent, we know that $A^{k+1} = 0$.

Proof of the last part

Using the Jordan normal form of A, we know that V can be written as a direct sum of $(k+1)$ $\mathbb{C}$-linear subspaces,

$$V = \oplus_{0 \le p \le k} V_p,$$

such that

$$A(V_p) \subset V_{p+1}, \ A(V_k) = \{0\}.$$

For each $0 \le p \le k-1$, let us denote the following $\mathbb{C}$-linear map

$$A : V_p \to V_{p+1},$$

by A_p. Put

$$A_{-1} = A_k = 0.$$

Proof of the last part

By a direct computation, we have

$$||[\theta_j^*, \theta_j]|| \geq \frac{\sum_{0\leq p\leq k} |\mathrm{Trace}(A_p^* A_p) - \mathrm{Trace}(A_{p-1}^* A_{p-1})|}{\sqrt{\mathrm{Rank}(H)}}.$$

Put $a_p = \mathrm{Trace}(A_p^* A_p)$. Then each a_p is non-negative, and

$$\begin{aligned} \sum |a_p - a_{p-1}| &= a_0 + \sum_{1\leq p\leq k-1} |a_p - a_{p-1}| + a_{k-1} \\ &\geq \max\{a_0, \cdots, a_{k-1}\} \geq \frac{1}{k} \sum a_p. \end{aligned}$$

Moreover, by definition, we have

$$||\partial/\partial t^j||_H^2 = \sum a_p.$$

Thus (7) is true. The proof is complete.

Corollary

The following Griffiths-Schmid theorem is a direct corollary of our main theorem.

Theorem (Griffiths-Schmid theorem)

Let $\pi : \mathcal{X} \to B$ be a proper holomorphic submersion from a Kähler manifold $\mathcal{X}$ to a complex manifold B with connected fibres X_t. Assume that the Higgs bundle

$$\mathcal{H}^k := \oplus_{p+q=k}\{H^{p,q}(X_t)\},$$

is admissible (i.e., the associated period map is an immersion). Then the holomorphic sectional curvature of B with respect to the Hodge metric is bounded above by a negative constant.

Final remark!

Remark (Calabi-Yau case)

In the above theorem, if we assume further that the canonical line bundle of each fibre is trivial then

- $\mathcal{H}^n$ ($n = \dim_{\mathbb{C}} X_t$) *is admissible iff the Kodaira-Spencer map is injective.*

Thus the above theorem can be used to study the curvature properties of the base manifold of a Calabi-Yau family.