Exceptionally simple super-PDE

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Geometric Structures & Supersymmetry

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Outline

1. Old and new models: Exceptionally simple (super-)PDE
2. Preliminaries
3. A G_2 story and its extensions
4. The odd-contact $F(4)$ story

Complex simple Lie (super-)algebras

Simple Lie alg $\mathfrak{g}$ were classified by Killing (1888), Cartan (1894):

- $A_\ell = \mathfrak{sl}_{\ell+1}(\mathbb{C})$, $B_\ell = \mathfrak{so}_{2\ell+1}(\mathbb{C})$, $C_\ell = \mathfrak{sp}_{2\ell}(\mathbb{C})$, $D_\ell = \mathfrak{so}_{2\ell}(\mathbb{C})$.
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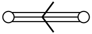

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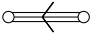
- Abstract classification via Dynkin diagrams, e.g. 

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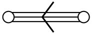
- Abstract classification via Dynkin diagrams, e.g. 

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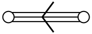
- Abstract classification via Dynkin diagrams, e.g. 

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- $[x, y] = -(-1)^{|x||y|}[y, x]$
- $[x, [y, z]] = [[x, y], z] + (-1)^{|x||y|}[y, [x, z]]$.

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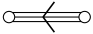

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where $|x|, |y| \in \mathbb{Z}_2$ (parities). From Kac (1977), we'll focus on:

$\mathfrak{g}$	dim	$\mathfrak{g}_{\bar{0}}$	$\mathfrak{g}_{\bar{1}}$	
$G(3)$	$(17\|14)$	$G_2 \times A_1$	$\mathbb{C}^7 \boxtimes \mathbb{C}^2$	
$F(4)$	$(24\|16)$	$B_3 \times A_1$	$\mathbb{S} \boxtimes \mathbb{C}^2$	(B_3-spin rep $\mathbb{S}$)

Cartan & Engel (1893): Structures with G_2 symmetry

Dim	Geometric structure	Model
7	Parabolic Goursat PDE $\mathcal{F}$	$9(u_{xx})^2 + 12(u_{yy})^2(u_{xx}u_{yy} - (u_{xy})^2)$ $+32(u_{xy})^3 - 36u_{xx}u_{xy}u_{yy} = 0$
6	Involutive pair of PDE $\mathcal{E}$	$u_{xx} = \frac{1}{3}(u_{yy})^3, \quad u_{xy} = \frac{1}{2}(u_{yy})^2$
5	$(2,3,5)$-distrib. $\overline{\mathcal{E}}$	$dU - PdX,$ $dP - QdX,$ $dZ - Q^2dX$ (a.k.a. Hilbert–Cartan: $Z' = (U'')^2$)
5	G_2-contact structure (contact twisted cubic field)	$dz + x_1dy_1 - y_1dx_1 + x_2dy_2 - y_2dx_2 = 0,$ $dx_2^2 + \sqrt{3}dy_1dy_2 = 0,$ $dx_2dy_2 - 3dx_1dy_1 = 0,$ $dy_2^2 + \sqrt{3}dx_1dx_2 = 0$

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- Given $(\mathbb{C}^m, \langle \cdot, \cdot \rangle)$, let $W = \mathcal{JS}_m = \mathbb{C}^m \oplus \mathbb{C}$ ("spin factor"). Given $t = (v, \lambda)$, we have $\mathfrak{C}(t^3) := \langle v, v \rangle \lambda$.

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NOTE: The Jordan algebra structure plays no role in this talk.

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Let $n-1 := \dim(W)$. Basis $\{\mathrm{w}_a\}$ on W, dual basis $\{\mathrm{w}^a\}$ on W^*. Let $\{x^i\}_{i=0}^{n-1}$ lin.coords on $\mathbb{C} \oplus W$.

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Theorem (Degenerate cases)

- $u_{ij} = 0$, $1 \leqslant i,j \leqslant n$: *point sym* $= A_{n+1}$. *(NOTE: $\mathfrak{sl}_2$ excluded!)*
- $u_{ijk} = 0$, $1 \leqslant i,j,k \leqslant n$: *contact sym* $= C_{n+1}$.

Example: D_4-symmetric PDE system

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This PDE system has contact symmetry algebra $\boxed{D_4 \cong \mathfrak{so}(8)}$.

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Here is a related, but quite different story:

Theorem (Santi, T. 2022)

Let u be even, and x^0, x^1, x^2, x^3 be odd. The 3rd order super-PDE

$$u_{0ab} = u_{ab}u_{123}, \quad 1 \leqslant a < b \leqslant 3$$

has contact symmetry superalgebra $F(4)$.

Supermanifolds

Definition

A supermanifold of dim $(m|n)$ is a pair $M = (M_o, \mathcal{A}_M)$, where

- M_o *is an m-manifold (the "body");*
- $\mathcal{A}_M$ *is a sheaf of superalgebras on* M_o *s.t. locally* $\mathcal{A}_M|_U \cong \Gamma(\Lambda^\bullet E^*)$, *where* $E \to U$ *is a vector bundle of rank n over* $U \subset M_o$.

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Remark

Superfunctions are sections of $\mathcal{A}_M$. Locally, they are of the form:

$$f = f_0(x) + f_{\alpha_1}(x)\theta^{\alpha_1} + \dots + f_{\alpha_1 \dots \alpha_n}(x)\theta^{\alpha_1} \wedge \dots \wedge \theta^{\alpha_n},$$

where $f_{\alpha_1 \dots \alpha_n} \in C^\infty(U)$ and $\{\theta^\alpha\}$ are generators for E^*. The latter are called "odd coordinates". They anticommute wrt $\wedge$, so are nilpotent.

What is a 2nd order super-PDE?

Global	Local
Contact supermanifold $(M^{2n+1}, \mathcal{C})$	$J^1(\mathbb{C}^n, \mathbb{C}) : \ (x^i, u, u_i), \, \sigma := du - (dx^i) u_i$ $\mathcal{C} = \{\sigma = 0\} = \mathrm{span}\{\partial_{x^i} + u_i \partial_u, \partial_{u_i}\}$

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Lagrange–Grassmann bundle $(\widetilde{M}=\mathrm{LG}(\mathcal{C}),\widetilde{\mathcal{C}})$	$J^2(\mathbb{C}^n,\mathbb{C})$: (x^i,u,u_i,u_{ij}) $\widetilde{\mathcal{C}}=\mathrm{span}\{\widetilde{D}_{x^i}:=\partial_{x^i}+u_i\partial_u+u_{ij}\partial_{u_j},\partial_{u_{ij}}\}$

What is a 2nd order super-PDE?

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Contact supermanifold $(M^{2n+1}, \mathcal{C})$	$J^1(\mathbb{C}^n, \mathbb{C}):\ (x^i, u, u_i)$, $\sigma := du - (dx^i)u_i$ $\mathcal{C} = \{\sigma = 0\} = \mathrm{span}\{\partial_{x^i} + u_i \partial_u, \partial_{u_i}\}$
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IDEA: contact supermanifold + structure ("filtered G_0-structure").

Contact gradings

The algebraic model for $(M^{2n+1}, \mathcal{C})$ is Heisenberg $\mathfrak{m} = \mathfrak{m}_{-2} \oplus \mathfrak{m}_{-1}$ (with $\dim(\mathfrak{m}_{\pm 2}) = 1$ & $\bigwedge^2 \mathfrak{m}_{-1} \to \mathfrak{m}_{-2}$ ndg).

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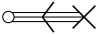

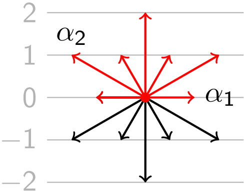

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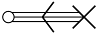

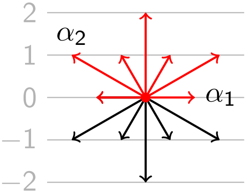

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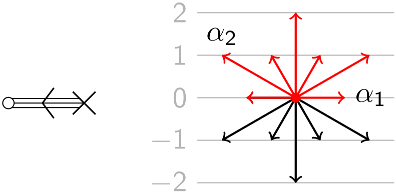

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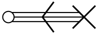

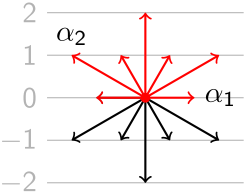

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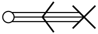

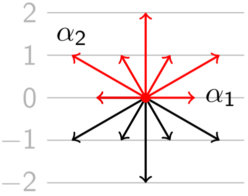

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Have $\mathfrak{der}_{gr}(\mathfrak{m})\cong\mathfrak{cspo}(\mathfrak{m})\supsetneq\mathfrak{g}_0$. What structure reduces it to $\mathfrak{g}_0$? How to relate it to Lagrangian subspaces and super-PDE?

Tanaka–Weisfeiler prolongation

Given $\mathfrak{m} = \mathfrak{m}_-$ and $\mathfrak{g}_0 \subset \mathfrak{der}_{gr}(\mathfrak{m})$, the Tanaka–Weisfeiler prolongation $\mathrm{pr}(\mathfrak{m}, \mathfrak{g}_0)$ is the graded LSA with:

(i) $\mathrm{pr}_{\leqslant}(\mathfrak{m}, \mathfrak{g}_0) = \mathfrak{m} \oplus \mathfrak{g}_0$

(ii) if $[X, \mathfrak{g}_{-1}] = 0$ for $X \in \mathrm{pr}_+(\mathfrak{m}, \mathfrak{g}_0)$, then $X = 0$;

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How to identify $\mathrm{pr}(\mathfrak{m}, \mathfrak{g}_0)$? Test a candidate $\mathfrak{g}$ via the criteria:

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Theorem (Kruglikov, Santi, T. 2022)

Consider a filtered G_0-structure $\mathcal{G}_0 \to M$ of type $(\mathfrak{m}, \mathfrak{g}_0)$ satisfying $\mathfrak{g} = \text{pr}(\mathfrak{m}, \mathfrak{g}_0)$. Then the symmetry superalgebra $\mathfrak{s}$ satisfies $\dim(\mathfrak{s}) \leqslant \dim(\mathfrak{g})$ (both even and odd parts).

Twisted cubic $\mathcal{V} \subset \mathbb{P}(V)$

G_2/P_2 suggests looking at $V := S^3\mathbb{C}^2 = \mathfrak{g}_{-1}$. The twisted cubic $\mathcal{V} = \{[v^3] : [v] \in \mathbb{P}^1\} \subset \mathbb{P}(V)$ is invariant wrt $G_0 = \mathrm{GL}_2$. On V,

$$\eta(f,g) := \frac{1}{3!}(f_{xxx}g_{yyy} - 3f_{xxy}g_{yyx} + 3f_{xyy}g_{yxx} - f_{yyy}g_{xxx}),$$

defines a GL_2-invariant CS-form $[\eta]$. Have $\dim(\mathrm{LG}(V)) = 3$.

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Example (Osculating filtration: differentiate $\gamma(t) = (\mathsf{x} + t\mathsf{y})^3$)

$$\begin{array}{ccccccc} \ell & \subset & \widehat{T}_\ell\mathcal{V} & \subset & \widehat{T}^{(2)}_\ell\mathcal{V} & \subset & \widehat{T}^{(3)}_\ell\mathcal{V} = V \\ \langle \mathsf{x}^3 \rangle & & \langle \mathsf{x}^3, \mathsf{x}^2\mathsf{y} \rangle & & \langle \mathsf{x}^3, \mathsf{x}^2\mathsf{y}, \mathsf{x}\mathsf{y}^2 \rangle & & \\ & & \text{Lagrangian!} & & & & \end{array}$$

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Wrt CS-basis, i.e. $\eta = \begin{pmatrix} 0 & \mathrm{id}_2 \\ -id_2 & 0 \end{pmatrix}$, have coords $\begin{pmatrix} c_{11} & c_{12} \\ c_{12} & c_{22} \end{pmatrix}$ on $\mathrm{LG}(V)$.

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Take CS-basis $(\mathsf{x}^3, -3\mathsf{x}^2\mathsf{y}, -6\mathsf{y}^3, -6\mathsf{x}\mathsf{y}^2)$ wrt $[\eta]$. Then:

$$(x+ty)^3 \leftrightarrow \left(1, -t, -\frac{t^3}{6}, -\frac{t^2}{2}\right).$$

Canonical objects associated to the twisted cubic

- $\hat{\mathcal{V}} := \{\hat{T}_\ell \mathcal{V} : \ell \in \mathcal{V}\} \subsetneq \mathrm{LG}(V)$: differentiate & row reduce:

$$\left.\begin{array}{c} (1, -t, -\frac{t^3}{6}, -\frac{t^2}{2}) \\ (0, -1, -\frac{t^2}{2}, -t) \end{array}\right\} \rightsquigarrow \begin{pmatrix} 1 & 0 & \frac{t^3}{3} & \frac{t^2}{2} \\ 0 & 1 & \frac{t^2}{2} & t \end{pmatrix}$$

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- $\tilde{\mathcal{V}} := \bigcup_{\ell\in\mathcal{V}} \{E \in \mathrm{LG}(V) : \ell \subset E\}$ (surface in $\mathrm{LG}(V)$).

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$$\rightsquigarrow \begin{pmatrix} c_{11} & c_{12}\\ c_{12} & c_{22}\end{pmatrix} = \begin{pmatrix} \frac{t^3}{3} & \frac{t^2}{2}\\ \frac{t^2}{2} & t\end{pmatrix} \quad (\text{curve in } \mathrm{LG}(V)).$$

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These all inherit $G_0 \cong GL_2$ invariance from $\mathcal{V}$.
(and $\mathfrak{g}_0 \subsetneq \mathfrak{csp}_4$ is a maximal subalgebra.)

Generalizations

For simple $G \neq \mathrm{SL}_2$, the adjoint variety $G/P \hookrightarrow \mathbb{P}(\mathfrak{g})$
$\rightsquigarrow$ contact grading
$\rightsquigarrow$ sub-adjoint variety: G_0-inv $\mathcal{V} \subset \mathbb{P}(V) = \mathbb{P}(\mathfrak{g}_{-1})$ ($\widehat{T}_\ell \mathcal{V}$ Lagrangian!)

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- Have unique max sym model: "flat" model, has sym $\mathfrak{g}$.
- Can efficiently compute syms of $\mathcal{E}, \mathcal{F}$ by using $\mathcal{V}$ instead. (In the flat cases, can do this uniformly and by-hand!)

Where does $\mathfrak{C}$ come from?

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Lemma

From L.-M., $\exists$CS-basis adapted to $V = \mathbb{C} \oplus W \oplus \mathbb{C} \oplus W^*$ s.t. $\mathcal{V} \subset \mathbb{P}(V)$ is given by $[\lambda, t^a] \to \left[\lambda^3, -\lambda^2 t^a, -\frac{\mathfrak{C}(t^3)}{2}, -\frac{3\lambda \mathfrak{C}_a(t^2)}{2}\right]$.

Using these as components wrt $\partial_{x^0} + u_0 \partial_u, \partial_{x^a} + u_a \partial_u, \partial_{u_0}, \partial_{u_a}$ leads to the asserted distinguished PDE.

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$\exists$ basis $\{\phi_i, \psi_i\}$ of $\mathbb{S}$ whose dual basis $\{\phi^i, \psi^i\}$ can be used to write:

$$\eta = \phi^0\psi^0 + \phi^1\psi^1 + \phi^2\psi^2 + \phi^3\psi^3$$

$$\begin{aligned} \mathrm{Q} = {} & \phi^0 \wedge \phi^1 \wedge \psi^0 \wedge \psi^1 + \phi^0 \wedge \phi^2 \wedge \psi^0 \wedge \psi^2 + \phi^0 \wedge \phi^3 \wedge \psi^0 \wedge \psi^3 \\ & - \phi^1 \wedge \phi^2 \wedge \psi^1 \wedge \psi^2 - \phi^1 \wedge \phi^3 \wedge \psi^1 \wedge \psi^3 - \phi^2 \wedge \phi^3 \wedge \psi^2 \wedge \psi^3 \\ & - 2\phi^0 \wedge \psi^1 \wedge \psi^2 \wedge \psi^3 + 2\phi^1 \wedge \phi^2 \wedge \phi^3 \wedge \psi^0 . \end{aligned}$$

The flat odd-contact $F(4)$-supergeometry

Definition

An odd-contact $F(4)$-supergeometry is $(M^{1|8}, \mathcal{C}, [\mathrm{Q}])$*, with* $\mathrm{Q} \in \Gamma(\bigodot^4 \mathcal{C}^*)$ *locally as above (for a conformal coframe of* $\mathcal{C}$*).*

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Locally, $(M^{1|8}, \mathcal{C})$ is $J^1(\mathbb{C}^{0|4}, \mathbb{C}^{1|0})$. For the frame $\partial_{x^i} + u_i \partial_u, \partial_{u_i}$ of $\mathcal{C}$, we have dual basis dx^i, du_i. Set $\phi^i = dx^i$, $\psi^i = du_i$ in Q, and call it the "flat" structure.

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Theorem

The flat odd-contact $F(4)$-supergeometry has symmetry $F(4)$.

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Locally, $(M^{1|8}, \mathcal{C})$ is $J^1(\mathbb{C}^{0|4}, \mathbb{C}^{1|0})$. For the frame $\partial_{x^i} + u_i \partial_u, \partial_{u_i}$ of $\mathcal{C}$, we have dual basis dx^i, du_i. Set $\phi^i = dx^i$, $\psi^i = du_i$ in Q, and call it the "flat" structure.

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The flat odd-contact $F(4)$-supergeometry has symmetry $F(4)$.

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The flat odd-contact $F(4)$-supergeometry

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Take inspiration from correspondence space / twistor theory. Abstractly, we are concerned with $F(4)/P^{\mathrm{I}}_{12} \to F(4)/P^{\mathrm{I}}_{1}$. How to geometrically "lift" the structure?

Lagrangian tangent spaces

Consider the quadric $\mathcal{V}^6 \hookrightarrow \mathbb{P}(\mathbb{S})$. Abstractly, $B_3/P_3 \cong \mathcal{V} \cong D_4/P_4$.

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canonical filtration	$\ell \subset L_\ell \mathcal{V} \subset \widehat{T}_\ell \mathcal{V} \subset \mathbb{S}$
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Consider the flag manifold $F(\mathcal{V}) = \{(\ell, L) : \ell \in \mathcal{V}, L = L_\ell \mathcal{V}\}$.

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A distinguished distribution on the ILG bundle

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For the flat odd-contact $F(4)$-supergeometry, we obtain:

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$\mathcal{D}$	$\partial_{u_{23}} - u_{12}\partial_{u_{02}} + u_{31}\partial_{u_{03}}$, $\partial_{u_{31}} - u_{23}\partial_{u_{03}} + u_{12}\partial_{u_{01}}$, $\partial_{u_{12}} - u_{31}\partial_{u_{01}} + u_{23}\partial_{u_{02}}$	$\widetilde{D}_{x0} - u_{23}\widetilde{D}_{x1} - u_{31}\widetilde{D}_{x2} - u_{12}\widetilde{D}_{x3}$
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From the ILG bundle to 3rd order super-PDE

Start with $\widetilde{M}^o \xrightarrow{\pi} M$, i.e. $J^2 \to J^1$. Construct $\check{M}^o \to \widetilde{M}^o$, i.e. $J^3 \to J^2$, as isotropic $E \subset \widetilde{\mathcal{C}}$ complementary to $\ker(\pi_*) = \mathrm{Ch}(\widetilde{\mathcal{C}}^2)$.

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Locally, we have coordinates $(x^i, u, u_i, u_{ij}, u_{ijk})$ with $\widetilde{\mathcal{C}} = \langle \widetilde{D}_{x^i}, \partial_{u_{ij}} \rangle$, $\ker(\pi_*) = \langle \partial_{u_{ij}} \rangle$, and $E = \langle \check{D}_{x^i} \rangle$ with

$$\check{D}_{x^i} := \widetilde{D}_{x^i} + \sum_{j<k} u_{ijk}\partial_{u_{jk}} = \partial_{x^i} + u_i\partial_u + u_{ij}\partial_{u_j} + \sum_{j<k} u_{ijk}\partial_{u_{jk}}.$$

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Now use $(\widetilde{M}^o, \widetilde{\mathcal{C}}, \mathcal{D})$ to geometrically construct a distinguished super-PDE $\Sigma \subset \check{M}^o \simeq_{\mathrm{loc}} J^3(\mathbb{C}^{0|4}, \mathbb{C}^{1|0})$:

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Require: $E \supset \mathcal{D}_{\bar{1}}$.

A simple computation

In $J^3(\mathbb{C}^{0|4}, \mathbb{C}^{1|0})$:
$$\begin{aligned} &\check{D}_{x^0} - u_{23}\check{D}_{x^1} - u_{31}\check{D}_{x^2} - u_{12}\check{D}_{x^3} \\ &= \widetilde{D}_{x^0} - u_{23}\widetilde{D}_{x^1} - u_{31}\widetilde{D}_{x^2} - u_{12}\widetilde{D}_{x^3} \\ &\qquad + \sum_{j<k}(u_{0jk} - u_{23}u_{1jk} - u_{31}u_{2jk} - u_{12}u_{3jk})\partial_{u_{jk}} \end{aligned}$$

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Require $\boxed{E \supset \mathcal{D}_{\bar{1}}}$ $\Rightarrow$ summation above must vanish!

$$\Rightarrow \boxed{u_{0ab} = u_{ab}u_{123}}$$

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