Partition Functions of Heterotic Potentials

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I: Introduction and Motivation

Physics / Pheno:

- The world is Quantum!

Mathematics:

- TQFT $\longleftrightarrow$ Geometric / enumerative / topological invariant theory.

Type II String: Topological A model and B model

Heterotic: All sectors are coupled!

I Stringy Moduli Problems

String theory/SUSY $\rightsquigarrow$ Geometries with
special structure
(Calabi-Yau, Instantons, ...)

Spacetime: $\quad M_{10} = M_4 \times X_6$

Supersymmetry: X_6 has special structure.

$$\left\{ \begin{array}{l} \text{deformations of } X_6 \\ \text{Moduli problems} \end{array} \right\} \leftrightsquigarrow \left\{ \begin{array}{l} \text{Interesting} \\ \text{physics in 4d.} \end{array} \right\}$$

3 levels of understanding moduli:

I) Infinitesimal massless spectrum:

- Geometry is described by BPS equations:

$$\mathrm{BPS} = 0 \leftarrow \text{SUSY equations}$$

- Infinitesimal deformation:

$$\delta(\mathrm{BPS}) = 0 \rightsquigarrow \mathcal{D}\alpha = 0$$

- Identity differential $\mathcal{D}$, $\mathcal{D}^2 = 0$.

- Massless fields / infinitesimal moduli:

$$TM = H^1_D = \frac{\{\text{deformations } \alpha \mid D\alpha = 0\}}{\{\text{symmetry transformations} \mid \alpha = D\gamma\}}$$

Infinitesimal symmetries

- D is usually part of an elliptic complex:

$$\dots \xrightarrow{D} \Omega^{-1} \xrightarrow{D} \Omega^0 \xrightarrow{D} \Omega^1 \to \Omega^2 \to \dots$$ $\rightsquigarrow$ Finite dimensional spectrum.

Exs: deformations of integrable complex structure:

μ: Beltrami differential $\rightsquigarrow$ $[\mu] \in H^{(0,1)}_{\bar\partial}(T^{(1,0)}X) \cong H^{(2,1)}_{\bar\partial}(X)$.

X is Calabi-Yau

II) Understand Geometry of moduli space $\mathcal{M}$:

- Geometric Structures on $\mathcal{M}$: Complex? Kähler?, ...

- Higher order def.'s; obstructions (Yukawa couplings), smooth directions, superpotential, ...

- Finite deformations: Solve Maurer-Cartan equation in associated L_∞-algebra:

$$\partial \alpha + \frac{1}{2}[\alpha, \alpha] \ (+ \ldots) = 0$$

Ex.: Finite def's of complex structure $\mu \in \Omega^{(0,1)}(T^{(1,0)}X)$ solve

$$\bar{\partial}\mu + \frac{1}{2}[\mu, \mu] = 0$$

$\rightsquigarrow$ Diff. graded Lie Algebra.

Tian - Todorov : X Calabi - Yau (or $\partial\bar{\partial}$-lemma)

$\Rightarrow$ infinitesimal complex structure moduli are unobstructed.

III) Understand Quantum moduli space :

- Quantize theory (BV - BRST, AKSZ, ...)

- Non-perturbative effects; Instantons, dualities, ...

associated to structure ...

- Compute Invariants : Knot invariants (CS-theory), Donaldson - Thomas, Gromov - Witten, ...

- Find topological theory governing geometric structure.

Exs:

Structure	"Target space"	"World-sheet"
Complex structure	Kodaira-Spencer theory $\leftrightarrow$	Wittens B-model
Kähler structure	Kähler-Gravity $\leftrightarrow$	Wittens A-model
Various Gauge theories	Chern-Simons Donaldson-Thomas (Hol. CS-theory) $\leftrightarrow$	Versions of topological open string

Mirror Symmetry

Open-Closed duality (Conifold transition)

II: Heterotic Modeli

The moduli problem of 6-dimensional heterotic geometries (Hull-Strominger solutions) are governed by the following action:

[Ashmore-delaOssa-Minasian-E.E.S - Strickland-Constable '18]

$$S(y) = \int_X \left(\langle y, \bar{D} y \rangle + \tfrac{1}{3} \langle y, [y,y] \rangle \right) \wedge \Omega$$

Holomorphic volume form

"Calabi-Yau"

- $y \in \Omega^1(\underbrace{T^{*1,0}X \oplus \mathrm{End}(V) \oplus T^{1,0}X}_{Q})$, $y = (x, \alpha, \mu)$

Hermitian gauge complex

- Natural differential $\bar{D}$; $\bar{D}^2 = 0$, pairing $\langle , \rangle$, and bracket $[,]$: Holomorphic Courant Algebroid

Similar theories: [Rosa et al '12, Costello-Li '15, 16, 19, Costello-Williams '21, ...]

Note:

- To preserve Supersymmetry: $\delta S(g) = 0$ and $S(g) = 0$
$\Rightarrow$ EOM is the MC-equation of an L_3-algebra.

- Infinitesimal EOM: $\bar{D} g = 0 \Rightarrow T\mathcal{M} \cong H^{(0,1)}_{\bar{D}}(Q)$.

- Note also similarity with holomorphic Chern-Simons. In fact:

$$x = \mu = 0 \Rightarrow \quad S = S_{CS}(\alpha) = \int_X tr(\alpha \bar{\partial} \alpha + \frac{1}{3}\alpha^3) \wedge \Omega$$

Note that the Partition function

$$Z = \int \mathcal{D}\alpha \, e^{-S_{CS}(\alpha)}$$

is the generating function for Donaldson-Thomas invariants.

Natural Question: Can we make sense of

$$Z = \int \mathcal{D}g \, e^{-S(g)}$$

and use it to compute invariants of heterotic geometries (holomorphic Courant algebroids)?

III: "Toy-Toy-Model": Chern-Simons

$$S_{CS}(A) = \int_{M_3} tr\left(A\,dA + \frac{2}{3}A^3\right)$$

$A \in \Omega^1(g)$, $\dim(M_3) = 3$.

EOM : $F(A_0) = dA_0 + A_0 \wedge A_0 = 0$.

1-loop Action: $A = A_0 + \alpha$, $\alpha \in \Omega^1(\mathrm{End}(V))$

$$\Rightarrow \quad S(\alpha) = \int_{M_3} \alpha\, d_0\, \alpha \quad , \quad d_0 = d + A_0 .$$

Partition function:

$$Z(M_3) = \frac{1}{\mathrm{Vol}(G)} \int \mathcal{D}\alpha \, e^{-S(\alpha)}$$

No metric $\rightsquigarrow$ Topological invariant of M_3.

$S(\alpha)$ has gauge symmetry: $\alpha \to \alpha + d_0 \epsilon$.

$\Rightarrow$ Quantise using BV-BRST formalism.

OR: Physical Approach [Pestun-Witten '05]

Often sufficient for quadratic actions...

Finite-dimensional case:

- S_{ij} symmetric pos. definite matrix

- $\int dx_1 dx_2 \ldots dx_n \, \mathrm{Exp}\left(-\frac{1}{2}\sum_{ij} x_i S_{ij} x_j\right) \propto \frac{1}{\det(S)^{\frac{1}{2}}}$

Generalisation to field theory:

Choose metric g on M_3:

$\Rightarrow$ Hodge decomposition: $\Omega^1 = d_0\Omega^0 \oplus d_0^{+}\Omega^2$

(we ignore finite-dimensional space of harmonic forms)

$$\Rightarrow Z = \frac{1}{\mathrm{Vol}(G)} \int_{\alpha \in d_0\Omega^0} \int_{\alpha \in d_0^{+}\Omega^2} \mathcal{D}\alpha \, \exp(-S(\alpha))$$

$$= \frac{1}{\mathrm{Vol}(G)} \frac{\mathrm{Vol}(d_0\Omega^0)}{\det(d_0 : d_0^+\Omega^2 \to d_0\Omega^1)^{\frac{1}{2}}}$$

Goal: Write answer in terms of determinants of elliptic operators (laplacians), whose determinants can be regularised.

Denominator: $\det(d_0 : d_0^+\Omega^2 \to d_0\Omega^1)^{\frac{1}{2}}$

We define:

$$\det(d_0|_{d_0^+\Omega^2}) := \det(d_0^+ d_0|_{d_0^+\Omega^2})^{\frac{1}{2}}$$

Note:

$$\det(d_0^+ d_0|_{d_0^+\Omega^2}) = \frac{\det(d_0^+ d_0|_{d_0^+\Omega^2})\det(d_0 d_0^+|_{d_0\Omega^0})}{\det(d_0 d_0^+|_{d_0\Omega^0})}$$

$$= \frac{\det(\Delta')}{\det\left(d_0 d_0^{\dagger}\big|_{d_0\Omega^0}\right)}$$

Consider eigenvector $d_0\alpha$ of $d_0 d_0^{\dagger}$ on $d_0\Omega^0$:

$$d_0 d_0^{\dagger} d_0 \alpha = \lambda d_0 \alpha$$

$$\Updownarrow \qquad (d_0 \text{ is invertible on } \mathrm{Im}(d_0^{\dagger}))$$

$$\underbrace{d_0^{\dagger} d_0}_{\Delta^0} \alpha = \lambda \alpha$$

$\Rightarrow \quad \det\left(d_0 d_0^{+}\big|_{d_0\Omega^0}\right) = \det(\Delta^0)$

$\Rightarrow \quad \det\left(d_0^{+} d_0\big|_{d_0^{+}\Omega^2}\right) = \dfrac{\det(\Delta^1)}{\det(\Delta^0)}$

Numerator: $\quad \mathrm{Vol}\left((d_0\Omega^0)\right).$

Recall: Given Linear operator $A : V \to W$

$$\Rightarrow \quad \mathrm{Vol}(AV) = \det(A) \frac{\mathrm{Vol}(V)}{\mathrm{Vol}(\ker(A))}$$

$$\Rightarrow \quad \mathrm{Vol}(d_0 \Omega^0) = \underbrace{\det(d_0|_{\Omega^0})}_{\det(\Delta^0)^{\frac{1}{2}}} \frac{\mathrm{Vol}(\Omega^0)}{\mathrm{Vol}[\underbrace{\ker(d_0|_{\Omega^0})}_{H^0(M_3)}] = 1}$$

$$\Rightarrow \quad \mathrm{Vol}(d_0 \Omega^0) = \det(\Delta^0)^{\frac{1}{2}} \underbrace{\mathrm{Vol}(\Omega^0)}_{\text{Volume of gauge-transformations}}$$

Collect:

$$Z(M_3) = \frac{1}{\cancel{\mathrm{Vol}(G')}} \frac{\det(\Delta^0)^{\frac{3}{4}}}{\det(\Delta^1)^{\frac{1}{4}}} \cancel{\mathrm{Vol}(\Omega^0)}$$

Regularise

$\Rightarrow \quad Z(M_3) = \dfrac{|\widetilde{\Delta^0}|^{\frac{3}{4}}}{|\widetilde{\Delta^1}|^{\frac{1}{4}}}$

This is the Ray-Singer torsion of M_3.

This is Topological!

IV: Heterotic Superpotential

Superpotential [Cardoso et al '03, Gurrieri et al '04, ...]:

$$W = \int_{X_6} (H + i\,d\omega) \wedge \Omega \rightsquigarrow S(y) \text{ above}$$

For us: - Background X_6 is Calabi-Yau.

- Large volume ($\alpha' \to 0$)

$\rightsquigarrow$ gauge sector decouples.

$\Rightarrow$ 1-loop action:

$$S = \int_{X_6} \left[x \wedge \bar\partial \chi + b \wedge \partial \chi + c \,\bar\partial b \right]$$

$x \in \Omega^{1,1}$, $\chi \in \Omega^{2,1}$, $b \in \Omega^{0,2}$, $c \in \Omega^{3,0}$.

$\Rightarrow$ 1-loop partition function:

$$Z_W(X_6) = \frac{|\tilde{\Delta}^{1,0}|\,|\tilde{\Delta}^{0,1}|}{|\tilde{\Delta}^{1,1}|^{\frac{1}{2}}\,|\tilde{\Delta}^{0,0}|^{\frac{3}{2}}}$$

Again, $Z_W(X_6)$ is topological!

Note:

$$Z^{1\text{-loop}}_{IIA}(X) = Z^{1\text{-loop}}_{IIB}(X) \text{ at large volume } (\alpha' \to 0).$$

Hermitian structure

complex structure

In heterotic, we might then expect:

$$Z_W^{\text{1-loop}}(X) = Z_K^{\text{1-loop}}(X) \qquad (\alpha' \to 0)$$

where K is the Kähler potential:

$$e^{-K} = i\int_X \Omega\wedge\bar{\Omega} + \frac{1}{6}\int_X \underbrace{\omega\wedge\omega\wedge\omega}_{\text{Kähler form}}$$

$$Z_\Omega(X) \qquad\qquad Z_\omega(X)$$

The 1-loop partition function of $Z_\Omega(X)$ has been computed in [Pestun-Witten '05]. We find

$$Z_W^{\text{1-loop}}(X) = Z_\Omega^{\text{1-loop}}(X)^2$$

BUT: Ordinary Mirror symmetry suggests

$$Z_\Omega^{\text{1-loop}}(X) = Z_\omega^{\text{1-loop}}(X) \qquad (\alpha' \to 0)$$

We would hence get

$$Z_W^{\text{1-loop}}(X) = Z_\Omega^{\text{1-loop}}(X)^2 = Z_\Omega^{\text{1-loop}}(X)\, Z_\omega^{\text{1-loop}}(X) = Z_{\mathcal{K}}^{\text{1-loop}}(X) \quad (*)$$

We are in the process of confirming this.

V : Conclusion / Outlook

- The 1-loop partition function of the heterotic superpotential (of the geometric sector) is Topological.

Outlook:

- Check $(*)$: Compute $Z_{\mathcal{K}}(X_0)$. (in progress)

- Turn on α': Couplings to gauge sector...
  (Het. Bianchi: $dH = \frac{\alpha'}{4}\left(\mathrm{tr}\, F\wedge F - \mathrm{tr}\, R\wedge R\right)$)

- Applications in (0,2)-Mirror Symmetry ?

- World-sheet models (holomorphic β-γ-system, holomorphic twist, etc)

- Beyond 1-loop: Higher order couplings / invariants

  Note: The cubic coupling in $S(\eta)$ has a holomorphic derivative $\rightsquigarrow$ treat as a coupling through the machinary of jet bundles ??

Thank You!