Monoidally Graded Manifolds

Shuhan Jiang

Max Planck Institute for Mathematics in the Sciences

Workshop on Geometric Structures and Supersymmetry
August 26, 2022

Contents

Parity Functions

Monoidally Graded Ringed Spaces

Monoidally Graded Manifolds

Applications

Contents

Parity Functions

Monoidally Graded Ringed Spaces

Monoidally Graded Manifolds

Applications

Let $(\mathcal{I}, 0, +)$ be a commutative monoid. A parity function is a (non-trivial) monoid homomorphism $p : \mathcal{I} \to \mathbb{Z}_2$, or equivalently, a $\mathbb{Z}_2$ grading on $\mathcal{I}$ which is compatible with the additive structure.

	0	a	b
0	0	a	b
a	a	b	a
b	b	a	b

Table 1.1: A commutative monoid of order 3.

p is defined by setting $\mathcal{I}_0 = \{0, b\}$ and $\mathcal{I}_1 = \{a\}$.

Proposition 1.1.

If $\mathcal{I}$ is cancellative, then $|\mathcal{I}_0| = |\mathcal{I}_1|$.

Given a cancellative $\mathcal{I}$, how can one construct p for $\mathcal{I}$?
If $\mathcal{I}$ is finite, then it is isomorphic to a direct product of cyclic groups of prime-power order. One of these cyclic groups must be $\mathbb{Z}_{2^k}$, $k \geq 1$. We can write

$$\mathcal{I} = \mathbb{Z}_{2^k} \times \cdots$$

and define p by sending $(x, \cdots) \in \mathcal{I}$ to $a - 1 \pmod 2$, where a is the order of $x \in \mathbb{Z}_{2^k}$.

If $\mathcal{I}$ is infinite, the construction of p is hard, perhaps not possible in general. However, the case of a free $\mathcal{I}$ is easy: one can choose $\mathcal{I}_0$ be the submonoid of elements generated by even number of generators, and $\mathcal{I}_1$ be the subset of elements generated by odd number of generators. As an example, let $\mathcal{I}$ be $\mathbb{N}$, the monoid of natural numbers under addition. p is then defined by sending even numbers to 0 and odd numbers to 1.

Let $K(\mathcal{I})$ denote the Grothendieck group of $\mathcal{I}$.

Proposition 1.2.

Let p be a parity function for $\mathcal{I}$. The map

$$\begin{aligned} p' : K(\mathcal{I}) &\to \mathbb{Z}_2 \\ [(a_1, a_2)] &\mapsto p(a_1) + p(a_2) \end{aligned}$$

is well-defined and gives a parity function for $K(\mathcal{I})$.

As an example, consider $K(\mathbb{N}) = \mathbb{Z}$, the monoid of integers under addition. The parity function p' induced from the parity function p for $\mathbb{N}$ again sends even numbers to 0 and odd numbers to 1.

Contents

Parity Functions

Monoidally Graded Ringed Spaces

Monoidally Graded Manifolds

Applications

Let R be a commutative ring. Let $\mathcal{I}$ be a countable cancellative commutative monoid equipped with a parity function p.

Definition 2.1.

An $\mathcal{I}$-graded R-module is an R-module V with a family of sub-modules $\{V_i\}_{i\in\mathcal{I}}$ indexed by $\mathcal{I}$ such that $V = \bigoplus_{i\in\mathcal{I}} V_i$. $v \in V$ is said to be homogeneous if $v \in V_i$ for some $i \in \mathcal{I}$.

Given two $\mathcal{I}$-graded R-modules V and W, we make $V \oplus W$ and $V \otimes W$ into $\mathcal{I}$-graded R-modules by setting

$$V \oplus W = \bigoplus_{i \in \mathcal{I}} (V_i \oplus W_i), \quad V \otimes W = \bigoplus_{k \in \mathcal{I}} \left(\bigoplus_{i+j=k} V_i \otimes W_j \right).$$

We can also make $\mathrm{Hom}(V, W)$ into a $K(\mathcal{I})$-graded R-module:

$$\mathrm{Hom}(V, W) = \bigoplus_{\alpha \in K(\mathcal{I})} \mathrm{Hom}(V, W)_\alpha,$$

$\mathrm{Hom}(V, W)_\alpha = \{f \in \mathrm{Hom}(V, W) | f(V_i) \subset W_j, [(j, i)] = \alpha\}$.
A morphism from V to W is just an element of $\mathrm{Hom}(V, W)_0$.

Suppose that $\mathcal{I}$ is also a semi-ring. We write ab as the multiplication of a and b in $\mathcal{I}$.

Definition 2.2.

An $\mathcal{I}$-graded R-module A is called an $\mathcal{I}$-graded R-algebra if A is a unital associative R-algebra and if the multiplication $\mu : A \otimes A \to A$ is a morphism of $\mathcal{I}$-graded R-modules. We write $xy = \mu(x \otimes y)$ as the shorthand notation. A is said to be commutative if

$$xy - (-1)^{p(d(x)d(y))} yx = 0 \tag{2.1}$$

for all homogeneous $x, y \in A$.

Remark 2.1.

We have to be careful about the sign factor appearing in the right hand side of (2.1). Although both of $\mathcal{I}$ and $\mathbb{Z}_2$ are semi-rings, p is not necessarily a semi-ring homomorphism and we do not have $p(d(x)d(y)) = p(x)p(y)$ in general. Bearing this in mind, we will use $(-1)^{p(x)p(y)}$ to replace the sign factor $(-1)^{p(d(x)d(y))}$ for simplicity.

Definition 2.3.
The tensor algebra $\mathrm{T}(V)$ is the $\mathcal{I}$-graded R-module $\mathrm{T}(V) = \bigoplus_{n\in\mathbb{N}} V^{\otimes^n}$, together with the tensor product $\otimes$ as the canonical multiplication. The symmetric algebra $\mathrm{S}(V)$ is the quotient algebra of $\mathrm{T}(V)$ by the $\mathcal{I}$-graded two-sided ideal generated by

$$v \otimes w - (-1)^{p(v)p(w)} w \otimes v,$$

where $v, w \in V \subset \mathrm{T}(V)$ are homogeneous.

$\mathrm{S}(V)$ has a canonical $\mathbb{N}$-grading inherited from $\mathrm{T}(V)$ which should not be confused with its $\mathcal{I}$-grading. We write

$$\mathrm{S}(V) = \bigoplus_{n\in\mathbb{N}} \mathrm{S}^n(V)$$

to indicate that fact.

$\mathrm{S}(V)$ is universal in the sense that, given a commutative $\mathcal{I}$-graded R-algebra A and a morphism $f : V \to A$. There exists a unique algebraic homomorphism $\tilde{f} : \mathrm{S}(V) \to A$ such that the following diagram commutes.

$$\begin{array}{ccc} V & \overset{\iota}{\hookrightarrow} & \mathrm{S}(V) \\ & \searrow^{f} & \downarrow{\scriptstyle \tilde{f}} \\ & & A \end{array}$$

Definition 2.4.
The $\mathcal{I}$-graded algebra of formal power series on V is the R-module

$$\mathrm{S}[V] = \prod_{n \in \mathbb{N}} S^n(V)$$

equipped with the canonical algebraic multiplication.

Let $I = \bigoplus_{n>0} S^n(V)$. One can equip $\mathrm{S}(V)$ with the so-called I-adic topology. The I-adic completion of $\mathrm{S}(V)$ is defined as the inverse limit

$$\widehat{\mathrm{S}(V)}_I := \varprojlim \mathrm{S}(V)/I^n.$$

By the universal property of the inverse limit, one has a morphism

$$\iota_I : \mathrm{S}(V) \to \widehat{\mathrm{S}(V)}_I$$

with kernel equal to $\bigcap_{n>0} I^n$. There is a canonical identification $\widehat{\mathrm{S}(V)}_I \cong \mathrm{S}[V]$ under which ι_I coincides with the canonical embedding of $\mathrm{S}(V)$ into $\mathrm{S}[V]$.

Lemma 2.1.

Let A be a commutative $\mathcal{I}$-graded R-algebra. Let J be an ideal of A such that A is J-adic complete. $S[V]$ is universal in the sense that, given a morphism $f : V \to A$ such that $f(V) \subset J$, there exists a unique (continuous) algebraic homomorphism $\tilde{f} : \mathrm{S}[V] \to A$ such that the following diagram commutes

$$\begin{array}{ccc} V & \overset{\iota}{\hookrightarrow} & \mathrm{S}[V] \\ & \searrow^{f} & \downarrow{\scriptstyle \tilde{f}} \\ & & A \end{array}$$

Let k be a field and R be a commutative k-algebra. Let A be a commutative $\mathcal{I}$-graded k-algebra.

Definition 2.5.

A k-algebra epimorphism $\epsilon : A \to R$ is called a body map of A if ϵ preserves the relevant $\mathcal{I}$-gradings.

Lemma 2.2.
Let V be an $\mathcal{I}$-graded R-module with $V_0 = 0$. Let ϵ be an R-linear body map of $\mathrm{S}[V]$. Then ϵ is unique.

Remark 2.2.
Let V be as above. Suppose $A \cong \mathrm{S}[V]$ as $\mathcal{I}$-graded k-algebras. In particular, this implies that A admits a decomposition $A = A' \oplus I$ where $A' \cong R$ and I is the ideal generated by homogeneous elements of non-zero degree. Let ϵ be a body map of A. Since $I \subset \ker \epsilon$, ϵ is determined by $\epsilon|_{A'}$. In other words, ϵ is determined by a k-algebra endomorphism of R.

More can be said if V is free.

Lemma 2.3.

Let V be a free $\mathcal{I}$-graded R-module with $V_0 = 0$. Let ϵ be a R-linear body map of $\mathrm{S}[V]$. (By Lemma 2.2, ϵ is the canonical one.) Let I denote the kernel of ϵ. Then there exists an R-algebra isomorphism

$$\mathrm{S}[V] \cong \mathrm{S}[I/I^2],$$

where I^2 is the square of the ideal I.

Lemma 2.4.

Let ϵ be the canonical body map of $\mathrm{S}[V]$. Then for $f \in \mathrm{S}[V]$, f is invertible if and only if $\epsilon(f)$ is invertible.

Corollary 2.1.

$\mathrm{S}[V]$ *is local if* R *is local.*

Remark 2.3.

As is in the case of $\mathcal{I} = \mathbb{Z}$, it is actually crucial to work with $\mathrm{S}[V]$ instead of $\mathrm{S}(V)$ when the even part of V is non-trivial. The former allows us to have a coordinate description of morphisms between "$\mathcal{I}$-graded domains", a notion of partition of unity for "$\mathcal{I}$-graded manifolds", and more.

Definition 2.6.

An $\mathcal{I}$-graded ringed space is a ringed space $(X, \mathcal{O})$ such that

1. $\mathcal{O}(U)$ is an $\mathcal{I}$-graded algebra for any open subset U of X;
2. the restriction morphism $\rho_{V,U} : \mathcal{O}(U) \to \mathcal{O}(V)$ is a morphism of $\mathcal{I}$-graded algebras.

A morphism between two $\mathcal{I}$-graded ringed spaces $(X_1, \mathcal{O}_1)$ and $(X_2, \mathcal{O}_2)$ is just a morphism $\varphi = (\tilde{\varphi}, \varphi^*)$ between ringed spaces such that $\varphi^*_U : \mathcal{O}_2(U) \to \mathcal{O}_1(\tilde{\varphi}^{-1}(U))$ preserves the $\mathcal{I}$-grading for any open subset U of X_2.

Let (X, C) be a ringed space where $C(U)$ are commutative rings.

Definition 2.7.

Let $\mathcal{F}$ be an $\mathcal{I}$-graded C-module. The formal symmetric power $\mathrm{S}[\mathcal{F}]$ of $\mathcal{F}$ is the sheafification of the presheaf

$$U \to \mathrm{S}[\mathcal{F}(U)],$$

where $\mathrm{S}[\mathcal{F}(U)]$ is the $\mathcal{I}$-graded algebra of formal power series on the $C(U)$-module $\mathcal{F}(U)$.

Lemma 2.5.

Let $\mathcal{A}$ be a commutative $\mathcal{I}$-graded C-algebra. Let $\mathcal{B}$ be a sub-sheaf of $\mathcal{A}$ such that $\mathcal{A}(U)$ is $\mathcal{B}(U)$-adic complete for all open subsets U. $\mathrm{S}[\mathcal{F}]$ is universal in the sense that, given a morphism of $\mathcal{I}$-graded C-modules $F : \mathcal{F} \to \mathcal{A}$ such that $F(\mathcal{F}(U)) \subset \mathcal{B}(U)$ for all open subsets U, there exists a unique morphism of $\mathcal{I}$-graded C-algebras $\tilde{F} : \mathrm{S}[\mathcal{F}] \to \mathcal{A}$ such that the following diagram commutes.

$$\begin{array}{ccc} \mathcal{F} & \overset{\iota}{\hookrightarrow} & \mathrm{S}[\mathcal{F}] \\ & \underset{F}{\searrow} & \downarrow \tilde{F} \\ & & \mathcal{A} \end{array}$$

Contents

Parity Functions

Monoidally Graded Ringed Spaces

Monoidally Graded Manifolds

Applications

Throughout this section, V is a real $\mathcal{I}$-graded vector space with $V_0 = 0$. $\dim V_i = m_i$.

Definition 3.1.

Let U be a domain of $\mathbb{R}^n$. An $\mathcal{I}$-graded domain $\mathcal{U}$ of dimension $n|(m_i)_{i\in\mathcal{I}}$ is an $\mathcal{I}$-graded ringed space $(U, \mathcal{O})$, where $\mathcal{O}$ is the sheaf of $\mathrm{S}[V]$-valued smooth functions.

Remark 3.1.

$\mathcal{U}$ is a locally ringed space by Corollary 2.1.

Remark 3.2.

Recall that we have a canonical body map $\epsilon : C^\infty(U) \otimes \mathrm{S}[V] \to C^\infty(U)$, which induces a monomorphism $U \hookrightarrow \mathcal{U}$.

Definition 3.2.

Let M be a n-dimensional manifold. An $\mathcal{I}$-graded manifold $\mathcal{M}$ of dimension $n|(m_i)_{i\in\mathcal{I}}$ is an $\mathcal{I}$-graded ringed space $(M, \mathcal{O}_M)$ which is locally isomorphic to an $\mathcal{I}$-graded domain of dimension $n|(m_i)_{i\in\mathcal{I}}$. That is, for each $x \in M$, there exist an open neighborhood U_x of x, an $\mathcal{I}$-graded domain $\mathcal{U}$, and an isomorphism of locally ringed spaces

$$\varphi = (\tilde{\varphi}, \varphi^*) : (U_x, \mathcal{O}_M|_{U_x}) \to \mathcal{U}.$$

φ is called a chart of $\mathcal{M}$ on U_x. (We often refer to U_x as a chart too.)

Remark 3.3.
A function on an $\mathcal{I}$-graded manifold of non-zero degree is not necessarily nilpotent as in the $\mathbb{Z}_2$-graded setting, which means that one cannot pass to the underlying manifold by simply modding out all nilpotent functions.

Remark 3.4.
The vector fields on an $\mathcal{I}$-graded manifold is in general not $\mathcal{I}$-graded, but $K(\mathcal{I})$-graded, where $K(\mathcal{I})$ is the Grothendieck group of $\mathcal{I}$.

Lemma 3.1.

Let $F : C^\infty \to C^\infty$ be an endomorphism of sheaves of commutative rings on M. Then F must be the identity.

Let U be an arbitrary open subset of M. We can choose a collection of charts $\{U_\alpha\}$ such that $U = \bigcup_\alpha U_\alpha$. For $f \in \mathcal{O}(U)$, one can apply the restriction morphisms to f to get a sequence of sections f_α in $\mathcal{O}(U_\alpha)$. Apply the canonical body map ϵ to each of them to get a sequence of smooth functions $\tilde{f}_\alpha$ in $C^\infty(U_\alpha)$. By Lemma 3.1 and Remark 2.2, ϵ is actually the unique sheaf morphism from $\mathcal{O}|_{U_\alpha}$ to $C^\infty|_{U_\alpha}$. Therefore, $\tilde{f}_\alpha$ are compatible with each other and can be glued together to give a smooth function $\tilde{f}$ over U. In this way, we construct a unique body map for every open subset of M, which are compatible with restrictions.

Theorem 3.1.

There exists a unique underlying manifold of $\mathcal{M}$.

Definition 3.3.

An $\mathcal{I}$-graded manifold $\mathcal{M}$ is called projected if there exists a splitting of the short exact sequence (of sheaves of rings)

$$0 \longrightarrow \mathcal{O}^1 \longrightarrow \mathcal{O} \xrightarrow{\epsilon} C^\infty \longrightarrow 0, \tag{3.1}$$

where $\mathcal{O}^1$ is the kernel of ϵ.

The structure sheaf $\mathcal{O}$ of a projected manifold is a C^∞-module.

Definition 3.4.

A projected $\mathcal{I}$-graded manifold $\mathcal{M}$ is called split if there exists a splitting of the short exact sequence (of C^∞-modules)

$$0 \longrightarrow \mathcal{O}^2 \longrightarrow \mathcal{O}^1 \xrightarrow{\pi} \mathcal{O}^1/\mathcal{O}^2 \longrightarrow 0, \tag{3.2}$$

where $\mathcal{O}^2$ is the square of $\mathcal{O}^1$.

Let $\mathcal{O}$ be the structure sheaf of a projected $\mathcal{I}$-graded manifold. Let $\mathcal{F}$ denote the sheaf $\mathcal{O}^1/\mathcal{O}^2$. $\mathcal{F}$ is an $\mathcal{I}$-graded C^∞-module and we can define its formal symmetric power $\mathrm{S}[\mathcal{F}]$. By construction, the ringed space $\mathcal{M}_S = (M, \mathrm{S}[\mathcal{F}])$ is also a projected $\mathcal{I}$-graded manifold.

Lemma 3.2.

Let $\mathcal{O}^1$ be the kernel of ϵ. $\mathcal{O}$ is $\mathcal{O}^1$-adic complete. That is, for any open subset U, $\mathcal{O}(U)$ is $\mathcal{O}^1(U)$-adic complete.

Using Lemma 3.2 and the universal property of $\mathrm{S}[\mathcal{F}]$, one can prove

Proposition 3.1.

Let $\mathcal{M} = (M, \mathcal{O})$ be a projected $\mathcal{I}$-graded manifold. $\mathcal{M}$ is split if and only if $\mathcal{M} \cong \mathcal{M}_S$.

Lemma 3.3.

Let $\mathcal{F}$ and $\mathcal{G}$ be two locally free C-modules. Then the obstruction to the existence of a splitting of the short exact sequence of C-modules

$$0 \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow \mathcal{F} \longrightarrow 0.$$

can be represented as an element in the first sheaf cohomology group $H^1(X, \mathrm{Hom}(\mathcal{F}, \mathcal{G}))$ of $\mathrm{Hom}(\mathcal{F}, \mathcal{G})$.

Proposition 3.2.

Every projected $\mathcal{I}$-graded manifold is split.

Proposition 3.3.

Every $\mathcal{I}$-graded manifold is projected.

Proof.

Let $\mathcal{O}_{(i)} = \mathcal{O}/\mathcal{O}^{i+1}$. Let $\phi_{(0)} : C^\infty \to \mathcal{O}_{(0)}$ be the identity morphism. One can construct by induction on i mappings $\phi_{(i+1)} : C^\infty \to \mathcal{O}_{(i+1)}$ such that $\pi_{i+1,i} \circ \phi_{(i+1)} = \phi_{(i)}$, where $\pi_{i+1,i} : \mathcal{O}_{i+1} \to \mathcal{O}_i$ is the canonical epimorphism. The obstruction to the existence of $\phi_{(i+1)}$ can be represented as an element $\omega(\phi_{(i)}) \in H^1(M, (\mathcal{T} \otimes \mathrm{S}^{i+1}(\mathcal{F}))_0)$, where $\mathcal{T}$ is the tangent sheaf of the underlying manifold M.

Proposition 3.3.

Every $\mathcal{I}$-graded manifold is projected.

Proof.

Due to the existence of a smooth partition of unity on M, $H^1(M, (\mathcal{T} \otimes \mathrm{S}^{i+1}(\mathcal{F}))_0) = 0$ and $\omega(\phi_{(i)}) = 0$. It follows that there exists a unique morphism $\phi : C^\infty \to \varprojlim \mathcal{O}_{(i)}$ such that $\pi_i \circ \phi = \phi_{(i)}$, where $\pi_i : \varprojlim \mathcal{O}_{(i)} \to \mathcal{O}_i$ is the canonical projection. By Lemma 3.2, ϕ can be seen as a morphism from C^∞ to $\mathcal{O}$. Note that $\pi_0 = \epsilon$ and $\pi_0 \circ \phi = \phi_{(0)} = \mathrm{id}$. ϕ splits (3.1). □

An $\mathcal{I}$-graded vector bundle $\pi : E \to M$ is a vector bundle such that $E = \bigoplus_{k\in\mathcal{I}} E_k$ where E_k are vector bundles whose fibers consist of elements of degree k. To any $\mathcal{I}$-graded vector bundle E we can associate an $\mathcal{I}$-graded ringed space with the underlying topological space being M and the structure sheaf being the sheaf of sections of $\mathrm{S}(\bigoplus_{k\in\mathcal{I}}(E_k)^*)$. (This is an $\mathcal{I}$-graded manifold in our sense if the fiber of E does not contain elements of degree 0.)

Theorem 3.2.

Every $\mathcal{I}$-graded manifold can be obtained from an $\mathcal{I}$-graded vector bundle.

Contents

Parity Functions

Monoidally Graded Ringed Spaces

Monoidally Graded Manifolds

Applications

Definition 4.1.

A QK-manifold is a bigraded manifold equipped with three vector fields Q, K and d of degree $(0,1)$, $(1,-1)$ and $(1,0)$, respectively, satisfying the following relations

$$Q^2 = 0, \quad d^2 = 0, \quad QK + KQ = d, \quad Kd + dK = 0.$$

K is known as the vector supersymmetry in the physics literature. It can be used to study the descent equations

$$Q(\mathcal{O}^{(p)}) = d(\mathcal{O}^{(p-1)}) \tag{4.1}$$

in a topological field theory.

Theorem 4.1.

Every solution to (4.1) *is a* K*-sequence up to some exact sequence.*

Thank you!

The preprints: https://arxiv.org/abs/2206.02586 and https://arxiv.org/abs/2202.12425.