Quasired | Let G be algebraic quasireductive supergroup
- G_0 be algebraic reductive
- Lie $\mathfrak{g} = \mathfrak{g}_0$ Lie superalgebra

Rep G = Rep $(\mathfrak{g}, G_0)$, finite dimensional representations completely reducible over G_0.

Hom. odd ~~vector~~ element | An ~~vector~~ odd element $x \in \mathfrak{g}_1$ is called homological if $[x, x] = 2x^2$ is a semisimple element of $\mathfrak{g}$
$x^2 = 0$ | $\mathfrak{g}_1^{hom}$ the set of (odd) homological elements.

DS_x Def | For any $x \in \mathfrak{g}_1^{hom}$ and a representation M we set $DS_x M = M^{x^2} \xrightarrow{x} M^{x^2}$ cohomology.
$= \mathrm{Ker}\, x / \mathrm{Im}\, x \cap \mathrm{Ker}\, x$.

$DS_x M$ is a vector superspace

Main properties | Property.
- $DS_x M^* \simeq DS_x M$
- $DS_x (M \oplus N) \simeq DS_x M \oplus DS_x N$
- $DS_x (M \otimes N) \simeq DS_x M \otimes DS_x N$
- $0 \to N \to M \to L \to 0$

$0 \to E \to DS_x N \to DS_x M \to DS_x L \to \Pi E \to 0$
- sdim $DS_x M$ = sdim M

tensor functor | $DS_x \mathfrak{g} = \mathfrak{g}_x$ is again a Lie superalgebra
Rep $\mathfrak{g} \to$ Rep $\mathfrak{g}_x$ is a symmetric monoidal functor.
holomorphic twist

Proj. modules | Projective modules: Direct summands of $\mathrm{Ind}_{\mathfrak{g}_0}^{\mathfrak{g}} M_0$.
Lemma. If $x \neq 0$ then $DS_x M \neq 0$. (and M is projective!)

One important property of projective modules: relation between Kostant cohomology and BWB theory in supercase.

$$\left[H^i(G/B, M) : L(\mu) \right] = Ext^i_{n^+}(M, P(\mu))$$

G/B

irreducible $\overset{sub}{\text{quotient}}$

$$= H^i(n^+, M^* \otimes P(\mu))$$

More about proj. modules

Restriction: Rep G $\longrightarrow$ Rep K maps projective to projective if K is quasireductive.

Def.

Support theory. M be a representation of G

$X_M = \text{proj}\{ x \in g_1^{hom} \mid DS_x M \neq 0 \}$

- $X_{M^*} = X_M$
- $X_{M \oplus N} = X_M \cup X_N$
- $X_{M \otimes N} = X_M \cap X_N$
- $X_M = \emptyset$ iff M is projective (want to invert this property)
- X_M is G_0-invariant subset of proj (g_1^{hom}).

Geometry. Let $\mathcal{X}$ be some affine supermanifold, and Q an odd vector field on $\mathcal{X}$, [Q,Q] is s.s on $\mathcal{O}(\mathcal{X})$ the space of regular functions

A. Sherman
D. Vaintrob

Reduction to the $Q^2=0$

Another important point ~~…~~ Γ

$$\mathcal{X}^{Q^2} = \mathrm{Spec}\left(\mathcal{O}(\mathcal{X}) / (Q\,\mathcal{O}(\mathcal{X}))\right)$$

descend to $\bar{Q}$, such that $\bar{Q}^2 = 0$ on $\mathcal{X}^{Q^2}$ is a nonsingular supermanifold

$\boxed{\text{Iversen theorem.}}$

$\boxed{\text{Localization theorem}}$ (A. Sherman, V. S.)

Let $\mathfrak{q} = \langle Q, \Upsilon Q, [Q,Q] \rangle$

$\mathcal{X}$ be a $\mathfrak{q}$-variety, $[Q,Q]$ acts semisimpl

Let $\mathcal{V}$ be some $\mathfrak{q}$-equivariant vector bundle on $\mathcal{X}$, $Z = Z(Q) \subseteq \mathcal{X}_0$ is the vanishing subvariety of Q and suppose Y is a smooth closed subvariety of $\mathcal{X}$

(1) $Z_0 \subset Y_0$

(2) $Q(I_Y) \subseteq I_Y$

Koszulity condition (3) $Q : ((N^*_Y)_p)_1 \to ((N^*_Y)_p)_0$ is an isomorphism for all $p \in Z_0$

void (4) For all $p \in Z_0$ $\exists$ Q stable open n-hood $\mathcal{U}_p$ $\Gamma(\mathcal{U}_p, \mathcal{V}) \to \mathcal{V}_p$ splits over $\mathfrak{q}$. Then Y is a $\mathfrak{q}$-subvariety

$$DS_Q\,\Gamma(\mathcal{X}, \mathcal{V}) = DS_Q\,\Gamma(Y, \mathcal{V}|$$

Important case $\Gamma(\mathcal{V})$, ~~…~~ $= \mathcal{O}(\mathcal{X})$
the condition (4) is automatic

Then $\mathcal{D}S_Q(\mathcal{O}(\mathcal{X})) = \mathcal{D}S_Q(\mathcal{O}(\mathcal{Y}))$.

Application to homogeneous spaces:

$G/K = GL(m|n)/GL(r|s)/GL(m-r/n-s)$

$G/K \subset Gr(r|s, m|n) \times Gr(m-r|n-r)$, Cotangent space to

Complexificatio of Grassmannian

sdim $G/K \geq 0$

(*) $r \geq s$, $m \geq n$, choose $Q \in (\mathrm{Lie}\ K)_1^{\mathrm{hom}}$ in

a generic position.

$$\mathcal{D}S_Q(G/K) = \coprod_{\binom{n}{s}} GL(m-n)/G(r-s) \times GL((m-r)-(n-s))$$

Cartesian product of $\binom{n}{s}$ copies.

Relations with volume theorem:

Vol $Gr(r|s, m|n) \neq 0$ iff sdim $Gr(r|s, m|n) \geq 0$

Vol $Gr(r|s, m|n) = \binom{n}{s}(2\pi)^A$ Vol $Gr(r-s|m-n)$.

A dim $Gr(r|s, m|n)_{odd}$

Schwarz-Zaboronsky localization formula.

Sam-Snowden result about cohomology of the structure sheaf on supergrassmannian.

Theorem. If sdim $\overbrace{Gr(r|s, m|n)}^{\mathcal{X} =} \geq 0$, $r \geq s$

then $H^\cdot(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \simeq H^*_{\mathrm{sing}}\ Gr(s, n)$ as a graded algebra.

Open problem. $\mathcal{D}S_x$ on non-affine varie

Theorem. M is projective iff $X_M = \emptyset$.

defect Proof. (1) $GL(m|n)$ $d = \min(m, n)$

$\mathcal{D} = SL(1|1)^d \subset GL(m|n)$ $Ext^i_G(\cdot,\cdot) \to Ext^i_D(\cdot,\cdot)$ inj

Unitary trick (2) Proof of the statement for $\mathcal{D}$ (Avramov).

(3) Proposition. $K \subset G$, $M \in Rep\, K$

~~If~~ $X^K_M = \emptyset \Rightarrow X^G_{Ind^G_K M} = \emptyset$

using localization theorem.

(4) G be some quasireductive group, $G \subset GL(m|n)$ (apply 3).

$$\boxed{X_M = \emptyset \Leftrightarrow Ext^1(\mathbb{C}, X_M) = 0}$$

$Ext^1(\mathbb{C}, Ind^{GL}_G M) = Ext^1(\mathbb{C}, M)$ Shapiro's le