”On Finite W-algebras for superalgebras and Super Yangians”

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1. Introduction

• For a finite-dimensional semi-simple Lie algebra $\mathfrak{g}$, the *Yangian* of $\mathfrak{g}$ is an infinite-dimensional *Hopf algebra* $Y(\mathfrak{g})$. It is a deformation of the universal enveloping algebra of the Lie algebra of polynomial currents of $\mathfrak{g}$: $\mathfrak{g}[t] = \mathfrak{g} \otimes \mathbb{C}[t]$.

• A *finite W-algebra* is a certain associative algebra attached to a pair $(\mathfrak{g}, e)$ where $\mathfrak{g}$ is a complex semi-simple Lie algebra and $e \in \mathfrak{g}$ is a nilpotent element.
It is a generalization of the universal enveloping algebra $U(\mathfrak{g})$.
For $e = 0$ it coincides with $U(\mathfrak{g})$.

`(A. Premet, Adv. Math., 2002)`

Theorem. `(B. Kostant, Invent. Math. 1978)`
For a reductive Lie algebra $\mathfrak{g}$ and a *regular* nilpotent element $e \in \mathfrak{g}$, the finite W-algebra coincides with the center of $U(\mathfrak{g})$.

• This theorem does not hold for Lie superalgebras, since the finite W-algebra has a non-trivial odd part, while the center of $U(\mathfrak{g})$ is even.
`(V. Kac, M. Gorelik, A. Sergeev)`

• J. Brown, J. Brundan and S. Goodwin proved that the finite W-algebra for $\mathfrak{g} = \mathfrak{gl}(m|n)$ associated with **regular (principal)** nilpotent element is a quotient of a shifted version of the super-Yangian $Y(\mathfrak{gl}(1|1))$ `(Algebra Namber Theory, 2013)`

2. The queer Lie superalgebra $\mathfrak{g} = \mathbf{Q(n)}$

• Equip $\mathbb{C}^{n|n}$ with the odd operator ζ such that $\zeta^2 = -\operatorname{Id}$.

$Q(n)$ is the centralizer of ζ in the Lie superalgebra $\mathfrak{gl}(n|n)$:

$$\zeta = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}, \quad Q(n) = \{ \left(\begin{array}{c|c} A & B \\ \hline B & A \end{array}\right) \mid A, B \text{ are } n \times n \text{ matrices}\}$$

• Supercommutator: $[X, Y] = XY - (-1)^{p(X)p(Y)} YX$.

• Standard bases in A and B respectively:

$$e_{i,j} = \left(\begin{array}{c|c} E_{ij} & 0 \\ \hline 0 & E_{ij} \end{array}\right), \quad f_{i,j} = \left(\begin{array}{c|c} 0 & E_{ij} \\ \hline E_{ij} & 0 \end{array}\right)$$

- $\mathfrak{g} = Q(n)$ admits an **odd** non-degenerate $\mathfrak{g}$-invariant super-symmetric bilinear form

$$(X|Y) := otr(XY) \text{ for } X, Y \in Q(n)$$

$$otr\left(\begin{array}{c|c} A & B \\ \hline B & A \end{array}\right) = trB$$

- $\mathfrak{g}^* \cong \Pi(\mathfrak{g})$, where Π is the parity functor.

3. The super-Yangian of $Q(n)$

The super-Yangian $Y(Q(n))$ was introduced by M. Nazarov.
(Lecture Notes in Math. 1992)

Let $\mathfrak{gl}(n|n)$ be the general linear Lie superalgebra

with the standard basis E_{ij},

where $i, j = \pm 1, \dots, \pm n$;

$p(i) = 0$ if $i > 0$ and $p(i) = 1$ if $i < 0$.

Define an involutive automorphism η of $\mathfrak{gl}(n|n)$ by

$$\eta(E_{ij}) = E_{-i,-j}$$

- $Q(n)$ is the *fixed point subalgebra* in $\mathfrak{gl}(n|n)$ relative to η.

Consider the *twisted polynomial current* Lie superalgebra

$$\mathfrak{g} = \{X(t) \in \mathfrak{gl}(n|n)[t] \mid \eta(X(t)) = X(-t)\}.$$

As a vector space, $\mathfrak{g}$ is spanned by the elements

$$E_{ij}t^m + E_{-i,-j}(-t)^m,$$

where $m = 0, 1, 2, \ldots$ and $i, j = \pm 1, \ldots, \pm n$.

- The enveloping algebra $U(\mathfrak{g})$ has a deformation, called the *Yangian* of $Q(n)$.

• M. Nazarov and A. Sergeev described the *centralizer construction* of the *Yangian* of $Q(n)$.

`(Studies in Lie Theory, 2006)`

Let A_n^m be the centralizer of $Q(n) \subset Q(n+m)$ in the associative superalgebra $U(Q(n+m))$ for each $m = 1, 2, \ldots$.

They constructed a sequence of surjective homomorphisms

$$U(Q(n)) \longleftarrow A_n^1 \longleftarrow A_n^2 \longleftarrow \ldots$$

and described the inverse limit of the sequence of centralizer algebras $A_n^1, A_n^2, \ldots$ in terms of the Yangian of $Q(n)$.

• $Y(Q(n))$ is the associative unital superalgebra over $\mathbb{C}$ with the countable set of generators.

$$T_{i,j}^{(m)} \text{ where } m = 1, 2, \dots \text{ and } i, j = \pm 1, \pm 2, \dots, \pm n.$$

• The $\mathbb{Z}_2$-grading of the algebra $Y(Q(n))$ is defined as follows:

$$p(T_{i,j}^{(m)}) = p(i) + p(j), \text{ where } p(i) = 0 \text{ if } i > 0 \text{ and } p(i) = 1 \text{ if } i < 0.$$

• To write down defining relations for these generators we employ the formal series in $Y(Q(n))[[u^{-1}]]$:

$$T_{i,j}(u) = \delta_{i,j} \cdot 1 + T_{i,j}^{(1)} u^{-1} + T_{i,j}^{(2)} u^{-2} + \dots.$$

$$\begin{aligned}
&(u^2 - v^2)[T_{i,j}(u), T_{k,l}(v)] \cdot (-1)^{p(i)p(k)+p(i)p(l)+p(k)p(l)} \\
&= (u+v)(T_{k,j}(u)T_{i,l}(v) - T_{k,j}(v)T_{i,l}(u)) \\
&- (u-v)(T_{-k,j}(u)T_{-i,l}(v) - T_{k,-j}(v)T_{i,-l}(u)) \cdot (-1)^{p(k)+p(l)}
\end{aligned} \tag{1}$$

$$T_{i,j}(-u) = T_{-i,-j}(u) \tag{2}$$

- $Y(Q(n))$ is a *Hopf superalgebra* with *comultiplication* given by

$$\Delta(T_{i,j}^{(r)}) = \sum_{s=0}^{r} \sum_{k} (-1)^{(p(i)+p(k))(p(j)+p(k))} T_{i,k}^{(s)} \otimes T_{k,j}^{(r-s)}.$$

The *evaluation homomorphism* $ev : Y(Q(n)) \to U(Q(n))$ is defined as follows

$$T_{i,j}^{(1)} \mapsto -e_{j,i}, \quad T_{-i,j}^{(1)} \mapsto -f_{j,i} \text{ for } i,j>0, \quad T_{i,j}^{(0)} \mapsto \delta_{i,j}, \quad T_{i,j}^{(r)} \mapsto 0 \text{ for } r>1.$$

$$\text{Basis in } Q(n) : e_{i,j} = \left(\begin{array}{c|c} E_{ij} & 0 \\ \hline 0 & E_{ij} \end{array}\right), \quad f_{i,j} = \left(\begin{array}{c|c} 0 & E_{ij} \\ \hline E_{ij} & 0 \end{array}\right)$$

4. The finite W-algebra for $Q(n)$

• We fix the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g} = Q(n) = \{\left(\begin{array}{c|c} A & B \\ \hline B & A \end{array}\right)\}$ to be the set of matrices with diagonal A and B:

$$\mathfrak{h} = \mathrm{Span}\{e_{i,i} \mid f_{i,i}\}, \qquad i = 1, \ldots, n$$

• $\mathfrak{n}^+$ (respectively, $\mathfrak{n}^-$) is the nilpotent subalgebra consisting of matrices with strictly upper triangular (respectively, low triangular) A and B.

• The Lie superalgebra $\mathfrak{g}$ has the triangular decomposition

$$\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$$

Set $\mathfrak{b} = \mathfrak{n}^+ \oplus \mathfrak{h}$.

- We define the finite W-algebra associated with the *regular* even nilpotent element φ in the coadjoint representation of $Q(n)$.
- Choose $\varphi \in \mathfrak{g}^*$ such that

$$\varphi(f_{i,j}) = 0, \quad \varphi(e_{i,j}) = \delta_{i,j+1}.$$

Remark. Let $E = \sum_{i=1}^{n-1} f_{i,i+1}$ (**odd**). Then $\varphi(x) = (x|E)$ for $x \in \mathfrak{g}$.

$$E = \left(\begin{array}{cccccc|cccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\
\hline
0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0
\end{array}\right)$$

φ is *regular* nilpotent $\Longleftrightarrow$ E has a *single* Jordan block

Let I_φ be the left ideal in $U(\mathfrak{g})$ generated by $x - \varphi(x)$ for all $x \in \mathfrak{n}^-$, and

$\pi : U(\mathfrak{g}) \to U(\mathfrak{g})/I_\varphi$ be the natural projection.

Definition. *The finite W-algebra* associated with φ is

$$W^n := \{\pi(y) \in U(\mathfrak{g})/I_\varphi \mid \mathrm{ad}(x)y \in I_\varphi \text{ for all } x \in \mathfrak{n}^-\}.$$

$$\pi(y_1)\pi(y_2) = \pi(y_1 y_2)$$

- We identify $U(\mathfrak{g})/I_\varphi$ with $U(\mathfrak{b})$, then W^n is a subalgebra of $U(\mathfrak{b})$.

Definition. The *Harish-Chandra homomorphism* is the natural projection

$$\vartheta : U(\mathfrak{b}) \to U(\mathfrak{h})$$

with the kernel $\mathfrak{n}^+ U(\mathfrak{b})$.

Proposition 1. `(P.--S., Adv. Math., 2016)` The restriction

$$\vartheta : W^n \longrightarrow U(\mathfrak{h})$$

is injective.

We consider W^n as a subalgebra of $U(\mathfrak{h})$.

5. W^n is a quotient of $YQ(1)$

Define $\Delta_n : YQ(1) \longrightarrow YQ(1)^{\otimes n}$ by

$$\Delta_n := \Delta_{n-1,n} \circ \cdots \circ \Delta_{2,3} \circ \Delta.$$

Let $\varphi_n : YQ(1) \to U(Q(1))^{\otimes n} \simeq U(\mathfrak{h})$ be

$$\varphi_n := ev^{\otimes n} \circ \Delta_n.$$

Proposition 2. `(P.-S., J. Math. Phys. 2017)`
The map φ_n is a surjective homomorphism from $YQ(1)$ onto W^n, realized as a subalgebra of $U(\mathfrak{h})$:

$$\varphi_n(YQ(1)) = \vartheta(W^n) \simeq W^n.$$

6. The structure of $U(\mathfrak{h})$

• The Cartan subalgebra of $\mathfrak{g} = Q(n)$ is

$$\mathfrak{h} = \mathrm{Span}\{e_{i,i} \mid f_{i,i}\}.$$

$$[f_{i,i}, f_{j,j}] = 0 \text{ if } i \neq j, [f_{i,i}, f_{i,i}] = 2e_{i,i}.$$

Set $\xi_i = (-1)^{i+1} f_{i,i}$, $x_i = \xi_i^2 = e_{i,i}$. Then

• $U(\mathfrak{h}) = \mathbb{C}[\xi_1, \dots, \xi_n]/(\xi_i\xi_j + \xi_j\xi_i)_{i<j\leq n}$.

• The center of $U(\mathfrak{h})$ coincides with $\mathbb{C}[x_1, \dots, x_n]$.

• The center $Z(U(\mathfrak{g}))$ was described by A.Sergeev.

• The center of W^n coincides with $W^n \bigcap \mathbb{C}[x_1, \dots, x_n] = \vartheta(Z(U(\mathfrak{g})))$.
`(P.-S., Adv. Math., 2016)`

7. The structure of W^n

- We define the following set of generators of W^n:

n **odd** generators ϕ_k and n **even** generators z_k.

Set

$$\phi_0 = \sum_{i=1}^{n} \xi_i, \quad \phi_k = T^k(\phi_0), \quad k = 0, \dots, n-1.$$

where the matrix of T in the standard basis $\xi_1, \dots, \xi_n$ has 0 on the diagonal and

$$t_{ij} = \begin{cases} x_j & if \quad i < j, \\ -x_j & if \quad i > j. \end{cases}$$

Even generators for *even* $0 \leq k < n$ are given by

$$z_k := [\phi_0, \phi_k] \in \text{ center of } W^n$$

Even generators for *odd* $0 \leq k < n$ are given by

$$z_k = [\sum_{i_1 \geq i_2 \geq \ldots \geq i_{k+1}} (x_{i_1} + (-1)^k \xi_{i_1}) \ldots (x_{i_k} - \xi_{i_k})(x_{i_{k+1}} + \xi_{i_{k+1}})]_{even},$$

Then

$$[\phi_i, \phi_j] = \begin{cases} (-1)^i 2 z_{i+j} \text{ if } i + j \text{ is even} \\ 0 \text{ if } i + j \text{ is odd} \end{cases}$$

• Elements $z_0, \ldots, z_{n-1}$ are algebraically independent in W^n and they commute with each other.

8. Irreducible representations of W^n

Now we give a classification of simple W^n-modules for $Q(n)$.
They are all finite-dimensional.

Restriction from $U(\mathfrak{h})$ to W^n.

Definition. Let $\mathbf{s} = (s_1, \dots, s_n) \in \mathbb{C}^n$. We call $\mathbf{s}$ *regular* if $s_i \neq 0$ for all $i \leq n$ and *typical* if $s_i + s_j \neq 0$ for all $i \neq j \leq n$.

• All irreducible representations of $U(\mathfrak{h})$ are enumerated by $\mathbf{s} \in \mathbb{C}^n$ up to change of parity.

Let V be an irreducible representation, then every x_i acts by scalar $s_i\mathrm{Id}$.

Let $I_{\mathbf{s}}$ be the ideal in $U(\mathfrak{h})$ generated by $x_i - s_i$.

Then the quotient algebra $U(\mathfrak{h})/I_{\mathbf{s}}$ is isomorphic to the Clifford algebra $C_{\mathbf{s}}$ associated with the quadratic form $B_{\mathbf{s}}$:

$$C_{\mathbf{s}} = \mathbb{C}[\xi_1, \dots, \xi_n]/(\xi_i\xi_j + \xi_j\xi_i - 2\delta_{i,j}s_i),$$

and V is a simple $C_{\mathbf{s}}$-module.

Let m be the number of non-zero coordinates of $\mathbf{s}$. Then

• $C_{\mathbf{s}}$ has **one** simple $\mathbb{Z}_2$-graded module $V(\mathbf{s})$ for **odd** m,

and **two** simple modules $V(\mathbf{s})$ and $\Pi V(\mathbf{s})$ for **even** m.

• The dimension of $V(\mathbf{s})$ equals 2^k, where $k = \lceil m/2 \rceil$.

• We denote by the same symbol $V(\mathbf{s})$ the restriction to W^n.

Proposition 3. If $\mathbf{s}$ is **typical**, then $V(\mathbf{s})$ is a simple W^n-module.

9. Simple W^2-modules for $Q(2)$

• The generators of W^2 are

Even: $z_0 = x_1 + x_2,\ z_1 = x_1 x_2 - \xi_1 \xi_2,$

Odd: $\phi_0 = \xi_1 + \xi_2,\ \phi_1 = x_2 \xi_1 - x_1 \xi_2.$

• $V(\mathbf{s})$ is simple as W^2-module if and only if $s_1 \neq -s_2$.

• If $s_1 = -s_2 \neq 0$, we have the following nonsplit exact sequence:

$$0 \to \Pi\Gamma_{-s_1^2+s_1} \to V(\mathbf{s}) \to \Gamma_{-s_1^2-s_1} \to 0,$$

where Γ_t is one-dimensional simple module on which ϕ_0, ϕ_1 and z_0 act by zero and z_1 acts by the scalar t.

10. General construction of simple W^n-modules

Let W^n be the finite W-algebra for $Q(n)$.

Let $i+j=n$. There is natural embedding of the Lie superalgebras:

$$Q(i)\oplus Q(j)\hookrightarrow Q(n).$$

This induces the isomorphism

$$U(\mathfrak{h})\simeq U(\mathfrak{h}_i)\otimes U(\mathfrak{h}_j),$$

where $\mathfrak{h}_r$ denotes the Cartan subalgebra of $Q(r)$.

Lemma. Let $i+j=n$. Then W^n is a subalgebra in the tensor product $W^i\otimes W^j$, where $W^r\subset U(\mathfrak{h}_r)$ denotes the W-algebra for $Q(r)$.

Corollary. If $i_1+\cdots+i_p=n$, then W^n is a subalgebra in $W^{i_1}\otimes\cdots\otimes W^{i_p}$.

Let $n = r + 2p + q$, where $r, p, q \geq 0$, and $\mathbf{t} = (t_1, \ldots, t_p) \in \mathbb{C}^p$, $t_1, \ldots, t_p \neq 0$, $\lambda = (\lambda_1, \ldots, \lambda_q) \in \mathbb{C}^q$, $\lambda_1, \ldots, \lambda_q \neq 0$, such that $\lambda_i + \lambda_j \neq 0$ for any $1 \leq i \neq j \leq q$.

We have an embedding

$$W^n \hookrightarrow W^r \otimes (W^2)^{\otimes p} \otimes W^q.$$

Set

$$S(\mathbf{t}, \lambda) := \mathbb{C} \boxtimes \Gamma_{t_1} \boxtimes \cdots \boxtimes \Gamma_{t_p} \boxtimes V(\lambda).$$

Theorem 1. `(P.-S., J. Algebra, 2021)`

(a) $S(\mathbf{t}, \lambda)$ is a simple W^n-module;

(b) Every simple W^n-module is isomorphic to $S(\mathbf{t}, \lambda)$ up to change of parity.

Proposition 4.
Two simple modules $S(\mathbf{t}, \lambda)$ and $S(\mathbf{t}', \lambda')$ are isomorphic if and only if $\mathbf{t}' = \sigma(\mathbf{t})$ and $\lambda' = \tau(\lambda)$ for some $\sigma \in S_p$ and $\tau \in S_q$.

11. The structure of the super Yangian of $Q(1)$

- $YQ(1)$ has generators $T_{1,1}^{(m)}$ (**even**) and $T_{1,-1}^{(m)}$ (**odd**)

Let

$$\eta_0 = T_{1,-1}^{(1)}, \quad \eta_i = (-\frac{1}{2})^i \operatorname{ad}^i T_{1,1}^{(2)}(T_{1,-1}^{(1)}), \quad Z_{2i} = \frac{1}{2}[\eta_0, \eta_{2i}].$$

- The surjective homomorphism $\varphi_n : YQ(1) \to W^n$ acts on generators by

$$\varphi_n(\eta_i) = \phi_i, \quad \varphi_n(Z_{2i}) = z_{2i}, \quad 0 \le i \le n-1.$$

Lemma.

(1) The following analogue of the relations in W^n holds:

$$[\eta_i, \eta_j] = \begin{cases} (-1)^i 2 Z_{i+j} & \text{if } i+j \text{ is even} \\ 0 & \text{if } i+j \text{ is odd} \end{cases}.$$

(2) The elements $\{Z_{2i} \mid i \in \mathbb{N}\}$ are algebraically independent generators of the center of $YQ(1)$.

(3) The elements η_0 and $\{T_{1,1}^{(2i)} \mid i \in \mathbb{N}\}$ generate $YQ(1)$.

12. Representations of the super Yangian of $Q(1)$

- Using the surjective homomorphism $\varphi_n : YQ(1) \to W^n$ we equip $V(\mathbf{s})$ with a $YQ(1)$-module structure.

- Let M be a simple $YQ(1)$-module. Then M admits a central character χ.

Set $\chi_{2i} = \chi(Z_{2i})$ and consider the generating function

$$\chi(u) = \sum_{i=0}^{\infty} \chi_{2i} u^{-2i-1}.$$

Proposition 5. Let M be a finite-dimensional simple $YQ(1)$-module admitting central character χ. Then $\chi(u)$ is a rational function of the form

$$\frac{a_0 u^{-1} + \cdots + a_{q-1} u^{-2q+1}}{1 + c_1 u^{-2} + \cdots + c_q u^{-2q}}.$$

Proposition 6. For any rational $\chi(u)$ there exist n and a regular typical $\mathbf{s} = (s_1, s_2, \ldots, s_n)$ such that $V(\mathbf{s})$ admits central character χ. $V(\mathbf{s})$ is a simple $YQ(1)$-module.

Lemma. $[T_{1,1}^{(2k)}, T_{1,1}^{(2l)}] = 0.$

Definition.

Let $\mathbf{A}$ be the commutative subalgebra in $YQ(1)$ generated by $T_{1,1}^{(2k)}$ for $k \geq 0$.

Proposition 7.

$YQ(1)/$ ideal generated by odd elements $\simeq \mathbf{A}$.

Hence $\mathbf{A}$ is a commutative cocommutative Hopf algebra with comultiplication

$$\Delta T_{1,1}(u^{-2}) = T_{1,1}(u^{-2}) \otimes T_{1,1}(u^{-2}),$$

where $T_{1,1}(u^{-2}) = \sum T_{1,1}^{(2k)} u^{-2k}$.

Let $f(u) = 1 + \sum_{k>0} f_{2k} u^{-2k}$. Let Γ_f be the one-dimensional $\mathbf{A}$-module, where the action of $T_{1,1}(u^{-2})$ is given by the generating function $f(u)$.

Lemma. The isomorphism classes of one-dimensional $YQ(1)$-modules are in bijection with the set $\{\Gamma_f\}$, and

$$\Gamma_f \otimes \Gamma_g \simeq \Gamma_{fg}.$$

Theorem 2. `(P.-S., J. Algebra 2021).`

(1) Any simple finite-dimensional $YQ(1)$-module is isomorphic to $V(\mathbf{s})\otimes\Gamma_f$ or $\Pi V(\mathbf{s})\otimes\Gamma_f$ for some regular typical $\mathbf{s}$ and $f(u)=1+\sum_{k>0} f_{2k}u^{-2k}$.

(2) $V(\mathbf{s})\otimes\Gamma_f$ and $V(\mathbf{s}')\otimes\Gamma_g$ are isomorphic up to change of parity if and only if $\mathbf{s}'$ is obtained from $\mathbf{s}$ by permutation of coordinates and $f(u)=g(u)$.

13. The relation between W^n-modules and $YQ(1)$-modules

The following diagram commutes:

$$\begin{array}{ccc} YQ(1) & \xrightarrow{\Delta} & YQ(1) \otimes YQ(1) \\ {\scriptstyle \varphi_{m+n}} \downarrow & & \downarrow {\scriptstyle \varphi_m \otimes \varphi_n} \\ W^{m+n} & \longrightarrow & W^m \otimes W^n \end{array}$$

Proposition 8. The simple $YQ(1)$-module $V(\mathbf{s}) \otimes \Gamma_f$ is lifted from some W^{m+n}-module if and only if $f \in \mathbb{C}[u^{-2}]$. The smallest m is equal to the degree of the polynomial f.

Proof. Note that $m = 2p$ is even. $S(t_1, \ldots, t_p, \lambda) \simeq V(\lambda) \otimes \Gamma_f$ where

$$f = \prod_{i=1}^{p} (1 + t_i u^{-2}).$$

Remark. Not all irreducible finite-dimensional representations of the Yangian $YQ(1)$ are obtained by lift from those of W-algebras and the classification is not a straightforward consequence of the classification for W-algebras.

14. The category $YQ(1)$–mod

Let $YQ(1)$–mod be the category of finite-dimensional $YQ(1)$-modules, and $(YQ(1))^\chi$–mod be the full subcategory of modules admitting generalized central character χ.

• What are the blocks in $(YQ(1))^\chi$–mod?

If there is a non-split short exact sequence $0 \longrightarrow M_i \longrightarrow M \longrightarrow M_j \longrightarrow 0$ with $\{i, j\} = \{1, 2\}$, then M_1 and M_2 belong to the same block, and we say that they are *linked*.

15. The subcategory $(YQ(1))^{\chi=0}$–mod

• The simple modules in the subcategory $(YQ(1))^{\chi=0}$–mod are exactly the 1-dimensional modules Γ_f up to change of parity.
Let Γ_f and Γ_g be two $YQ(1)$-modules, where

$$f(u) = \sum_{k\geq 0} a_{2k}u^{-2k}, \quad g(u) = \sum_{k\geq 0} b_{2k}u^{-2k}, \quad a_0 = b_0 = 1.$$

Let $x_k = \frac{1}{2}(a_{2k} - b_{2k})$.

Theorem 3. $\mathrm{Ext}^1(\Pi(\Gamma_g), \Gamma_f) \neq 0$ if and only if x_1 is an arbitrary complex number and x_k for $k > 1$ satisfies the recurrence relation

$$x_{k+1} = (x_1x_k - x_k + a_{2k})x_1$$

Conjecture. Let S be a simple finite-dimensional $YQ(1)$-module.

Let $n \geq 1$, $\mathbf{s} = (s_1, s_2, \ldots, s_n)$ be regular typical and $f(u)$ and $g(u)$ be given by

$$f(u) = \sum_{k \geq 0} a_{2k} u^{-2k}, \quad g(u) = \sum_{k \geq 0} b_{2k} u^{-2k}, \quad a_0 = b_0 = 1, \quad x_k = \frac{1}{2}(a_{2k} - b_{2k}).$$

Then

$$Ext^1(S, V(\mathbf{s}) \otimes \Gamma_f) \neq 0,$$

if and only if $S \simeq V(\mathbf{s}) \otimes \Pi(\Gamma_g)$, where x_k satisfies the recurrence relation

$$x_{k+1} = (x_1 x_k - x_k + a_{2k}) x_1.$$

Remark. The short exact sequences

$$0 \longrightarrow \Gamma_f \longrightarrow \mathbb{C}^{1|1} \longrightarrow \Pi(\Gamma_g) \longrightarrow 0$$

is non-split.
If $n \geq 2$, then the short exact sequence

$$0 \longrightarrow V(s) \otimes \Gamma_f \longrightarrow V(s) \otimes \mathbb{C}^{1|1} \longrightarrow V(s) \otimes \Pi(\Gamma_g) \longrightarrow 0$$

is non-split.
If $n = 1$, then it is non-split if and only if $x_1 \neq s_1$.

16. The category (W^n)–mod

Let W^n–mod be the category of finite-dimensional W^n-modules, and
$(W^n)^\chi$–mod be the full subcategory of modules admitting generalized central character χ.

Recall that simple W^n-modules are $S(\mathbf{t}; \lambda_1, \dots, \lambda_q)$.

If $q = 0$, we use the notation $S(\mathbf{t})$.

Simple modules in the subcategory $(W^n)^{\chi=0}$–mod are exactly the 1-dimensional modules $S(\mathbf{t})$ up to change of parity.

17. The subcategory $(W^n)^{\chi=0}$–mod

Let σ_k be the k-th elementary symmetric polynomial.

Theorem 4. Fix $\mathbf{t} = (t_1, \ldots, t_p)$ and $\mathbf{t}' = (t'_1, \ldots, t'_q)$, where $p, q \leq \frac{n}{2}$.
Consider the W^n-modules $S(\mathbf{t})$ and $S(\mathbf{t}')$.
Define $a_{2k} = \sigma_k(t_1, \ldots, t_p)$ for $k = 1, \ldots, p$, $a_{2k} = 0$ for $k > p$.
Similarly, define $b_{2k} = \sigma_k(t'_1, \ldots, t'_q)$ for $k = 1, \ldots, q$, $b_{2k} = 0$ for $k > q$.
Let $x_k = \frac{1}{2}(a_{2k} - b_{2k})$.

(a) If $S(\mathbf{t})$ is a nontrivial W^n-module, then $\mathrm{Ext}^1(\Pi(S(\mathbf{t}')), S(\mathbf{t})) \neq 0$ if and only if $x_1 \neq 0$ and x_k satisfy the recurrence relation

$$x_{k+1} = (x_1 x_k - x_k + a_{2k})x_1.$$

or $S(\mathbf{t}')$ is isomorphic to $S(\mathbf{t})$ and $n > 2p$.

(b) If $S(\mathbf{t}) = \mathbb{C}^{1|0}$ is the trivial W^n-module, then $\mathrm{Ext}^1(\Pi(S(\mathbf{t}')), S(\mathbf{t})) \neq 0$ if and only if $S(\mathbf{t}') = \mathbb{C}^{1|0}$ or $\mathbf{t}' = (t'_1)$ with $t'_1 = -2$.

18. Blocks in the category W^2–mod

• Every $\mathbf{s} = (s_1, \dots, s_n)$ defines the central character $\chi_{\mathbf{s}} : Z^n \longrightarrow \mathbb{C}$.

Theorem 5.
(1) Each simple W^2-module $V(s_1, s_2)$ for $s_1 \neq -s_2, s_1, s_2 \neq 0$ forms a block in $(W^2)^{\chi_{\mathbf{s}}}$-mod.

(2) Each simple W^2-module $V(s, 0)$ for $s \neq 0$ forms a block in $(W^2)^{\chi_{\mathbf{s}}}$-mod.

(3) The blocks in the subcategory $(W^2)^{\chi=0}$-mod are described as follows.
Let $a \in \mathbb{C}$. Define

$$a_n = a - n^2 + n\sqrt{1-4a} \text{ for } n = 0, \pm 1, \pm 2, \dots$$

Then Γ_a lies in the block formed by Γ_{a_n} if n is even and $\Pi\Gamma_{a_n}$, if n is odd. $\Pi\Gamma_a$ lies in the block formed by $\Pi\Gamma_{a_n}$ if n is even and Γ_{a_n}, if n is odd.

Example. Let $a = 0$, then $a_n = n(1-n)$ and Γ_0 lies in the block

$$\dots, \Gamma_{-30}, \Pi\Gamma_{-20}, \Gamma_{-12}, \Pi\Gamma_{-6}, \Gamma_{-2}, , \Pi\Gamma_0, \Gamma_0, \Pi\Gamma_{-2}, \Gamma_{-6}, \Pi\Gamma_{-12}, \Gamma_{-20}, \Pi\Gamma_{-30}, \dots$$

19. About non-regular case

Theorem 5. `(P.-S., J. Math. Phys. 2017).`

Let W^n be the finite W-algebra for $Q(n)$ associated with the *non-regular* even nilpotent coadjoint orbit in the case when the corresponding nipotent element has Jordan blocks each of size l. Then W^n is isomorphic to the image of $YQ(\frac{n}{l})$ under the homomorphism

$$ev^{\otimes l} \circ \Delta_l : YQ(\frac{n}{l}) \longrightarrow U(Q(\frac{n}{l}))^{\otimes l}$$

The *regular* case is $l = n$.

20. Open Problems

- Describe the structure of finite W-algebra for $Q(n)$ associated with an *arbitrary* nilpotent φ.

- Classify the simple finite-dimensional modules over the Yangian for $Q(n)$ for $n > 1$.

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