Twisted Covariant Form Hierarchies

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Material based on papers with PhD students
Edgar Peréz-Bolaños, Loukas Grimanellis and Jake Phillips
Extracts are used from papers with Ulf Gran and Jan Gutowski

A review

Integrability has a long history and a naive approach is

- Classical: A system is integrable provided all classical solutions can be expressed in closed form in terms of some simple functions
- Quantum: A system is integrable provided that the eigenvalues and eigenfunctions of a set of characteristic observables can be evaluated in a closed form

Opinion 1: Integrability is too restrictive: theories that describe nature cannot be integrable

Opinion 2: Indispensable: investigating integrable theories will lead to a deeper understanding of nature

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Separability

The Hamiltonian equations of motion give rise to a first order non-linear system for positions and momenta

- Separability: This means that there is a coordinate system on the phase space such that Hamiltonian system becomes a set of first order non-linear equations with each equation depending of a single unknown. Typically this happens upon substitution of the solution of the previous equations into the next one.
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Liouville Integrability

Liouville: A Hamiltonian system with phase space P, $\dim P = 2n$, is (completely) integrable provided that it admits n independent observables Q (including the Hamiltonian) in involution.

- Independent: The map $Q\ : P \to \mathbb{R}^n$ has rank n.
- In involution: $\{Q_r, Q_s\}_{\mathrm{PB}} = 0$

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Liouville Integrability

However if a Hamiltonian system is integrable, in the coordinates of the phase space

$$Q_r\ ,\quad \Phi_r\ ;\quad X_{Q_r}(f)=\frac{\partial}{\partial\Phi_r}f$$

where X_{Q_r}, $X_{Q_r}f=\{Q_r,f\}_{\rm PB}$, is the Hamiltonian vector field of Q_r. The classical solutions of the theory are

- $\Phi_H=(\Phi_H)_0+t\ ,$ with Q_r and rest of Φ_r constant
- Thus the classical trajectories in this coordinate system are straight lines. The system is integrable, even in the naive sense, and separable.

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Geodesic flow

Consider a spacetime (or equivalently a manifold M) with metric g. The action (or energy functional) of the geodesic flow is

$$E = \frac{1}{2} \int dt \, g_{\mu\nu} \, \dot{x}^{\mu} \, \dot{x}^{\nu}$$

Then $Q_{(k)} = d_{\mu_1 \ldots \mu_k} \dot{x}^{\mu_1} \cdots \dot{x}^{\mu_k}$ are constants of motion provided that

$$\nabla_{(\mu_1} d_{\mu_2 \ldots \mu_{k+1})} = 0$$

- For $k = 1$, d is a Killing vector field
- For $k \geq 2$, the Killing-Stäckel (KS) symmetric $(0, k)$ tensors d generate "hidden" symmetries for the system.

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Geodesic flow

An example that includes all supersymmetric brane and extreme black hole solutions with rotational symmetry is

$$g = A(|y|)g(\mathbb{R}^{p,1}) + B(|y|)g(\mathbb{R}^{k+1})$$

It is well known that the geodesic flow is separable in angular coordinates. It is also integrable.

- Isometry Lie algebra is $(\mathfrak{so}(p,1) \oplus_s \mathbb{R}^{p,1}) \oplus \mathfrak{so}(k+1)$.
- The independent observables which are in involution are the Hamiltonian, the conserved charges associated to translations $\mathbb{R}^{p,1}$ and the charges associated with the quadratic Casimir operators of the Lie alebras $\mathfrak{so}(2) \subset \cdots \subset \mathfrak{so}(k) \subset \mathfrak{so}(k+1)$
- Similar conclusion holds for all spherically symmetric black holes (Schwarzschild)
- Famously the geodesic flow of Kerr black hole is integrable [Carter]

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Conformal Killing-Yano forms

Def: Let M^n be a spacetime (manifold) with metric g. A k-form ϕ on M is conformal Killing-Yano (CKY), iff

$$\nabla_X \phi = \frac{1}{k+1} i_X d\phi - \frac{1}{n-k+1} X \wedge \delta\phi$$

- If $\delta\phi = 0$, then ϕ is Killing-Yano (KY).
- If $d\phi = 0$, then ϕ is closed CKY (CCKY)
- If ϕ is KY, then $^\star\phi$ is CCKY and vice-versa
- If ϕ is KY, then $d(X,Y) \equiv (i_X\phi, i_Y\phi)$ is a KS tensor
- Famously the KS (0,2) symmetric tensor associated with the integrability of the Kerr geodesic flow is constructed as above by a KY 2-form. [Floyd, Penrose]

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Killing-Yano forms and symmetries

A supersymmetric generalisation of the energy functional for the geodesic flow [Berezin, Marinov; Brink, Deser, Zumino, Di Vecchia, Howe] is

$$E = -\frac{i}{2}\int dtd\theta\, g_{\mu\nu}Dx^{\mu}\partial_t x^{\nu}$$

where (t, θ) are the worldline superspace coordinates, $D^2 = i\partial_t$.

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Therefore KY forms generate symmetries in spinning particle probes [Gibbons, Rietdijk, van Holten]

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Killing-Yano forms and symmetries

- There are many generalisations of CKY forms, e.g ∇ can be chosen not to be the Levi-Civita connection
- There are many different (supersymmetric) spinning particle probes with additional couplings to that of the metric [Coles, GP]
- Can one match the generalisations of KY tensors with the conditions for symmetries in these more general particle probes?

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Lichnerowicz Theorem

Let M^n be a spin, closed manifold with metric g. It is a consequence of the formula

$$\int_{M^n} \| \not\nabla \epsilon \|^2 = \int_{M^n} \| \nabla \epsilon \|^2 + \frac{1}{4} \int_{M^n} R \| \epsilon \|^2$$

where ϵ is a spinor, that

- If $R \geq 0$, then $\mathrm{Ker} \not\nabla = \{0\}$
- if $R = 0$, then $\mathrm{Ker} \not\nabla = \mathrm{Ker} \nabla$
- if λ is an eigenvalue of $\not\nabla$, then $|\lambda|^2 \geq \frac{1}{4} \inf_{M^n} R$

The geometry of simply connected manifolds with parallel spinors can be described using the Berger classification theorem of irreducible Riemannian manifolds leading to hyper-Kähler, Calabi-Yau, G_2 and $Spin(7)$ manifolds

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Remarkably there are Lichnerowicz type of theorems for connections

$$\nabla^{\mathcal{F}}_X = \nabla_X + c(i_X\mathcal{F}) + c(X \wedge \mathcal{F})$$

on the spin bundle S of M^n, where $\mathcal{F}$ is a multi-form on M^n consisting on forms of various degrees.

- In supergravity theories with form field strengths $\mathcal{F}$, $\nabla^{\mathcal{F}}$ can be identified with the supercovariant connection which arises in the supersymmetry transformation of the gravitino
- Lichnerowicz type of theorems for $\nabla^{\mathcal{F}}$ connections have been established as part of the investigation of AdS solutions and the near horizon geometries of black holes [Gran, Gutowski, Beck, GP]
- What is the geometry of spacetimes (manifolds) with $\nabla^{\mathcal{F}}$ parallel spinors?

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Twisted Covariant Form Hierarchies

Def: A twisted covariant form hierarchy is a collection of forms described by a multi-form Ω and a connection $\mathcal{D}^{\mathcal{F}}$ in the space of forms on M^n, such that

$$\mathcal{D}^{\mathcal{F}}_X \Omega = i_X \mathcal{P} + X \wedge \mathcal{Q}$$

where $\mathcal{F}$ is a multi-form on M^n describing the fluxes of the connection and $\mathcal{P}$ and $\mathcal{Q}$ are multi-forms on M^n

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Twisted Covariant Form Hierarchies

Th: All manifolds with $\nabla^{\mathcal{F}}$-parallel spinors admit a TCFH, where Ω is a collection of form bilinears (Dirac currents) [GP]

Proof: Given a solution ϵ of $\nabla^{\mathcal{F}}\epsilon = 0$, the form bilinears are spanned by the forms

$$\chi_p = \frac{1}{p!}\langle\epsilon, \Gamma_{A_1\dots A_p}\epsilon\rangle_s \; e^{A_1}\wedge\cdots\wedge e^{A_p} = \frac{1}{p!}\langle\epsilon, \Gamma_{A_1\dots A_p}\epsilon\rangle_s \; e^{A_1\dots A_p}$$

Suppose $\mathcal{F}$ is an ℓ-form F-linearity will extend the proof to multi-forms. Then

$$\begin{aligned}\nabla_X\chi_p \quad = \quad & -\frac{1}{p!}(\langle c(i_XF)\epsilon, \Gamma_{A_1\dots A_p}\epsilon\rangle_s + (\langle\epsilon, \Gamma_{A_1\dots A_p}c(i_XF)\epsilon\rangle_s \, e^{A_1\dots A_p}\\ & -\frac{1}{p!}(\langle c(X\wedge F)\epsilon, \Gamma_{A_1\dots A_p}\epsilon\rangle_s + (\langle\epsilon, \Gamma_{A_1\dots A_p}c(X\wedge F)\epsilon\rangle_s \, e^{A_1\dots A_p}\end{aligned}$$

Twisted Covariant Form Hierarchies

After using the Hermiticity properties of Clifford algebra elements w.r.t. to $\langle \cdot, \cdot \rangle_s$ and some Clifford algebra, the terms in the list line can be written as

$$\Big(\sum_q \tilde{c}_q \, i_X F \cdot \chi_q\Big)\big|_p$$

and can be interpreted as terms that contribute in the connection $\mathcal{D}^{\mathcal{F}}$ of the TCFH, where $\tilde{c}$ are numerical coefficients and

$$(\psi_m \cdot \chi_q)|_p = \frac{1}{p!(m-k)!} \psi^{B_1 \dots B_{m-k}}{}_{A_1 \dots A_k} \chi_{B_1 \dots B_{m-k} A_{k+1} \dots A_p} \; e^{A_1 \dots A_p}$$

While the terms in the second line give

$$X \wedge \Big(\sum_q c'_q \, F \cdot \chi_q\Big)\big|_{p-1} + \Big(\sum_q c''_q \, F \cdot i_X \chi_q\Big)\big|_p$$

Twisted Covariant Form Hierarchies

The first term determines the multi-form $\mathcal{Q}$ of TCFH. The last term above can be re-arranged as

$$\Big(\sum_q c''_q F \cdot i_X \chi_q\Big)|_p = \Big(\sum \hat{c}''_q i_X F \cdot \chi_q\Big)|_p + \Big(i_X \sum \check{c}''_q F \cdot \chi_q\Big)|_p$$

The second term determines $\mathcal{P}$ of the TCFH while the first term contributes to the TCFH connection $\mathcal{D}^{\mathcal{F}}$ connection. This completes the proof.

All form bilinears satisfy a generalisation of the CKY equation with respect to the TCFH connection $\mathcal{D}^{\mathcal{F}}$

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Twisted Covariant Form Hierarchies

- The TCFH of some 4- and 5-dimensional [Peréz-Bolaños, GP] and all 10- and 11-dimensional supergravities have been computed [Peréz-Bolaños, GP; Grimanellis, Phillips, GP]
- The TCFH on the internal spaces of all warped AdS backgrounds of the above theories have been computed
- There is a TCFH connection with respect to which all form bilinears are parallel. This is given by extending the action of $\nabla^{\mathcal{F}}$ on the sections of $S \otimes S$, S spin bundle, and using the decomposition of two spin representations in terms of forms. Ω in this case will be spanned by the form bilinears and their Hodge duals. And the TCFH connection $\mathcal{D}^{(\mathcal{F},{}^*\mathcal{F})}$ will depend on both $\mathcal{F}$ and the Hodge dual ${}^*\mathcal{F}$ of $\mathcal{F}$.

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Supersymmetry and Integrability

The presence of a TCFH in all supersymmetric backgrounds and the relation of the former to KY forms raises the question on whether superymmetry and integrability are closely related. In particular how much supersymmetry is it required for integrability?

The question posed in this way is rather general. There are several issues

- Which theory is supposed to be integrable? Even if one considers particle probes, there are several and choices have to be made
- How can one specify the number of supersymmetries required for integrability?

There are some partial answers to these questions but require a case by case examination

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Consider the geodesic flow on supersymmetric backgrounds.

- Is the geodesic flow of all supersymmetric backgrounds that preserve more than $1/2$ of supersymmetry integrable? There are no counter examples in 10- and 11-dimensional supergravity theories
- The geodesic flow of many backgrounds that preserve strictly $1/2$ of supersymmetry is not integrable

All $\mathrm{AdS}_n \times S^{D-n}$, $D = 10, 11$ and plane wave solutions have integrable geodesic flows. Moreover all supersymmetric backgrounds that preserve more than $1/2$ of supersymmetry are homogeneous spaces [Figueroa-O'Farrill, Hustler]. However not all homogeneous spaces have integrable geodesic flows. Therefore the statement remains inconclusive.

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Supersymmetry and Integrability

Consider the Dp-brane backgrounds which preserve 1/2 of supersymmetry

$$g = h^{-\frac{1}{2}} g(\mathbb{R}^{p,1}) + h^{\frac{1}{2}} g(\mathbb{R}^{9-p}) \ , \quad h = 1 + \sum_k \frac{q_k}{|y - y_k|^{7-p}}$$

The solutions with one centre y_k are spherically symmetric and so the geodesic flow is integrable.

However one does not expect this to be the case for multi-centred solutions with centres at generic points.

- The KY forms responsible for the integrability of the geodesic flow in the spherically symmetric solutions are different from those constructed as Killing spinor bilinears
- In this context supersymmetry and integrability seem to point to different directions.

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Supersymmetry and Integrability

Nevertheless the form bilinears generate symmetries in a variety of probe actions like spinning particles and strings. These include the following

- All the form bilinears of the common sector [Howe, GP]. These give W-type of symmetries for sigma models with target spaces with appropriate G-structures, e.g. CY, G_2 and so on.
- Many of the form bilinears of D-branes in type II supergravities and M-brane backgrounds are either KY or CCKY forms [Peréz-Bolaños, Grimanellis, Phillips, GP]
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Conclusions

- Every spacetime that admits a $\nabla^{\mathcal{F}}$-parallel spinor is associated with a TCFH. These include all supersymmetric backgrounds, even those of theories with higher order curvature corrections and on spacetimes of any signature. It also includes manifolds that admit generalisations of the Lichnerowicz theorem.
- Form bilinears can generate symmetries for many probes propagating on supersymmetric backgrounds. However their contribution to the integrability of the dynamics of probes is rather tenuous.
- Open question: Can one match the TCFH conditions of a background with the invariance conditions of a (particle) probe under symmetries generated by Ω?

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