- Let $C(n,n)$ be the associative super-algebra generated by $2n$ odd elements θ^a, K_a $(a = 1, 2, \dots, n)$ modulo

  $$[\theta^a, \theta^b] = [K_a, K_b] = 0, \quad [K_a, \theta^b] = \delta_a^b$$

  $$[K_a, \theta^b] = K_a\theta^b + \theta^b K_a$$

  (also Clifford algebra of signature (n,n))

- $\Lambda(n) = \langle \theta^{a_1} \cdots \theta^{a_p} \rangle \subset C(n,n)$

  $(p = 0, 1, \dots, n)$

- Lie superalgebras of Cartan type obtained as subalgebras of the commutator algebra of $C(n,n)$:
  - $W(n) = \Lambda(n)\langle K_a \rangle = \langle \theta^{a_1} \cdots \theta^{a_p} K_b \rangle$

  $\mathbb{Z}$-grading:

level	basis of $W(n)$	- of $sl(n\|1)$
1	K_a	K_a
0	$\theta^a K_b$	$K^a{}_b$
-1	$\theta^a\theta^b K_c$	K^b
$\vdots$	$\vdots$	
$-n+1$		

- $S(n)$: subalgebra of $W(n)$ consisting of traceless elements

  $$\mathrm{tr}(xK_a) = [x, K_a] \quad (x \in \Lambda(n))$$

- Lie algebras appearing as subalgebras at level zero:

  $W(n)_0 = gl(n)$

  $S(n)_0 = sl(n) = A_{n-1}$

Can be extended to a contragredient Lie superalgebra $A(0, n-1) = sl(n|1)$ with the same level 0, 1 as $W(n)$.

Generalise:

- A_{n-1} to g, any Kac-Moody algebra
- Λ_1 to λ, any dominant integral weight of g
- $sl(n|1)$ to $B(g, \lambda)$. $\longrightarrow$ contragredient

→

- $W(n)$ to $W(g,\lambda)$
- $S(n)$ to $S(g,\lambda)$

} tensor hierarchy algebras !

- can be defined by a modified Chevalley-Serre presentation [Carbone, Cederwall, P 2018]

- $C(n,n)$ to ... ? (restrict to the local part)

Set $K^a = \theta^a$. Then the bracket in $sl(n|1)$ differs from the commutator in $C(n,n)$:

$$[\![K_a, \theta^b]\!] = -\theta^b K_a + \delta_a{}^b \underbrace{\theta^c K_c}_{K}$$

$$[K_a, \theta^b] = \delta_a{}^b$$

Starting from $sl(n|1)$, we can re-construct the products $\theta^b K_a$ and $K_a \theta^b$ in $C(n,n)$ by

$$\left.\begin{aligned} \theta^b K_a &= -[\![K_a,\theta^b]\!] + \delta_a{}^b K \\ K_a\theta^b &= [\![K_a,\theta^b]\!] - \delta_a{}^b K + \delta_a{}^b \end{aligned}\right\} (*)$$

Extend the local part $B_{-1} \oplus B_0 \oplus B_1$ of B to $C^\ell = C_{-1} \oplus C_0 \oplus C_1 \subset U(B)$, where

$$g \oplus \langle K \rangle = B_0$$

$$C_1 = B_1 \otimes U(B_0)$$

$$C_0 = U(B_0)$$

$$C_{-1} = B_{-1} \otimes U(B_0)$$

- Products C_0C_0, $C_0C_{\pm1}$, $C_{\pm1}C_0$ from $U(B)$.
- Products $C_{\pm1}C_{\mp1}$ by $(ux)(yv) = u(xy)v$, where $x \in B_{\pm1}$, $y \in B_{\mp1}$, $u,v \in U(B_0)$ and xy is given by $(*)$.

Together with these products, C^ℓ is a local algebra. Restricted associativity:

$$\mathrm{Ass}(C_{\pm1}, C_{\mp1}, C_{\pm1}) \neq 0$$

From C^ℓ, we can in many cases construct the local part of W by taking the (local) subalgebra of C^ℓ generated by $B_1 \cup B_{-1}B_0$ and factoring out the maximal ideal intersecting C_0 trivially !