

- Let  $C(n, n)$  be the associative super-algebra generated by  $2n$  odd elements  $\theta^a, K_a$  ( $a=1, 2, \dots, n$ ) modulo

$$[\theta^a, \theta^b] = [K_a, K_b] = 0, \quad [K_a, \theta^b] = \delta_a^b \theta^b$$

(also Clifford algebra of signature  $(n, n)$ )

- $\Lambda(n) = \langle \theta^{a_1} \dots \theta^{a_p} \rangle \subset C(n, n)$   
( $p=0, 1, \dots, n$ )

- Lie superalgebras of Cartan type obtained as subalgebras of the commutator algebra of  $C(n, n)$ :

$$W(n) = \Lambda(n) \langle K_a \rangle = \langle \theta^{a_1} \dots \theta^{a_p} K_b \rangle$$

$\mathbb{Z}$ -grading:

level	basis of $W(n)$	- of $sl(n 1)$
1	$K_a$	$K_a$
0	$\theta^a K_b$	$K_a^b$
-1	$\theta^a \theta^b K_c$	$K^b_c$
$\vdots$	$\vdots$	
$-n+1$		

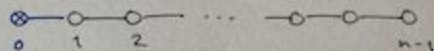
- $S(n)$ : subalgebra of  $W(n)$  consisting of traceless elements

$$\text{tr}(x K_a) = [x, K_a] \quad (x \in \Lambda(n))$$

- Lie algebras appearing as subalgebras at level zero:

$$W(n)_0 = gl(n)$$

$$S(n)_0 = sl(n) = A_{n-1}$$



Can be extended to a contragredient Lie superalgebra  $A(0, n-1) = sl(n|1)$  with the same level 0, 1 as  $W(n)$ .

Generalise:

- $A_{n-1}$  to  $g$ , any Kac-Moody algebra
- $\Lambda_1$  to  $\lambda$ , any dominant integral weight of  $g$
- $sl(n|1)$  to  $B(g, \lambda)$ , contragredient  $\rightarrow$

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- $W(n)$  to  $W(g, \lambda)$
  - $S(n)$  to  $S(g, \lambda)$
- } tensor hierarchy algebras!
- can be defined by a modified Chevalley-Serre presentation [Carbone, Cederwall, P 2018]
- $C(n, n)$  to ... ? (restrict to the local part)

Set  $K^a = \theta^a$ . Then the bracket in  $sl(n|1)$  differs from the commutator in  $C(n, n)$ :

$$[K_a, \theta^b] = -\theta^b K_a + \delta_a^b \underbrace{\theta^c K_c}_K$$

$$[K_a, \theta^b] = \delta_a^b$$

Starting from  $sl(n|1)$ , we can reconstruct the products  $\theta^b K_a$  and  $K_a \theta^b$  in  $C(n, n)$  by

$$\left. \begin{aligned} \theta^b K_a &= -[K_a, \theta^b] + \delta_a^b K \\ K_a \theta^b &= [K_a, \theta^b] - \delta_a^b K + \delta_a^b \end{aligned} \right\} (*)$$

$g \oplus \langle K \rangle$

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Extend the local part  $B_{-1} \oplus B_0 \oplus B_1$  of  $B$  to  $C^\lambda = C_{-1} \oplus C_0 \oplus C_1 \subset U(B)$ , where

$$C_1 = B_1 \oplus U(B_0)$$

$$C_0 = U(B_0)$$

$$C_{-1} = B_{-1} \oplus U(B_0)$$

- Products  $C_0 C_0, C_0 C_{\pm 1}, C_{\pm 1} C_0$  from  $U(B)$ .
- Products  $C_{\pm 1} C_{\pm 1}$  by  $(ux)(yv) = u(xy)v$ , where  $x \in B_{\pm 1}, y \in B_{\mp 1}, u, v \in U(B_0)$  and  $xy$  is given by (\*).

Together with these products,  $C^\lambda$  is a local algebra. Restricted associativity:

$$\text{Ass}(C_{\pm 1}, C_{\mp 1}, C_{\pm 1}) \neq 0$$

From  $C^\lambda$ , we can in many cases construct the local part of  $W$  by taking the (local) subalgebra of  $C^\lambda$  generated by  $B_1 \cup B_{-1} B_0$  and factoring out the maximal ideal intersecting  $C_0$  trivially!