Whittaker categories and Fock space categorification for Lie superalgebras

Volodymyr Mazorchuk

(Uppsala University)

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Based on and coauthors

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Coauthors:

Chih-Whi Chen — National Central University, Chung-Li, Taiwan

Shun-Jen Cheng — Academia Sinica and National Center of Theoretical Sciences, Taipei, Taiwan

Lie algebra story: setup

Let $\mathfrak{g}$ be a semisimple (or reductive) finite dimensional complex Lie algebra.

Fix a triangular decomposition (essentially unique):

$$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+.$$

Example. If $\mathfrak{g} = \mathfrak{sl}_n$, then we can take:

- as $\mathfrak{n}_+$, strictly upper triangular matrices;
- as $\mathfrak{h}$, diagonal matrices (of trace 0);
- as $\mathfrak{n}_-$, strictly lower triangular matrices.

Algebraic properties of the ingredients:

- $\mathfrak{h}$ is commutative;
- both $\mathfrak{n}_+$ and $\mathfrak{n}_-$ are nilpotent.

Recall: Lie's Theorem: all simple finite dimensional modules over $\mathfrak{h}$, $\mathfrak{n}_+$ and $\mathfrak{n}_-$ have dimension 1.

Lie algebra story: modules

Terminology for $\mathfrak{g}$-modules:

- Weight module means $\mathfrak{h}$-semisimple.
- Highest weight module means weight and generated by an element killed by $\mathfrak{n}_+$.
- Whittaker module means $\mathfrak{n}_+$ acts locally finitely.

Note: by this definition, highest weight modules are Whittaker modules.

General "principle": to get a reasonable family of modules, one has to kill one half of the parameters.

How it manifests: generic highest weight modules are simple and $\mathfrak{n}_-$-free.

These are (generic) Verma modules:

$$M(\lambda) := U(\mathfrak{g}) \bigotimes_{U(\mathfrak{h}\oplus\mathfrak{n}_+)} \mathbb{C}_\lambda, \qquad \lambda \in \mathfrak{h}^*.$$

Lie algebra story: category $\mathcal{O}$

Highest weight modules belong to the Bernstein-Gelfand-Gelfand category $\mathcal{O}$ (introduced in early 70's).

Formal definition: $\mathcal{O}$ is the full subcategory of $U(\mathfrak{g})$-mod consisting of all weight Whittaker modules.

Simples in $\mathcal{O}$ are simple highest weight modules (= simple quotients of Verma modules), they are indexed by $\mathfrak{h}^*$.

Structural properties of category $\mathcal{O}$:

- Splits into blocks, each being equivalent to the category of finite dimensional modules over a finite dimensional associative algebra.
- Each block is a highest weight category (i.e. the corresponding fin.dim algebra is quasi-hereditary).
- Has finite global dimension.
- Each block is Koszul.
- Has a simple preserving duality.
- Combinatorics of category $\mathcal{O}$ (e.g. multiplicities of simples in Vermas) is described by the Kazhdan-Lusztig combinatorics.

Lie algebra story: Whittaker modules

Introduced in the late 70's by Kostant.

Let $\eta : \mathfrak{n}_+ \to \mathbb{C}$ be a character.

We have the corresponding 1-dimensional simple $\mathfrak{n}_+$-module $\mathbb{C}_\eta$.

Observation: If a $\mathfrak{g}$-module V is generated by some $U(\mathfrak{n}_+)$-submodule isomorphic to $\mathbb{C}_\eta$, then each simple subquotient of $\mathrm{Res}^{\mathfrak{g}}_{\mathfrak{n}_+} V$ is isomorphic to $\mathbb{C}_\eta$.

Consequence: the category of finite length Whittaker modules splits into blocks indexed by η.

Note: those split further by the action of the center (the central character).

If $\eta = 0$, we get a "thick version" of category $\mathcal{O}$.

For all η, the corresponding simple Whittaker modules can be constructed inside $\mathrm{Hom}_{\mathbb{C}}(M(\lambda), \mathbb{C})$.

Note: For $\eta \neq 0$, these are not weight modules.

Lie algebra story: $\mathfrak{sl}_2$-example

Example. (Arnal and Pinczon, early 70's):

Consider $\mathfrak{sl}_2$ with the usual basis

$$F := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad H := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad E := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Let $\eta, \theta \in \mathbb{C}$ with $\eta \neq 0$.

Set $V_{\eta,\theta} = \mathbb{C}[H]$ with the regular action of $\mathbb{C}[H]$.

Claim. Setting

$$E \cdot g(H) := \eta g(H-2) \quad \text{and} \quad F \cdot g(H) = \frac{1}{4\eta}\big(\theta - (H+1)^2\big)g(H+2)$$

defines on $V_{\eta,\theta}$ the structure of a simple $\mathfrak{sl}_2$-module.

Note: This is a Whittaker module as E does not increase the H-degree of the polynomial g.

Lie algebra story: projective functors

For any $\mathfrak{g}$-module V, we have the corresponding endofunctor $V \otimes_{\mathbb{C}} \_$ of $\mathfrak{g}$-Mod.

If V is finite dimensional, such functors preserve both $\mathcal{O}$ and, more generally, the category $\mathcal{W}$ of finite length Whittaker modules.

Direct sums and summands of such endofunctors of $\mathcal{O}$ and $\mathcal{W}$ are called projective functors.

There is a classification of indecomposable projective functors (Bernstein and S. Gelfand, 1980).

In a certain sense, these are in bijection with indecomposable projectives in category $\mathcal{O}$.

The monoidal bicategory of projective functors can be described using Harish-Chandra $\mathfrak{g}$-$\mathfrak{g}$-bimodules.

Lie algebra story: Whittaker categories vs $\mathcal{O}$

Consider the category $\mathcal{W}$ of finite length Whittaker modules.

It decomposes as $\mathcal{W} \cong \bigoplus_{\theta,\eta} \mathcal{W}_{\theta,\eta}$,

where θ is a central character,

and η is a character of $\mathfrak{n}_+$.

If η is generic (non-zero on all simple roots), then $\mathcal{W}_{\theta,\eta}$ has a unique simple module and it is easy to describe its self-extensions (Kostant).

If $\eta = 0$, we get a "thick" version of category $\mathcal{O}$, i.e. the condition on the action of $\mathfrak{h}$ relaxes from semisimple to locally finite.

In between we have a family of degenerations of η when it is zero on some simple roots.

These degenerations are hence indexed by subsets of simple roots, that is, by parabolic subalgebras $\mathfrak{p}$ of $\mathfrak{g}$ containing $\mathfrak{h} \oplus \mathfrak{n}_+$.

Lie algebra story: in between

Let W be the Weyl group of $\mathfrak{g}$.

It controlls the central characters of Verma modules: two Verma modules have the same central character iff their highest weights are in the same orbit of the dot-action of W on $\mathfrak{h}^*$.

Central character blocks of $\mathcal{O}$ are thus given by the cosets W/G, where G is the dot-stabilizer of the dominant weight.

Let $\mathfrak{p}$ be a parabolic subalgebra of $\mathfrak{g}$.

Then we have the corresponding parabolic subgroup $W_\mathfrak{p}$ of W.

Centralizer subalgebra: If A is an associative and unital algebra and $e \in A$ is an idempotent, then eAe is the centralizer subalgebra of e.

In this case eAe-mod is a full subcategory of A-mod which can be very explicitly described (in several ways).

Observation. (McDowell, Miličić-Soergel, Backelin)

If the degeneration of η corresponds to $\mathfrak{p}$, then, for integral θ, the category $\mathcal{W}_{\theta,\eta}$ can be described inside the thick version of $\mathcal{O}_\theta$ via the centralizer corresponding to the longest coset rep. in $W_\mathfrak{p} \setminus W/G$.

Lie algebra story: consequences

Let $\overline{\mathcal{W}}_{\theta,\eta}$ be the "category $\mathcal{O}$-counterpart" of $\mathcal{W}_{\theta,\eta}$, i.e. not thick.

Important: $\overline{\mathcal{W}}_{\theta,\eta}$ is stable under the action of projective functors.

Structural properties of the category $\overline{\mathcal{W}}_{\theta,\eta}$:

- It is equivalent to the category of modules over a finite dimensional associative algebra.
- Has a simple preserving duality.
- The corresponding fin dim algebra is properly stratified.
- This means that instead of standard (Verma) modules, we now have two families: standard and proper standard modules.
- Combinatorics of this category (e.g. multiplicities of simples in proper standard modules) is described by (some part of) the Kazhdan-Lusztig combinatorics.

Remark. Also, both for $\mathcal{O}$ and $\overline{\mathcal{W}}$ we have interesting homological properties related to tilting modules, in particular, Ringel self-duality.

Remark. No longer finite global dimension, nor Koszulity, in general.

Lie algebra story: an alternative realization

If A is a finite dimensional associative algebra

and $e \in A$ is an idempotent,

then eAe-mod is equivalent to the Serre quotient of A-mod

modulo the Serre subcategory of A-mod generated by

all simple modules killed by e.

Therefore we can realize $\overline{\mathcal{W}}_{\theta,\eta}$ as the quotient of $\mathcal{O}_\theta$ by the Serre subcategory generated by all simples that do not correspond to longest representatives in $W_{\mathfrak{p}} \setminus W/G$.

In particular, we have the corresponding Serre quotient functor

$$\pi : \mathcal{O}_\theta \to \overline{\mathcal{W}}_{\theta,\eta}$$

which commutes with the action of the bicategory of projective functors.

Superalgebra story: setup

Let $\mathfrak{s}$ be a finite dimensional Lie superalgebra with a compatible $\mathbb{Z}$-grading:

$$\mathfrak{s} = \mathfrak{s}_{-1} \oplus \mathfrak{s}_0 \oplus \mathfrak{s}_1, \qquad \text{where} \quad \mathfrak{s}_0 = \mathfrak{g}.$$

Note: Such $\mathfrak{s}$ is quasireductive.

Examples: type I Lie superalgebras: $\mathfrak{gl}(m|n)$, $\mathfrak{sl}(m|n)$, $\mathfrak{psl}(n|n)$, $\mathfrak{osp}(2|2n)$, $\mathfrak{p}(n)$ and $[\mathfrak{p}(n), \mathfrak{p}(n)]$.

Assume that $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ extends to a triangular decomposition

$$\mathfrak{s} = \mathfrak{m}_- \oplus \mathfrak{h} \oplus \mathfrak{m}_+,$$

where $\mathfrak{m}_\pm = \mathfrak{n}_\pm \oplus \mathfrak{s}_{\pm 1}$.

Example: The standard triangular decomposition of $\mathfrak{gl}(m) \oplus \mathfrak{gl}(n)$ (i.e. upper triangular, diagonal and lower triangular matrices) extends to the standard triangular decomposition of $\mathfrak{gl}(m|n)$.

Superalgebra story: simple modules

Note: the problem of classification of simple $\mathfrak{g}$-modules is equivalent to the problem of classification of simple $\mathfrak{s}$-supermodules.

Theorem. (Chen-M., preprint 2018, published 2021)

There is a natural $1:2$ correspondence between simple $\mathfrak{g}$-modules and simple $\mathfrak{s}$-supermodules.

The correspondence is defined in terms of taking the unique simple top of the Kac module.

The $1:2$ ratio corresponds to the parity change.

At the same time: Categories $\mathcal{O}$ for $\mathfrak{g}$ and $\mathfrak{s}$ might be very different!

For example: Indecomposable blocks of the category $\mathcal{O}$ for $\mathfrak{s}$ might have infinitely many simples.

Superalgebra story: various kinds of modules and categories

We have the restriction functor $\mathrm{Res}^{\mathfrak{s}}_{\mathfrak{g}}$.

Terminology for $\mathfrak{s}$-supermodules:

- A supermodule V is weight provided that $\mathrm{Res}^{\mathfrak{s}}_{\mathfrak{g}} V$ is weight.
- A supermodule V is highest weight provided that it is weight and generated by an element killed by $\mathfrak{m}_+$.
- A supermodule V is Whittaker provided that $\mathrm{Res}^{\mathfrak{s}}_{\mathfrak{g}} V$ is Whittaker.

We also have the induction functor $\mathrm{Ind}^{\mathfrak{s}}_{\mathfrak{g}}$.

Note: $\mathrm{Res}^{\mathfrak{s}}_{\mathfrak{g}} \circ \mathrm{Ind}^{\mathfrak{s}}_{\mathfrak{g}}$ is a projective endofunctor of $\mathfrak{g}$-Mod.

Since both $\mathcal{O}$ and $\mathcal{W}$ (for $\mathfrak{g}$) are stable under projective functors, we can define the analogous categories for $\mathfrak{s}$ as full subcategories in $\mathfrak{s}$-sMod which restrict to the corresponding categories for $\mathfrak{g}$.

Then we have the usual (bi)adjunction given by induction and restriction between the corresponding categories for $\mathfrak{g}$ and $\mathfrak{s}$.

Superalgebra story: equivalence of categories

Observation. Just like for Lie algebras, the category of Whittaker supermodules splits w.r.t. the characters of $\mathfrak{n}_+$.

If we fix some character η, it does not vanish on some (possibly empty) set of simple even roots.

Let $\mathcal{I}_\eta$ be the Serre subcat. of $\mathcal{O}$ generated by all simples whose highest weights are dominant integral for at least one of these simple roots.

Let $\overline{\mathcal{W}}$ be the "category $\mathcal{O}$" analogue of $\mathcal{W}$.

Theorem. (Chen-Cheng-M.)

There is an equivalence between $\overline{\mathcal{W}}(\eta)$ and $\mathcal{O}/\mathcal{I}_\eta$.

This equivalence commutes with the action of projective functors and hence factors through the corresponding category of Harish-Chandra bimodules.

In fact the equivalence is proved via such a factorization.

Superalgebra story: structural properties

Note: Indecomposable blocks of $\overline{\mathcal{W}}(\eta)$ might have infinitely many simple objects.

So they are not described by finite dimensional algebras, in general.

However, they have enough projectives and those projective have finite length.

So, they are described by "locally finite dimensional" algebras.

Also, in general, these blocks are not highest weight categories.

In particular, the family of Verma modules is substituted by two families: standard and proper standard modules.

Theorem. (Chen-Cheng-M.)

Blocks of $\overline{\mathcal{W}}(\eta)$ are described by (possibly infinite dimensional) properly stratified algebras.

This means that all projective objects have a filtration with standard subquotients and a filtration with proper standard subquotients.

Superalgebra story: homological features

Note: $\overline{\mathcal{W}}(\eta)$ has a simple preserving duality.

We have: four families of modules: standard, costandard, proper standard and proper costandard.

Two homologically orthogonal pairs: standard and proper costandard; proper standard and costandard.

Tilting modules: have a standard and a proper costandard filtrations.

Cotilting modules: have a proper standard and a costandard filtrations.

Results:

- Tilting = cotilting.
- Classification of indecomposable (co)tilting modules.
- Ringel self-duality (the category of tilting objects is equivalent to the category of projective objects).

Superalgebra story: Fock space

Consider the quantum group $U_q(\mathfrak{gl}_\infty)$.

We have the natural $U_q(\mathfrak{gl}_\infty)$-module $\mathbb{V}$

and its dual $\mathbb{W}$.

Consider the Fock space $\mathbb{T}^{m|n} := \mathbb{V}^{\otimes m} \otimes \mathbb{W}^{\otimes n}$.

There is a natural bar-involution on $\mathbb{T}^{m|n}$.

A certain completion $\hat{\mathbb{T}}^{m|n}$ of $\mathbb{T}^{m|n}$ has:

- a bar-involution;
- a canonical basis (which, in fact, belongs to $\mathbb{T}^{m|n}$);
- a dual canonical basis.

Superalgebra story: q-symmetrizer

Schur-Weyl duality: there is an action of a Hecke algebra (of type A) on the right of $\mathbb{T}^{m|n}$.

Let W_η be a parabolic subgroup of $S_m \times S_n$.

Consider $\mathbb{T}^{m|n}_\eta := \mathbb{T}^{m|n} \sum_{\sigma \in W_\eta} q^{\ell(w_0^{W_\eta}) - \ell(\sigma)} H_\sigma$.

Observation. The bar involution on $\hat{\mathbb{T}}^{m|n}$ restricts to $\hat{\mathbb{T}}^{m|n}_\eta$.

Consequence: $\hat{\mathbb{T}}^{m|n}_\eta$ has analogues of canonical and dual canonical bases.

Superalgebra story: categorification

Let $\mathrm{Gr}(\_)$ denote the Gorthendieck group.

Then $\mathrm{Gr}(\mathcal{O}_{\mathrm{int}}) \cong \mathbb{T}^{m|n}$ by sending the classes of Verma modules to the standard basis.

Theorem. (Chen-Cheng-M.)

For $\mathfrak{gl}(m|n)$, there is an isomorphism between $\mathrm{Gr}(\widehat{\overline{\mathcal{W}}(\eta)}_{\mathrm{int}})$ and $\hat{\mathbb{T}}_{\eta}^{m|n}$ which sends the classes of standard modules to the standard basis.

This isomorphism sends the classes of indecomposable tilting module to the canonical basis

and the classes of simple modules to the dual canonical basis.

This allows one to describe the combinatorics of $\overline{\mathcal{W}}(\eta)$ in terms of Brundan-Kazhdan-Lusztig polynomials.

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THANK YOU!!!

Check out: Uppsala Algebra on YouTube:
https://www.youtube.com/channel/UCPWnhR29VHTAk7rZUEDQdDQ