Jordan meets Freudenthal
A Black Hole Exceptional Story

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Summary

Maxwell-Einstein-Scalar Theories

Attractor Mechanism

Duality Orbits and **Attractor Moduli Spaces**

Freudenthal Map

Groups of type E_7

Pre-Homogeneous Vector Spaces

Hints for the Future…

Maxwell-Einstein-Scalar Theories

$$\mathcal{L}=-\frac{R}{2}+\frac{1}{2}g_{ij}\left(\varphi\right)\partial_{\mu}\varphi^{i}\partial^{\mu}\varphi^{j}+\frac{1}{4}I_{\Lambda\Sigma}\left(\varphi\right)F_{\mu\nu}^{\Lambda}F^{\Sigma|\mu\nu}+\frac{1}{8\sqrt{-G}}R_{\Lambda\Sigma}\left(\varphi\right)\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}^{\Lambda}F_{\rho\sigma}^{\Sigma}$$

$$H:=\left(F^{\Lambda},G_{\Lambda}\right)^{T};$$

$$^{*}G_{\Lambda|\mu\nu}:=2\frac{\delta\mathcal{L}}{\delta F^{\Lambda|\mu\nu}}.$$

D=4 Maxwell-Einstein-scalar system (with no potential)

[may be the bosonic sector of D=4 (ungauged) sugra]

Abelian 2-form field strengths

static, spherically symmetric, asymptotically flat, **extremal black hole**

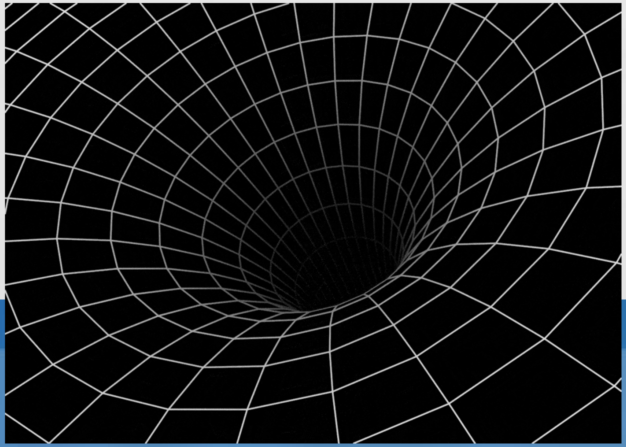
$$ds^{2}=-e^{2U(\tau)}dt^{2}+e^{-2U(\tau)}\left[\frac{d\tau^{2}}{\tau^{4}}+\frac{1}{\tau^{2}}\left(d\theta^{2}+\sin\theta d\psi^{2}\right)\right]$$

$$\tau:=-1/r$$

$$\mathcal{Q}\ :=\int_{S^{2}_{\infty}}H=\left(p^{\Lambda},q_{\Lambda}\right)^{T};$$

$$p^{\Lambda}\ :=\frac{1}{4\pi}\int_{S^{2}_{\infty}}F^{\Lambda},\ q_{\Lambda}=\frac{1}{4\pi}\int_{S^{2}_{\infty}}G_{\Lambda}.$$

dyonic vector of electric-magnetic fluxes
(**black hole charges**)



$$S_{D=1} = \int [(U')^2 + g_{ij}\varphi'^i\varphi'^j + e^{2U}V_{BH}(\varphi(\tau), \mathcal{Q})]d\tau$$

$$' \equiv \frac{d}{d\tau}$$

reduction D=4 →D=1 :effective 1-dimensional (radial) Lagrangian

$$V_{BH}(\varphi, \mathcal{Q}) := -\frac{1}{2}\mathcal{Q}^T\mathcal{M}(\varphi)\mathcal{Q},$$

BH effective potential

Ferrara,Gibbons,Kallosh

Euler-Lagrange Eqs.

$$\begin{cases} \frac{d^2U}{d\tau^2} = e^{2U}V_{BH}; \\ \frac{d^2\varphi^i}{d\tau^2} = g^{ij}e^{2U}\frac{\partial V_{BH}}{\partial\varphi^j}. \end{cases}$$

Attractor Mechanism : $\partial_\varphi V_{BH} = 0 \Leftrightarrow lim_{\tau\to-\infty}\varphi^a(\tau) = \varphi^a_H(\mathcal{Q})$

conformally flat geometry $AdS_2 \times S^2$ near the horizon

$$ds^2_{\text{B-R}} = \frac{r^2}{M^2_{\text{B-R}}}dt^2 - \frac{M^2_{\text{B-R}}}{r^2}(dr^2 + r^2d\Omega)$$

near the horizon, the scalar fields are **stabilized** purely in terms of charges

$$S = \frac{A_H}{4} = \pi\, V_{BH}|_{\partial_\varphi V_{BH}=0} = -\frac{\pi}{2}\mathcal{Q}^T\mathcal{M}_H\mathcal{Q}$$

Bekenstein-Hawking entropy-area formula for extremal dyonic black hole

Symmetric Scalar Manifolds

Let's specialize the discussion to theories with scalar manifolds (*target spaces*) which are **symmetric cosets G/H**

[**N>2** : general, **N=2** : particular, **N=1** : special cases]

H = isotropy group; *linearly* realized : scalar fields sit in an **H**-repr.

G = (global) electric-magnetic duality group [in string theory (over **Z**): U-duality];
Abelian 1-form sit in a **G**-repr; *non-linearly* realized on scalar fields

G is an *on-shell* symmetry of the Lagrangian,
i.e. it is a simmetry of the equations of motion **only**

The vector of 2-form field strengths (F,G), as well as the BH e.m. charges sit in a **G**-repr. **R** which is **symplectic** :

$$\exists ! \mathbb{C}_{[MN]} \equiv \mathbf{1} \in \mathbf{R} \times_a \mathbf{R};$$

$$\langle \mathcal{Q}_1, \mathcal{Q}_2 \rangle \equiv \mathcal{Q}_1^M \mathcal{Q}_2^N \mathbb{C}_{MN} = - \langle \mathcal{Q}_2, \mathcal{Q}_1 \rangle$$

symplectic product

$$\mathbb{C} = \begin{pmatrix} \mathbf{0}_n & \mathbb{I}_n \\ -\mathbb{I}_n & \mathbf{0}_n \end{pmatrix}$$

$$\begin{array}{rcl} G & \subset & Sp(2n, \mathbb{R}); \\ \mathbf{R} & = & \mathbf{2n} \end{array}$$

Gaillard-Zumino embedding
(generally maximal, but not symmetric)

Dynkin, Gaillard-Zumino

❖ symmetric scalar manifolds of N=2, D=4 sugra [all but T^3 model]

all **special Kaehler** of projective type	$\frac{G_V}{H_V}$	r	$dim_{\mathbb{C}} \equiv n_V$
quadratic sequence $n \in \mathbb{N}$	$\frac{SU(1,n)}{U(1)\otimes SU(n)}$	1	n
$\mathbb{R} \oplus \mathbf{\Gamma}_n,\ n \in \mathbb{N}$	$\frac{SU(1,1)}{U(1)} \otimes \frac{SO(2,n)}{SO(2)\otimes SO(n)}$	$2\ (n=1)$ $3\ (n \geqslant 2)$	$n+1$
$J_3^{\mathbb{O}}$	$\frac{E_{7(-25)}}{E_{6(-78)}\otimes U(1)}$	3	27
$J_3^{\mathbb{H}}$	$\frac{SO^*(12)}{U(6)}$	3	15
$J_3^{\mathbb{C}}$	$\frac{SU(3,3)}{S(U(3)\otimes U(3))} = \frac{SU(3,3)}{SU(3)\otimes SU(3)\otimes U(1)}$	3	9
$J_3^{\mathbb{R}}$	$\frac{Sp(6,\mathbb{R})}{U(3)}$	3	6

Pascual Jordan
(1902-1980)

$$R_{i\bar{j}k\bar{l}} = -g_{i\bar{j}}g_{k\bar{l}} - g_{i\bar{l}}g_{k\bar{j}} + C_{ikm}\overline{C}_{\bar{j}\bar{l}\bar{p}}g^{m\bar{p}}$$

symmetric scalar manifolds **G/H** (including symmetric SK spaces of N=2, D=4 sugra)

The **G**-representation space **R** of the BH e.m. charges gets **stratified**, under the action of **G**, in **U-orbits** (*non-symmetric* cosets **G/$\mathcal{H}$**). Ferrara, Gunaydin

$\mathcal{H}$ is the **stabilizer** (isotropy) group of the **U-orbit** = symmetry of the charge configurations, it relates *physically equivalent* BH charge configurations

each **U-orbit** supports a class of critical points of V_{BH} , corresponding to specific ***SUSY-preserving properties*** of the near-horizon geometry

When $\mathcal{H}$ is **non-compact**, there is a **residual compact symmetry** linearly acting on scalars, such that the scalars belonging to the
***"moduli space"* $\mathcal{H}$/mcs($\mathcal{H}$)** (symmetric **submanifold** of **G/H**)
are **not** stabilized in terms of BH charges at the event horizon of the extremal BH

Ferrara, AM

The Attractor Mechanism is **inactive** on these **unstabilized** scalar fields, which are ***flat directions*** of V_{BH} at its critical points.

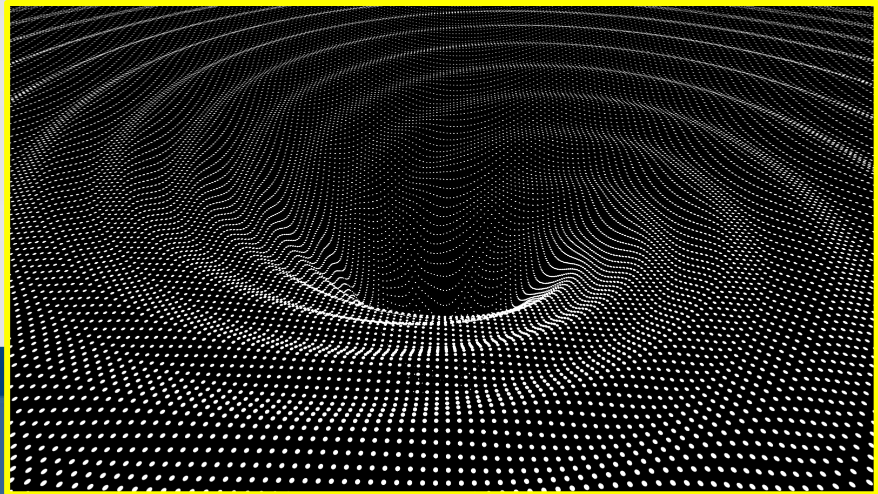
symmetric scalar manifolds **G/H** (cont'd)

The **absence** of flat directions at **N=2 ½-BPS attractors** can thus be explained by the fact that the stabilizer of the ½-BPS orbit is **compact** : **$\mathcal{H}$=H/U(1)**, where **H** is the stabilizer of the scalar manifold **G/H** itself

The **massless Hessian modes**, ubiquitous at non-BPS crit pts of V_{BH}, are actually ***all flat directions*** of V_{BH} itself at the considered class of crit. pts.

Black hole entropy is independent on unstabilized scalar fields

Thus, the ***flat directions*** of V_{BH} at its critical points span various "***moduli spaces***", corresponding to each class of **extremal black** hole solutions



❖ Duality Orbits of *symmetric* N=2, D=4 supergravities [all but T^3 model]

	$\frac{1}{2}$-BPS orbits $\mathcal{O}_{\frac{1}{2}-BPS} = \frac{G}{H_0}$	non-BPS, $Z \neq 0$ orbits $\mathcal{O}_{non-BPS,Z\neq 0} = \frac{G}{\hat{H}}$	non-BPS, $Z = 0$ orbits $\mathcal{O}_{non-BPS,Z=0} = \frac{G}{\tilde{H}}$
Quadratic Sequence $(n = n_V \in \mathbb{N})$	$\frac{SU(1,n)}{SU(n)}$	$-$	$\frac{SU(1,n)}{SU(1,n-1)}$
$\mathbb{R} \oplus \Gamma_n$ $(n = n_V - 1 \in \mathbb{N})$	$\frac{SU(1,1)\otimes SO(2,n)}{SO(2)\otimes SO(n)}$	$\frac{SU(1,1)\otimes SO(2,n)}{SO(1,1)\otimes SO(1,n-1)}$	$\frac{SU(1,1)\otimes SO(2,n)}{SO(2)\otimes SO(2,n-2)}$
$J_3^{\mathbb{O}}$	$\frac{E_{7(-25)}}{E_6}$	$\frac{E_{7(-25)}}{E_{6(-26)}}$	$\frac{E_{7(-25)}}{E_{6(-14)}}$
$J_3^{\mathbb{H}}$	$\frac{SO^*(12)}{SU(6)}$	$\frac{SO^*(12)}{SU^*(6)}$	$\frac{SO^*(12)}{SU(4,2)}$
$J_3^{\mathbb{C}}$	$\frac{SU(3,3)}{SU(3)\otimes SU(3)}$	$\frac{SU(3,3)}{SL(3,\mathbb{C})}$	$\frac{SU(3,3)}{SU(2,1)\otimes SU(1,2)}$
$J_3^{\mathbb{R}}$	$\frac{Sp(6,\mathbb{R})}{SU(3)}$	$\frac{Sp(6,\mathbb{R})}{SL(3,\mathbb{R})}$	$\frac{Sp(6,\mathbb{R})}{SU(2,1)}$

Bellucci, Ferrara, Gunaydin, AM

in **N=2** susy : $\{Q_\alpha^A, Q_\beta^B\} = \epsilon_{\alpha\beta} Z^{[AB]} = \epsilon_{\alpha\beta}\epsilon^{AB} Z$

❖ non-BPS Z\neq0 ***moduli spaces*** of *symmetric* **N=2, D=4 supergravities**

Ferrara,AM

	$\frac{\widehat{H}}{\widehat{h}}$	r	$dim_{\mathbb{R}}$
$\mathbb{R} \oplus \Gamma_n$ $(n = n_V - 1 \in \mathbb{N})$	$SO(1,1) \otimes \frac{SO(1,n-1)}{SO(n-1)}$	$1(n=1)$ $2(n \geqslant 2)$	n
$J_3^{\mathbb{O}}$	$\frac{E_{6(-26)}}{F_{4(-52)}}$	2	6
$J_3^{\mathbb{H}}$	$\frac{SU^*(6)}{USp(6)}$	2	14
$J_3^{\mathbb{C}}$	$\frac{SL(3,C)}{SU(3)}$	2	8
$J_3^{\mathbb{R}}$	$\frac{SL(3,\mathbb{R})}{SO(3)}$	2	5

$\widehat{h}$ =mcs $\widehat{H}$

Nota Bene : these ***moduli spaces*** are nothing but the **scalar manifolds** of the corresponding theories uplifted to **D=5 [real special** geometry] let's reconsider the starting **Maxwell-Einstein-scalar Lagrangian density**

$$\mathcal{L} = -\frac{R}{2} + \frac{1}{2} g_{ij}(\varphi)\,\partial_\mu \varphi^i \partial^\mu \varphi^j + \frac{1}{4} I_{\Lambda\Sigma}(\varphi)\, F^{\Lambda}_{\mu\nu} F^{\Sigma|\mu\nu} + \frac{1}{8\sqrt{-G}} R_{\Lambda\Sigma}(\varphi)\, \epsilon^{\mu\nu\rho\sigma} F^{\Lambda}_{\mu\nu} F^{\Sigma}_{\rho\sigma}$$

…and introduce the following real 2n x 2n matrix (n = number of Abelian 1-forms)

$$\mathcal{M} = \begin{pmatrix} \mathbb{I} & -R \\ 0 & \mathbb{I} \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & I^{-1} \end{pmatrix} \begin{pmatrix} \mathbb{I} & 0 \\ -R & \mathbb{I} \end{pmatrix} = \begin{pmatrix} I + R I^{-1} R & -R I^{-1} \\ -I^{-1} R & I^{-1} \end{pmatrix}$$

$$\mathcal{M}^T = \mathcal{M} \qquad \mathcal{M}\mathbb{C}\mathcal{M} = \mathbb{C}$$

By virtue of this matrix, one can introduce a (scalar fields dependent) **anti-involution** in ***any*** Maxwell-Einstein-scalar gravity theory with **symplectic structure** :

$$\mathfrak{F}(\varphi) := -\mathbb{C}\mathcal{M}(\varphi)$$

$$\mathfrak{F}^2(\varphi) = \mathbb{C}\mathcal{M}(\varphi)\mathbb{C}\mathcal{M}(\varphi) = \mathbb{C}^2 = -Id$$

Hans Freudenthal (1905-1990)

This **anti-involution** is named (**scalar fields dependent**) **Freudenthal (duality) map**

Let us evaluate the action of the **Freudenthal map** on e.m. charges (vector *Q*) at the **event horizon** of the extremal black hole

Attractor Mechanism $\partial_\varphi V_{BH} = 0 \Leftrightarrow lim_{\tau \to -\infty} \varphi^a(\tau) = \varphi^a_H(\mathcal{Q})$

Bekenstein-Hawking entropy $S = \frac{A_H}{4} = \pi V_{BH}|_{\partial_\varphi V_{BH}=0} = -\frac{\pi}{2}\mathcal{Q}^T \mathcal{M}_H \mathcal{Q}$

By defining the matrix *M* at the horizon : $\lim_{\tau \to -\infty} \mathcal{M}(\varphi(\tau)) = \mathcal{M}_H(\mathcal{Q})$

one can define the **horizon limit** of the action of the **Freudenthal map** on *Q* as

$$\lim_{\tau \to -\infty} \mathfrak{F}(\mathcal{Q}) =: \mathfrak{F}_H(\mathcal{Q}) = -\mathbb{C}\mathcal{M}_H\mathcal{Q} = \frac{1}{\pi}\mathbb{C}\frac{\partial S_{BH}}{\partial \mathcal{Q}} =: \tilde{\mathcal{Q}},$$

$$\mathfrak{F}^2_H(\mathcal{Q}) = \mathfrak{F}_H(\tilde{\mathcal{Q}}) = -\mathcal{Q}$$

This is a **non-linear (scalar fields independent) anti-involutive map** on Q (hom. deg. = 1)

Bekenstein – Hawking entropy is **invariant** under its own **symplectic gradient** :

$$S(\mathcal{Q}) = S(\mathfrak{F}_H(\mathcal{Q})) = S\left(\frac{1}{\pi}\mathbb{C}\frac{\partial S}{\partial \mathcal{Q}}\right) = S(\tilde{\mathcal{Q}})$$

This can be extended to include *at least* **all quantum corrections** with **homogeneity 2** or **0** in the black hole charges Q

Ferrara, AM, Yeranyan (and late ***Raymond Stora***)

Lie groups of type E_7 : (G,**R**)

Brown (1967);
Garibaldi; Krutelevich;
Borsten,Dahanayake,Duff,Rubens;
Ferrara,Kallosh,AM;
AM,Orazi,Riccioni

- the (ir)repr. **R** is **symplectic** :

$$\exists! \mathbb{C}_{[MN]} \equiv \mathbf{1} \in \mathbf{R} \times_a \mathbf{R}; \quad \langle Q_1, Q_2 \rangle \equiv Q_1^M Q_2^N \mathbb{C}_{MN} = -\langle Q_2, Q_1 \rangle ;$$

symplectic product

- the (ir)repr. admits a completely symmetric **invariant rank-4** tensor

$$\exists!\ K_{MNPQ} = K_{(MNPQ)} \equiv \mathbf{1} \in [\mathbf{R} \times \mathbf{R} \times \mathbf{R} \times \mathbf{R}]_s$$

(*K-tensor*)

G-invariant quartic polynomial

$$I_4 := K_{MNPQ} Q^M Q^N Q^P Q^Q =: \epsilon \, |I_4| \, , \quad \rightarrow \quad S_{BH} = \pi \sqrt{|I_4|}$$

- defining a triple map in **R** as

$$T \colon \mathbf{R} \times \mathbf{R} \times \mathbf{R} \rightarrow \mathbf{R} \quad \langle T(Q_1, Q_2, Q_3), Q_4 \rangle \equiv K_{MNPQ} Q_1^M Q_2^N Q_3^P Q_4^Q$$

it holds $\langle T(Q_1, Q_1, Q_2), T(Q_2, Q_2, Q_2) \rangle = \langle Q_1, Q_2 \rangle K_{MNPQ} Q_1^M Q_2^N Q_2^P Q_2^Q$

[the 3rd axiom makes a **group of type E_7** definable in terms of **Freudenthal triple systems**]

All electric-magnetic (**U-**)**duality groups** of **D=4** sugras with **symmetric** scalar manifolds and *at least* **N=2** supercharges are **of type E_7**

$N = 2$

G	R
$U(1,n)$	$(\mathbf{1+n})$
$SL(2,\mathbb{R}) \times SO(2,n)$	$(\mathbf{2}, \mathbf{2+n})$
$SL(2,\mathbb{R})$	$\mathbf{4}$
$Sp(6,\mathbb{R})$	$\mathbf{14'}$
$SU(3,3)$	$\mathbf{20}$
$SO^*(12)$	$\mathbf{32}$
$E_{7(-25)}$	$\mathbf{56}$

N	G	R
3	$U(3,n)$	$(\mathbf{3+n})$
4	$SL(2,\mathbb{R}) \times SO(6,n)$	$(\mathbf{2}, \mathbf{6+n})$
5	$SU(1,5)$	$\mathbf{20}$
6	$SO^*(12)$	$\mathbf{32}$
8	$E_{7(7)}$	$\mathbf{56}$

"degenerate" groups of type E_7

$$I_4(p,q) = (I_2(p,q))^2$$

$$S_{BH} = \pi\sqrt{|I_4(p,q)|} = \pi\,|I_2(p,q)|\,.$$

In supegravities with electric-magnetic duality group **of type E_7**, the **G**-invariant **K-tensor** determines the Bekenstein-Hawking entropy of extremal black holes

$$S_{BH} = \pi\sqrt{|I_4|}$$

$$I_4 := K_{MNPQ} \mathcal{Q}^M \mathcal{Q}^N \mathcal{Q}^P \mathcal{Q}^Q =: \epsilon\, |I_4|\,,$$

The **K-tensor** can generally be expressed as adjoint-trace of the product of **G**-generators (dim **R** = 2n, and dim **Adj** = d) :

$$K_{MNPQ} = -\frac{n\,(2n+1)}{6d}\left[t^{\alpha}_{MN} t_{\alpha|PQ} - \frac{d}{n(2n+1)}\mathbb{C}_{M(P}\mathbb{C}_{Q)N}\right]$$

The **horizon Freudenthal (duality) map** can be expressed in terms of the **K-tensor**

$$\mathfrak{F}_H(\mathcal{Q})_M = \tilde{\mathcal{Q}}_M = \frac{\partial\sqrt{|I_4(\mathcal{Q})|}}{\partial \mathcal{Q}^M} = \epsilon\frac{2}{\sqrt{|I_4(\mathcal{Q})|}} K_{MNPQ}\mathcal{Q}^N\mathcal{Q}^P\mathcal{Q}^Q$$

Borsten,Dahanayake,Duff,Rubens

In this class of theories, the **invariance** of the Bekenstein-Hawking black hole entropy under **horizon Freudenthal duality map** reads

$$I_4\,(\mathcal{Q}) = I_4(\mathbb{C}\tilde{\mathcal{Q}}) = I_4\left(\mathbb{C}\frac{\partial\sqrt{|I_4(\mathcal{Q})|}}{\partial \mathcal{Q}}\right)$$

The non-transitive action of the split form $\mathbf{E}_{7(7)}$ on its 56-dim. repr. **56** gives rise to a generic (open) orbit $\frac{E_{7(7)}}{E_{6(2)}} \times \mathbb{R}^+$ which is a non-compact real form of $\frac{E_7}{E_6} \times GL(1)$

regular **pre-homogeneous vector space** (**PVS**) of type (29) in the classification by Sato and Kimura ('77) :

(29) $(GL(1) \times E_7,\ \Box \otimes \Lambda_6,\ V(1) \otimes V(56)).$

(i) $H \sim E_6$, (ii) $\deg f = 4$, (iii) $f(X) = T(x^\sharp, y^\sharp) - \xi N(x) - \eta N(y) - \frac{1}{4}(T(x,y) - \xi\eta)^2$ (see (1.16), or Proposition 52 in § 5).

A **PVS** is a finite-dimensional vector space ***V*** together with a subgroup ***G*** of **GL(*V*)** such that ***G*** has an **open**, **dense orbit** in ***V*** [Sato,Kimura; Knapp]

PVS are subdivided into two types, according to whether there exists a *homogeneous* polynomial **f** on ***V*** which is **invariant** under the semisimple part of ***G***.

In this case : **V** = **56** (fundamental irrep. of $\mathbf{G=E_7}$), **f** = **quartic** invariant polynomial $\mathbf{I_4}$
H= isotropy (stabilizer) group = $\mathbf{E_6}$

Manifestly $\mathbf{E_6}$ **-invariant** expression of the quartic invariant $\mathbf{I_4}$ of the **56** of $\mathbf{E_7}$:
well before (**'77** = almost contemporary to sugra) the expression introduced by Ferrara & Gunaydin (**'97**)**!**

$$I_4\left(p^0,p^i,q_0,q_i\right) = -\left(p^0q_0 + p^iq_i\right)^2 + 4\left[q_0I_3\left(p\right) - p^0I_3\left(q\right) + \left\{\frac{\partial I_3\left(p\right)}{\partial p}, \frac{\partial I_3\left(q\right)}{\partial q}\right\}\right]$$

simple groups **of type E_7 of sugra** almost saturate list of **irr. PVS** with **invariant** deg **4**

G	V	n	Isotropy algebra	Degree	
$SL(2,\mathbb{C})$	$S^3\mathbb{C}^2$	1	0	4	N=2, T^3 model
$SL(6,\mathbb{C})$	$\Lambda^3\mathbb{C}^6$	1	$\mathfrak{sl}(3,\mathbb{C}) \times \mathfrak{sl}(3,\mathbb{C})$	4	N=2 magic on **R**
$SL(7,\mathbb{C})$	$\Lambda^3\mathbb{C}^7$	1	$\mathfrak{g}_2^{\mathbb{C}}$	7	
$SL(8,\mathbb{C})$	$\Lambda^3\mathbb{C}^8$	1	$\mathfrak{sl}(3,\mathbb{C})$	16	
$SL(3,\mathbb{C})$	$S^2\mathbb{C}^3$	2	0	6	
$SL(5,\mathbb{C})$	$\Lambda^2\mathbb{C}^3$	3,4	$\mathfrak{sl}(2,\mathbb{C}), 0$	5,10	
$SL(6,\mathbb{C})$	$\Lambda^2\mathbb{C}^3$	2	$\mathfrak{sl}(2,\mathbb{C}) \times \mathfrak{sl}(2,\mathbb{C}) \times \mathfrak{sl}(2,\mathbb{C})$	6	
$SL(3,\mathbb{C}) \times SL(3,\mathbb{C})$	$\mathbb{C}^3 \otimes \mathbb{C}^3$	2	$\mathfrak{gl}(1,\mathbb{C}) \times \mathfrak{gl}(1,\mathbb{C})$	6	
$Sp(6,\mathbb{C})$	$\Lambda_0^3\mathbb{C}^6$	1	$\mathfrak{sl}(3,\mathbb{C})$	4	N=2 magic on **C**
$Spin(7,\mathbb{C})$	$\mathbb{C}^8$	1,2,3	$\mathfrak{g}_2^{\mathbb{C}}, \mathfrak{sl}(3,\mathbb{C}) \times \mathfrak{so}(2,\mathbb{C}), \mathfrak{sl}(2,\mathbb{C}) \times \mathfrak{so}(3,\mathbb{C})$	2,2,2	
$Spin(9,\mathbb{C})$	$\mathbb{C}^{16}$	1	$\mathfrak{spin}(7,\mathbb{C})$	2	
$Spin(10,\mathbb{C})$	$\mathbb{C}^{16}$	2,3	$\mathfrak{g}_2^{\mathbb{C}} \times \mathfrak{sl}(2,\mathbb{C}), \mathfrak{sl}(2,\mathbb{C}) \times \mathfrak{so}(3,\mathbb{C})$	2,4	
$Spin(11,\mathbb{C})$	$\mathbb{C}^{32}$	1	$\mathfrak{sl}(5,\mathbb{C})$	4	?
$Spin(12,\mathbb{C})$	$\mathbb{C}^{32}$	1	$\mathfrak{sl}(6,\mathbb{C})$	4	N=2 magic on **H**, N=6
$Spin(14,\mathbb{C})$	$\mathbb{C}^{64}$	1	$\mathfrak{g}_2^{\mathbb{C}} \times \mathfrak{g}_2^{\mathbb{C}}$	8	
$G_2^{\mathbb{C}}$	$\mathbb{C}^7$	1,2	$\mathfrak{sl}(3,\mathbb{C}), \mathfrak{gl}(2,\mathbb{C})$	2,2	
$E_6^{\mathbb{C}}$	$\mathbb{C}^{27}$	1,2	$\mathfrak{f}_4^{\mathbb{C}}, \mathfrak{so}(8,\mathbb{C})$	3,6	
$E_7^{\mathbb{C}}$	$\mathbb{C}^{56}$	1	$\mathfrak{e}_6^{\mathbb{C}}$	4	N=2 magic on **O**, N=8

In sugra, **n** can be associated to the # of centers of the **multi-centered** black holes

[Nota Bene : here only *irreducible* **PVS** (with **G** simple and **complex** Lie group) are considered]

→ classification of groups **of type E_7** ? *in progress….*

Some advances in rather recent papers,
e.g. [Garibaldi, Guralnick]

G	V	$\dim V$	char k	G	V	$\dim V$	char k
B_n	λ_1	$2n+1$	$\neq 2$	A_1	$\lambda_1 + p^i\lambda_1$ $(i \geq 1)$	4	$= p \neq 0$
D_n	λ_1	$2n$	all	A_2	$\lambda_1 + \lambda_2$	7	3
A_1	$2\lambda_1$	3	$\neq 2$	A_3	λ_2	6	all
A_5	λ_3	20	2	B_4	λ_4	16	all
B_3	λ_3	8	all	B_5	λ_5	32	2
C_3	λ_3	8	2	C_3	λ_2	13	3
D_6	half-spin	32	2	G_2	λ_1	7	$\neq 2$
E_7	λ_7	56	2	F_4	λ_4	25	3

p=2 : T^3 model

?

known simple Lie groups **of type E_7** occurring in **D=4 Maxwell-Einstein (super)gravity theories**
[exactly the ones occurring in the Sato-Kimura table !]

Some Hints for the Future…

- **Freudenthal duality map** for **non-symmetric** proj. **special Kaehler** manifolds
  [deWit, Van Proeyen; Alekseevsky, Cortes, …]
  and relation to *cubic* **T-Algebras** [Vinberg, Cecotti] recent devs. : [Alekseevsky, AM, Spiro]

- **D=5 : «Jordan duality map» for black holes and black strings**,
  groups **of type E_6, PVS** , and **D=5 Maxwell-Einstein (super)gravity**
  Borsten, Duff *et al.*

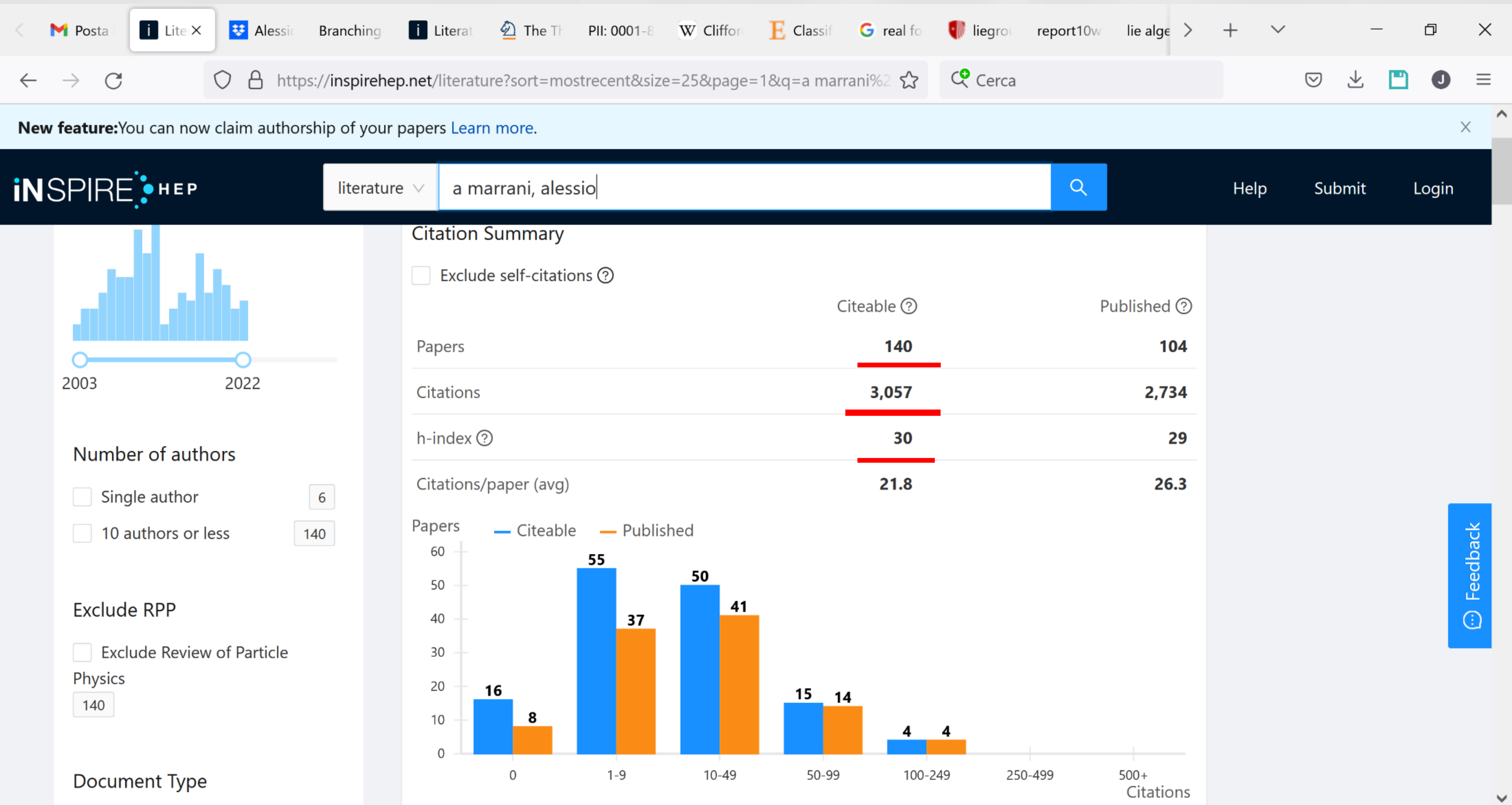
- **Freudenthal duality map** for intrinsically *quantum* black holes
  («small» orbits)

- extension to **multi-centered (extremal) black hole solutions**:
  work in progress [Yeranyan; Ferrara,AM,Shcherbakov,Yeranyan]

- **new** groups of type E_7 and exceptional/orthosymplectic Lie superalgebras

- into the ***quantum regime*** of gravity [**U-duality groups** is over **Z**] :
  Freudenthal duality map for **integer, quantized charges ?** Borsten, Duff *et al.*,

- (super)string/M- theoretical realization/interpretation of the
  Freudenthal dual map : *still, a mystery….*



Despite my academic seniority (17 years of post-doctoral appointments all over the world), I still don't have any tenured, nor tenure-track, job in academic research.

Any help/advice in finding a tenured job will be very much welcome !

my CV is available upon request : if interested,send me a mail to **alessio.marrani@um.es**

KEEP
CALM
AND
STUDY GEOMETRIC STRUCTURES
AND SUPERSYMMETRY

Tusen Takk !