The geometry and DSZ quantization of four-dimensional supergravity

Calin Lazaroiu (with C. S. Shahbazi)

DFT, IFIN-HH

Outline

Main points

- For simplicity, I present only the universal bosonic sector and ignore the vertical complex structure. Spinors are easy to add if the spacetime manifold has $\mathrm{Spin}_0(3,1)$ structure, much harder if it only admits a $\mathrm{Spin}_0^c(3,1)$ structure.
- We define the classical and semiclassical theories on any paracompact, connected and oriented four-manifold M, so $\pi_1(M)$ can be "wild" (for example we can have $\pi_1(M) = \mathbb{Q}$).
- Our formulation of the classical and semiclassical theories are manifestly U-duality invariant.
- We prove that the DSZ quantization is implemented using certain principal bundles whose structure group is the group of affine transformations of certain integral symplectic tori. This includes a local and unconstrained realization of electromagnetic duality at the level of gauge connections.
- The globally-defined solutions of the classical or semiclassical theory are (classical or semiclassical) locally geometric U-folds. Our formulation gives a global geometric description of such objects in four dimensions as global solutions of a coupled system of geometric PDEs.
- The formulation relies on a much more general theory of quasi-Abelian principal bundles and connections which I will not discuss.

Let M be an oriented and connected four-manifold. The universal bosonic sector of classical 4d supergravity defined on M is a **generalized Einstein-Scalar-Maxwell (GESM) theory**, which is parameterized by:

- A **scalar bundle** $(\pi, \mathcal{H}, \mathcal{G})$, where $\pi \colon X \to M$ is a surjective submersion equipped with a complete flat Ehresmann connection $\mathcal{H}$ and a vertical Euclidean metric $\mathcal{G}$, i.e. a Euclidean metric on the vertical bundle $\mathcal{V}$ which is invariant under the parallel transport of $\mathcal{H}$.
- A **duality structure** $\Delta = (\mathcal{S}, \omega, \mathcal{D})$ i.e. a flat symplectic vector bundle defined on X.
- A **vertical taming** $\mathcal{J}$ of Δ, i.e. a taming of $(\mathcal{S}, \omega)$ which is invariant under the extended parallel transport induced by $\mathcal{H}$ and $\mathcal{D}$.

For simplicity, I take the scalar potential of the theory to vanish identically. The configuration space is $\mathrm{Lor}(M) \times \Gamma(\pi) \times \Omega^2_{\mathrm{d}_{\mathcal{D}}\text{-cl}}(M, \mathcal{S})$.

Definition

Let $\Phi = (\pi, \mathcal{H}, \mathcal{G}, \Delta, \mathcal{J})$ be a scalar-electromagnetic bundle. The **fundamental field** of Φ is:

$$\Psi \stackrel{\mathrm{def.}}{=} \mathcal{D}^{\mathrm{ad}}(\mathcal{J})^{\sharp_h} \in \Gamma(X, \mathcal{V} \otimes \mathit{End}(\mathcal{S}))\,,$$

where h is the Kaluza-Klein metric induced by g and $\mathcal{H}$ on X.

Let $Q \stackrel{\text{def.}}{=} \omega \circ (\mathcal{J} \otimes \mathrm{id}_{\mathcal{S}})$ be the Euclidean metric induced on $\mathcal{S}$. The GESM theory parameterized by $\Phi = (\pi, \mathcal{H}, \mathcal{G}, \Delta, \mathcal{J})$ is defined by:

- The Einstein equations:

$$\mathrm{Ric}(g) - \frac{1}{2}\mathrm{R}(g)g = \frac{1}{2}\mathrm{Tr}_g(s_v^*\mathcal{G})\, g - s_v^*\mathcal{G} + 2\mathcal{F} \oslash_{Q^s} \mathcal{F}\,, \tag{1}$$

  where $\mathrm{Ric}(g)/\mathrm{R}(g)$ are the Ricci tensor/scalar of g, $\mathrm{Tr}_g : \otimes^2 T^*M \to \underline{\mathbb{R}}$ is the g-trace, $s_v^*\mathcal{G} = \mathcal{G} \circ (\mathrm{d}^v s \otimes \mathrm{d}^v s)$ is the **vertical first fundamental form** of s and $\oslash_{Q^s}$ is the **twisted inner contraction** of $\mathcal{S}^s$-valued two-forms. Here $\mathrm{d}^v s = P_V \circ \mathrm{d}s : T\mathcal{M} \to \mathcal{V}$.

- The scalar equations:

$$\mathrm{Tr}_g(\nabla^v \mathrm{d}^v s) = \frac{1}{2}(*\mathcal{F}, \Psi^s \mathcal{F})_{g,Q^s}\,, \tag{2}$$

  where ∇^v is a certain connection on $TM \otimes \mathcal{V}^s$ and $(\ ,\ )_{g,Q^s}$ is the **twisted exterior pairing** on $\wedge T^*M \otimes \mathcal{S}^s$.

- The twisted Maxwell equations:

$$\star_{g,\mathcal{J}^s} \mathcal{F} = \mathcal{F}\,, \tag{3}$$

  where $\star_{g,\mathcal{J}^s} = *_g \otimes \mathcal{J}^s$ is the $\mathcal{J}^s$-*polarized Hodge operator* on $\wedge T^*M \otimes \mathcal{S}^s$.

The globally-defined solutions are classical locally geometric U-folds.

Duality and electromagnetic structures

Let $(\mathcal{S}, \mathcal{D})$ be a flat vector bundle defined on a manifold X and $H^*_{\mathcal{D}}(X, \mathcal{S})$ be the cohomology of the twisted de Rham complex:

$$0 \to \mathcal{C}^\infty(X, \mathcal{S}) \xrightarrow{\mathcal{D}} \Omega^1(X, \mathcal{S}) \xrightarrow{\mathrm{d}_{\mathcal{D}}} \ldots \xrightarrow{\mathrm{d}_{\mathcal{D}}} \Omega^d(X, \mathcal{S}) \to 0 \quad .$$

We have a natural isomorphism of graded vector spaces:

$$H^*_{\mathcal{D}}(X, \mathcal{S}) \simeq H^*(X, \mathcal{C}^\infty_{\mathrm{flat}}(\mathcal{S})) \quad .$$

Proposition

For any $x \in X$, the set of isomorphism classes of duality structures of rank $2n$ on X is in bijection with the character variety:

$$\mathfrak{R}(\pi_1(X, x), \mathrm{Sp}(2n, \mathbb{R})) \stackrel{\mathrm{def.}}{=} \mathrm{Hom}(\pi_1(X, x), \mathrm{Sp}(2n, \mathbb{R}))/\mathrm{Sp}(2n, \mathbb{R}) \, ,$$

where $\mathrm{Sp}(2n, \mathbb{R})$ acts through its adjoint representation.

Definition

An **electromagnetic structure** defined on X is a pair $\Xi = (\Delta, \mathcal{J})$, where $\Delta = (\mathcal{S}, \omega, \mathcal{D})$ is a duality structure on X and $\mathcal{J}$ is a taming of the symplectic vector bundle $(\mathcal{S}, \omega)$. The **rank** of Ξ is the rank of Δ.

Let (M, g) be an oriented and connected Lorentzian four-manifold and $\Xi = (\mathcal{S}, \omega, \mathcal{D}, \mathcal{J})$ be an electromagnetic structure on M. Let $\Delta \stackrel{\mathrm{def.}}{=} (\mathcal{S}, \omega, \mathcal{D})$.

Definition

The $\mathcal{J}$**-polarized Hodge operator** $\star_{g,\mathcal{J}} \in \mathrm{Aut}_b(\wedge^*(M, \mathcal{S}))$ is defined through:

$$\star_{g,\mathcal{J}} \stackrel{\mathrm{def.}}{=} *_g \otimes \mathcal{J}.$$

The polarized Hodge operator restricts to an involution of $\wedge^2(M, \mathcal{S})$.

Definition

The space of **field strength configurations** of $\Delta = (\mathcal{S}, \omega, \mathcal{D})$ is:

$$\mathrm{Conf}(M, \Delta) \stackrel{\mathrm{def.}}{=} \Omega^2_{\mathrm{d}_{\mathcal{D}}\text{-cl}}(M, \mathcal{S}) = \left\{ \mathcal{V} \in \Omega^2(M, \mathcal{S}) | \mathrm{d}_{\mathcal{D}} \mathcal{V} = 0 \right\} .$$

The **Maxwell equation** defined by $\Xi = (\Delta, \mathcal{J})$ is:

$$\star_{g,\mathcal{J}} \mathcal{V} = \mathcal{V} \qquad (\mathrm{where} \;\; \mathcal{V} \in \mathrm{Conf}(M, \Delta)) \;\; .$$

The solutions are called **classical field strengths** and form the vector space:

$$\mathrm{Sol}(M, g, \Xi) = \mathrm{Sol}(M, g, \Delta, \mathcal{J}) \stackrel{\mathrm{def.}}{=} \{ \mathcal{V} \in \mathrm{Conf}(M, \Delta) \,|\, \star_{g,\mathcal{J}} \mathcal{V} = \mathcal{V} \} .$$

Definition

An **integral symplectic space** is a triple (V, ω, Λ) such that:

- (V, ω) is a finite-dimensional symplectic vector space over $\mathbb{R}$.
- $\Lambda \subset V$ is full lattice in V, i.e. a lattice in V such that $V = \Lambda \otimes_{\mathbb{Z}} \mathbb{R}$.
- ω is integral with respect to Λ, i.e. we have $\omega(\Lambda, \Lambda) \subset \mathbb{Z}$.

Define:

$$\mathrm{Div}^n \stackrel{\text{def.}}{=} \{(t_1, \dots, t_n) \in \mathbb{Z}^n_{>0} \mid t_1 | t_2 | \dots | t_n\} \ .$$

Definition

An **integral symplectic basis** of a $2n$-dimensional integral symplectic space (V, ω, Λ) is a basis $\mathcal{E} = (\xi_1, \dots, \xi_n, \zeta_1, \dots, \zeta_n)$ of Λ such that:

$$\omega(\xi_i, \xi_j) = \omega(\zeta_i, \zeta_j) = 0 \,, \qquad \omega(\xi_i, \zeta_j) = -t_i \delta_{ij} \ , \ \ \forall \ i, j = 1, \dots, n \,,$$

where $t \in \mathrm{Div}^n$.

Every integral symplectic space admits integral symplectic bases. The **type** $\mathfrak{t}(V, \omega, \Lambda)$ does not depend on the choice of such a basis. (V, ω, Λ) is called **principal** if:

$$\mathfrak{t}(V, \omega, \Lambda) = \delta(n) \stackrel{\text{def.}}{=} (1, \dots, 1) \in \mathrm{Div}^n \ .$$

Modified Siegel modular groups

Proposition

The type gives a bijection between the set of isomorphism classes of integral symplectic spaces and the set Div^n.

For any $\mathfrak{t} \in \mathrm{Div}^n$, let $\Lambda_\mathfrak{t} \subseteq \mathbb{R}^{2n}$ be the full lattice:

$$\Lambda_\mathfrak{t} \stackrel{\text{def.}}{=} \mathbb{Z}^n \oplus (\oplus_{i=1}^n t_i \mathbb{Z})$$

Then $(\mathbb{R}^{2n}, \omega_{2n}, \Lambda_\mathfrak{t})$ is the **standard integral symplectic space of type** $\mathfrak{t}$. Let:

$$\mathrm{Sp}(V, \omega, \Lambda) \stackrel{\text{def.}}{=} \{T \in \mathrm{Sp}(V, \omega) \mid T(\Lambda) = \Lambda\} \ .$$

Definition

The **modified Siegel modular group** of type $\mathfrak{t} \in \mathrm{Div}^n$ is:

$$\mathrm{Sp}_\mathfrak{t}(2n, \mathbb{Z}) \stackrel{\text{def.}}{=} \mathrm{Sp}(\mathbb{R}^{2n}, \omega_{2n}, \Lambda_\mathfrak{t}) \subset \mathrm{Sp}(2n, \mathbb{R})$$

Div^n is a lattice with bottom $\delta(n)$ under the partial order:

$$(t_1, \ldots, t_n) \leq (t'_1, \ldots, t'_n) \ \text{ iff } \ t_i | t'_i \ \ \forall i = 1, \ldots, n \ .$$

We have $\mathrm{Sp}_{\delta(n)}(2n, \mathbb{Z}) = \mathrm{Sp}(2n, \mathbb{Z})$ and $\mathrm{Sp}_\mathfrak{t}(2n, \mathbb{Z}) \subseteq \mathrm{Sp}_{\mathfrak{t}'}(2n, \mathbb{Z})$ when $\mathfrak{t} \leq \mathfrak{t}'$. Hence $(\mathrm{Sp}_\mathfrak{t}(2n, \mathbb{Z}))_{t \in \mathrm{Div}^n}$ is a direct system of overgroups of $\mathrm{Sp}(2n, \mathbb{Z})$.

Definition

An **affine torus** is a principal homogeneous space $\mathfrak{A}$ for a torus group A. The **standard affine d-torus** is the affine torus $\mathfrak{A}_d$ given by the right action of $\mathrm{U}(1)^d$ on itself. The **toral affine group** Aff_d is the group $\mathrm{Aff}(\mathfrak{A}_d)$.

Definition

A **special affine symplectic torus** is a pair $\mathbb{A} = (\mathfrak{A}, \Omega)$, where $\mathfrak{A}$ is an even-dimensional affine torus and Ω is a translation-invariant symplectic form on $\mathfrak{A}$ which has integer periods, i.e. such that $(H_1(\mathfrak{A}, \mathbb{R}), H_1(\mathfrak{A}, \mathbb{Z}), \omega)$ is an integral symplectic space, where $\omega = [\Omega] \in H^2(\mathfrak{A}, \mathbb{R}) \simeq \wedge^2 H_1(\mathfrak{A}, \mathbb{R})^\vee$.

Let $\Omega_\mathfrak{t}$ be the symplectic form induced by ω_{2n} on the torus group $\mathbb{R}^{2n}/\Lambda_\mathfrak{t}$.

Definition

The **standard special symplectic torus group** of type $\mathfrak{t} \in \mathrm{Div}^n$ is:

$$\mathrm{A}_\mathfrak{t} \stackrel{\mathrm{def.}}{=} \left(\mathbb{R}^{2n}/\Lambda_\mathfrak{t}, \Omega_\mathfrak{t}\right) \quad .$$

The **standard special affine symplectic torus** $\mathbb{A}_\mathfrak{t}$ is obtained from $\mathrm{A}_\mathfrak{t}$ by forgetting the origin.

Proposition

Any special affine symplectic torus $\mathbb{A} = (\mathfrak{A}, \Omega)$ *is affinely symplectomorphic to a unique standard special affine symplectic torus* $\mathbb{A}_\mathfrak{t}$*, whose type* $\mathfrak{t}$ *is called the* **type** *of* $\mathbb{A}$*.*

Definition

The **standard special toral affine group** $\mathrm{Aff}_\mathfrak{t}$ of type $\mathfrak{t} \in \mathrm{Div}^n$ is:

$$\mathrm{Aff}_\mathfrak{t} \stackrel{\mathrm{def.}}{=} \mathrm{Aut}(\mathbb{A}_\mathfrak{t}) \quad .$$

For any $\mathfrak{t} \in \mathrm{Div}^n$, we have $\mathrm{Aff}_\mathfrak{t} = A_n \rtimes \mathrm{Sp}_\mathfrak{t}(2n, \mathbb{Z})$, where $A_n \simeq \mathrm{U}(1)^{2n}$ is the underlying torus group of $\mathbb{A}_\mathfrak{t}$.

Let $\Delta = (\mathcal{S}, \omega, \mathcal{D})$ be a duality structure defined on M.

Definition

A **Dirac system** for Δ is a smooth fiber sub-bundle $\mathcal{L} \subset \mathcal{S}$ of full symplectic lattices in $(\mathcal{S}, \omega)$ which is preserved by the parallel transport of $\mathcal{D}$. A pair $\boldsymbol{\Delta} \stackrel{\mathrm{def.}}{=} (\Delta, \mathcal{L})$ consisting of a duality structure Δ and a choice of Dirac system $\mathcal{L}$ for Δ is called an **integral duality structure**. A duality structure Δ of rank $2n$ is called **semiclassical** if it admits a Dirac system, i.e. if its holonomy group can be conjugated to lie inside $\mathrm{Sp}_{\mathfrak{t}}(2n, \mathbb{Z})$ for some $\mathfrak{t} \in \mathrm{Div}^n$.

Definition

The **type** $\mathfrak{t}_{\boldsymbol{\Delta}} \in \mathrm{Div}^n$ of an integral duality structure $\boldsymbol{\Delta} = (\mathcal{S}, \omega, \mathcal{D}, \mathcal{L})$ is the common type of the integral symplectic spaces $(\mathcal{S}_m, \omega_m, \mathcal{L}_m)$, where $m \in M$.

Proposition

For any $m \in M$, the set of isomorphism classes of integral duality structures of type $\mathfrak{t}$ defined on M is in bijection with the character variety:

$$\mathfrak{R}_{\mathfrak{t}}(\pi_1(M, m), \mathrm{Sp}_{\mathfrak{t}}(2n, \mathbb{Z})) \stackrel{\mathrm{def.}}{=} \mathrm{Hom}(\pi_1(M, m), \mathrm{Sp}_{\mathfrak{t}}(2n, \mathbb{Z}))/\mathrm{Sp}_{\mathfrak{t}}(2n, \mathbb{Z}) \quad .$$

Definition

A **Siegel system** of rank $2n$ is a local system of free Abelian groups of rank $2n$ defined on M whose structure group reduces to a subgroup of $\mathrm{Sp}_{\mathfrak{t}}(2n,\mathbb{Z})$ for some $\mathfrak{t} \in \mathrm{Div}^n$. The smallest $\mathfrak{t}$ with this property is called the **type** $\mathfrak{t}_Z$ of Z.

Proposition

Let Z be a bundle of free Abelian groups of rank $2n$ defined on M. T.a.e.:

- (a) *Z is a Siegel system of type $\mathfrak{t}$ defined on M.*
- (b) *The vector bundle $\mathcal{S} \stackrel{\mathrm{def.}}{=} Z \otimes_{\mathbb{Z}} \mathbb{R}$ admits a symplectic pairing ω which makes $(\mathcal{S}, \omega, \mathcal{D}, Z)$ into an integral duality structure of type $\mathfrak{t}$, where $\mathcal{D}$ is the flat connection induced from Z.*

Let $\ell_{\mathfrak{t}} : \mathrm{Sp}_{\mathfrak{t}}(2n,\mathbb{Z}) \to \mathrm{Aut}_{\mathbb{Z}}(\mathbb{Z}^{2n})$ be the left action of $\mathrm{Sp}_{\mathfrak{t}}(2n,\mathbb{Z})$ on $\mathbb{Z}^{2n}$. For any principal $\mathrm{Sp}_{\mathfrak{t}}(2n,\mathbb{Z})$-bundle Q defined on M, let $Z(Q) \stackrel{\mathrm{def.}}{=} Q \times_{\ell_{\mathfrak{t}}} \mathbb{Z}^{2n}$. For any Siegel system Z, let $\mathrm{Fr}(Z)$ be its principal bundle of frames.

Proposition

The correspondences $Q \mapsto Z(Q)$ and $Z \mapsto \mathrm{Fr}(Z)$ extend to quasi-inverse equivalences between the groupoids of principal $\mathrm{Sp}_{\mathfrak{t}}(2n,\mathbb{Z})$-bundles and Siegel systems of type $\mathfrak{t}$ defined on M.

Let $\mathbf{\Delta} = (\Delta, Z)$ be an integral duality structure on M, where $\Delta = (\mathcal{S}, \omega, \mathcal{D})$. Consider the flat bundle of torus groups $\mathcal{A} \stackrel{\text{def.}}{=} \mathcal{S}/Z$. The exact sequence of sheaves of Abelian groups:

$$1 \to \mathcal{C}(Z) \xrightarrow{j} \mathcal{C}^{\infty}_{\text{flat}}(\mathcal{S}) \xrightarrow{\exp} \mathcal{C}^{\infty}_{\text{flat}}(\mathcal{A}) \to 1$$

induces a long exact sequence in sheaf cohomology:

$$\ldots \to H^1(M, \mathcal{A}_{\text{disc}}) \xrightarrow{\delta_1} H^2(M, Z) \xrightarrow{j_*} H^2_{\mathcal{D}}(M, \mathcal{S}) \xrightarrow{\exp_*} H^2(M, \mathcal{A}_{\text{disc}}) \to \ldots$$

where δ_1 is the connecting morphism.

Definition

The **lattice of charges** of the integral duality structure $\mathbf{\Delta}$ is:

$$L_{\mathbf{\Delta}} \stackrel{\text{def.}}{=} j_*(H^2(M, Z)) \subset H^2_{\mathcal{D}}(M, \mathcal{S})$$

A field strength configuration $\mathcal{V} \in \mathrm{Conf}(M, \Delta) = \Omega^2_{\mathrm{d}_{\mathcal{D}}\text{-cl}}(M, \mathcal{S})$ is called **integral** if $[\mathcal{V}]_{\mathcal{D}} \in L_{\mathbf{\Delta}}$.

The condition $[\mathcal{V}]_{\mathcal{D}} \in L_{\mathbf{\Delta}}$ is the twisted Dirac integrality condition.

Definition

A **Siegel bundle** P of type $\mathfrak{t} \in \mathrm{Div}^n$ is a principal bundle on M with structure group $\mathrm{Aff}_\mathfrak{t}$.

Notice that $G \stackrel{\mathrm{def.}}{=} \mathrm{Aff}_\mathfrak{t}$ is a split weakly Abelian Lie group:

$$G \simeq A \rtimes_{\rho_\mathfrak{t}} \Gamma_\mathfrak{t} \ \text{ where } \ A = \mathbb{R}^{2n}/\Lambda_\mathfrak{t} \simeq \mathrm{U}(1)^{2n} \ \text{ and } \ \Gamma_\mathfrak{t} = \mathrm{Sp}_\mathfrak{t}(2n, \mathbb{Z}) \ ,$$

with fundamental lattice $\Lambda_\mathfrak{t} \simeq \pi_1(G) \simeq \mathbb{Z}^{2n}$. By our general results on weakly Abelian principal bundles, Siegel bundles of type $\mathfrak{t}$ are classified by their remnant bundle $\Gamma_\mathfrak{t}(P)$ and their **twisted Chern class** $c(P) \in H^2(M, \Lambda_\mathfrak{t}(P))$. In this case:

- The local system $\Lambda_\mathfrak{t}(P) \stackrel{\mathrm{def.}}{=} P \times_{\mathrm{Ad}_0} \Lambda_\mathfrak{t}$ is a Siegel system of type $\mathfrak{t}$, which we denote by $Z(P)$.
- $\mathfrak{g} = \mathbb{R}^{2n}$ and $\mathrm{ad}(P) = Z(P) \otimes_\mathbb{Z} \underline{\mathbb{R}} \stackrel{\mathrm{def.}}{=} \mathcal{S}$ supports the duality structure $(\mathcal{S}, \omega, \mathcal{D})$, induced by $Z(P)$.
- $A(P) = P \rtimes_{\mathrm{Ad}_G^A} A$ coincides with the bundle of symplectic torus groups $\mathcal{A}(P) \stackrel{\mathrm{def.}}{=} \mathcal{S}/Z(P)$.
- The characteristic lattice $L(P) = L_0(P) \in H^2(M, \mathrm{ad}(P)) = H^2(M, \mathcal{S})$ of P coincides with the lattice of charges $L_{\boldsymbol{\Delta}_P}$ of $\boldsymbol{\Delta}_P \stackrel{\mathrm{def.}}{=} (\mathcal{S}, \omega, \mathcal{D}, Z(P))$.

Let $\Delta_P \stackrel{\mathrm{def.}}{=} (\mathcal{S}, \omega, \mathcal{D})$.

Theorem

Consider the set:

$$\Sigma(M) \stackrel{\mathrm{def.}}{=} \Big\{(Z,c)\,|\,Z \ \text{ is a Siegel system on } M \ \& \ c \in H^2(M,Z)\Big\}/\sim,$$

where $(Z,c) \sim (Z',c')$ *iff there exists an isomorphism* $\varphi : Z \xrightarrow{\sim} Z'$ *s.t.* $\varphi_*(c) = c'$. *Then the map* $P \mapsto (Z(P), c(P))$ *induces a bijection between the set of isomorphism classes of Siegel bundles defined on* M *and the set* $\Sigma(M)$.

Let $\mathfrak{c}(P) \in H^2_{\mathcal{D}}(M, \mathrm{ad}(P)) = H^2_{\mathcal{D}}(M, \mathcal{S})$ be the twisted **real** Chern class of P.

Proposition

We have $\mathfrak{c}(P) = j_*(c(P))$, *where* $j_* : H^*(M,Z) \to H^*(M,\mathcal{S})$ *is induced by the inclusion* $Z = \Lambda_{\mathfrak{t}}(P) \hookrightarrow \mathcal{S} = \mathrm{ad}(P)$.

Since $\mathfrak{c}(P) \in L(P) = L_{\boldsymbol{\Delta}}$, it follows that the adjoint curvature $\mathcal{V}_{\mathcal{A}}$ of any principal connection $\mathcal{A} \in \mathrm{Conn}(P)$ satisfies the Dirac integrality condition $[\mathcal{V}]_{\mathcal{D}} \in L_{\boldsymbol{\Delta}}$ of the integral duality structure $\boldsymbol{\Delta}_P$ determined by P.

Theorem

Let Z be a Siegel system on M and $\mathbf{\Delta} = (\Delta, Z)$ be its integral duality structure, where $\Delta = (\mathcal{S}, \omega, \mathcal{D})$. For any $\mathfrak{c} \in L_{\mathbf{\Delta}}$, there exists a Siegel bundle P on M s.t. $\mathbf{\Delta}_P = \mathbf{\Delta}$ and $\mathfrak{c} = \mathfrak{c}(P)$. Thus any $\mathcal{V} \in \mathrm{Conf}(M, \Delta) = \Omega^2_{\mathrm{d}_{\mathcal{D}}\text{-cl}}(M, \mathcal{S})$ which satisfies the Dirac integrality condition $[\mathcal{V}]_{\mathcal{D}} \in L_{\mathbf{\Delta}}$ coincides with the adjoint curvature $\mathcal{V}_{\mathcal{A}}$ of some principal connection $\mathcal{A} \in \mathrm{Conn}(P)$.

Definition

A **polarized Siegel bundle** is a pair $\mathrm{P} = (P, \mathcal{J})$, where P is a Siegel bundle and $\mathcal{J}$ is a taming of the duality structure Δ_P defined by P.

Definition

Let $\mathrm{P} = (P, \mathcal{J})$ be a polarized Siegel bundle on M. The space of **gauge configurations** defined by P on M is the affine space:

$$\mathfrak{Conf}(M, P) \stackrel{\mathrm{def.}}{=} \mathrm{Conn}(P) \quad .$$

The **quasi-Abelian gauge theory** defined by P on (M, g) has e.o.m:

$$\star_{g,\mathcal{J}} \mathcal{V}_{\mathcal{A}} = \mathcal{V}_{\mathcal{A}} \quad \text{where} \quad \mathcal{A} \in \mathrm{Conn}(P)\,.$$

Let M be an oriented and connected 4-manifold, $(\pi, \mathcal{H}, \mathcal{G})$ a scalar bundle, P a Siegel bundle of type $\mathfrak{t} \in \mathrm{Div}^{n_v}$ on X. Let $\mathbf{\Delta}(P) = (\mathcal{S}, \omega, Z, \mathcal{D})$ be the integral duality structure determined by P. Given a vertical taming $\mathcal{J}$ of $\mathbf{\Delta}(P)$, the pair $(P_{\mathfrak{t}}, \mathcal{J})$ is called a **polarized Siegel bundle**. Let:

$$\Xi(P, \mathcal{J}) \stackrel{\text{def.}}{=} (\mathbf{\Delta}(P), \mathcal{J})$$

be the integral electromagnetic bundle determined by $(P, \mathcal{J})$. Given $s \in \Gamma(\pi)$, the pullback of P^s is a Siegel bundle over M. Let $\mathbf{\Delta}(P^s_{\mathfrak{t}})$ and $\Xi(P^s, \mathcal{J}^s)$ be the associated integral duality and electromagnetic bundles, which coincide with the s-pullbacks of the corresponding bundles defined by $(P, \mathcal{J})$ on X. The adjoint curvature of a connection $\mathcal{A} \in \mathrm{Conn}(P^s_{\mathfrak{t}})$ is denoted by $\mathcal{F}_{\mathcal{A}} \in \Omega^2(M, \mathcal{S}^s)$ and is $\mathrm{d}_{\mathcal{D}^s}$-closed by the Bianchi identity, because all connections on P^s induce the same connection on $\mathcal{S}^s$, which coincides with $\mathcal{D}^s$:

$$\mathrm{d}^s_{\mathcal{D}} \mathcal{F}_{\mathcal{A}} = 0\,.$$

Definition

A **scalar-Siegel bundle** of type $\mathfrak{t} \in \mathrm{Div}^{n_v}$ over M is a system $\zeta \stackrel{\text{def.}}{=} (\pi, \mathcal{H}, \mathcal{G}, P)$, where $(\pi, \mathcal{H}, \mathcal{G})$ is a scalar bundle over M with submersion $\pi : X \to M$ and P is a Siegel bundle of type $\mathfrak{t} \in \mathrm{Div}^{n_v}$ defined on X. Given a vertical taming $\mathcal{J}$ of $\Delta(P)$, the system $\boldsymbol{\zeta} \stackrel{\text{def.}}{=} (\zeta, \mathcal{J})$ is called a **polarized scalar-Siegel bundle** of type $\mathfrak{t}$ over M.

Definition

Let $\zeta \stackrel{\mathrm{def.}}{=} (\pi, \mathcal{H}, \mathcal{G}, P, \mathcal{J})$ be a polarized scalar-Siegel bundle of type $\mathfrak{t}$ over M. The **semiclassical configuration space** of the bosonic supergravity defined by ζ is the set:

$$\mathfrak{Conf}(\zeta) = \{(g, s, \mathcal{A}) \mid g \in \mathrm{Lor}(M)\,,\; s \in \Gamma(\pi)\,,\; \mathcal{A} \in \mathrm{Conn}(P^s)\}\ .$$

The theory is defined by the following eom for $(g, s, \mathcal{A}) \in \mathfrak{Conf}(\zeta)$:

- The Einstein equations:

$$\mathrm{Ric}^g - \frac{g}{2}\mathrm{R}^g = \frac{1}{2}\mathrm{Tr}_g(s_v^*\mathcal{G})\, g - s_v^*\mathcal{G} + 2\mathcal{F}_{\mathcal{A}} \oslash_{Q^s} \mathcal{F}_{\mathcal{A}}\ .$$

- The scalar equations:

$$\mathrm{Tr}_g(\nabla^v \mathrm{d}^v s) = \frac{1}{2}(*\mathcal{F}_{\mathcal{A}}, \Psi^s \mathcal{F}_{\mathcal{A}})_{g,Q^s}\ .$$

- The Maxwell equations:

$$\star_{g,\mathcal{J}^s} \mathcal{F}_{\mathcal{A}} = \mathcal{F}_{\mathcal{A}}\ ,$$

Globally-defined solutions of the theory are semiclassical locally geometric U-folds.