From differential crossed modules to tensor hierarchies

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(with Jim Stasheff, arXiv:2003.07838)

Gauging procedures (Supergravity, Exceptional field theory)

Classical gauge theories : gauge fields A_μ take values in the adjoint representation of the gauge Lie algebra $\mathfrak{g}$.

Supergravity theories : they take values in the defining/fundamental representation V of the Lie algebra of global symmetries $\mathfrak{g}$.

The other fields ϕ live in a representation of $\mathfrak{g}$. Gauge fields act on them via an embedding tensor $\Theta : V \longrightarrow \mathfrak{g}$ so that

$$A_\mu \circlearrowright \phi = \Theta(A_\mu) \cdot \phi$$

Two consequences :

1. The gauge algebra is *not* $\mathfrak{g}$ but the Lie subalgebra $\mathfrak{h} = \mathrm{Im}(\Theta)$
2. V can be equipped with a Leibniz algebra structure

Gauging procedures (Supergravity, Exceptional field theory)

The field strength takes values in V and now reads $F = dA + \frac{1}{2} A \circ A$
$\Longrightarrow$ the *Palatini identity* does not hold anymore : $\delta F \neq D(\delta A)$ but :

$$\delta F = D(\delta A) + \text{symm. tensor in } A, \delta A$$

Then, physicists add :

1. a 2-form $B \in \Omega^2(M, W)$ taking values in a $\mathfrak{g}$-**module** W,
2. a linear map $\partial_{-1} : W \longrightarrow V$, such that :

$$F' = F + \partial_{-1}(B) \quad \text{satisfies} \quad \delta F' = D(\delta A)$$

$\Longrightarrow$ its corresponding field strength H is a 3-form taking values in W. But then H does not transform 'covariantly', i.e. $\delta H \neq D(\delta B + A \wedge \delta A)$.

$\Longrightarrow$ add a 3-form $C \in \Omega^3(M, X)$ taking values in a $\mathfrak{g}$-**module** X compensating the discrepancy to covariance.

$$\begin{array}{c} X \\ \downarrow \partial_{-2} \\ W \\ \downarrow \partial_{-1} \\ V \\ \downarrow \Theta \\ \mathfrak{h} \subset \mathfrak{g} \end{array}$$

Gauging procedures (Supergravity, Exceptional field theory)

We end up with what physicists call a tensor hierarchy, i.e. a tower of $\mathfrak{g}$-modules forming a **chain complex** (not a resolution !), in which the successive p-forms gauge fields $A, B, C, D, \ldots$ take values :

$$\ldots \xrightarrow{\partial_{-4}} Y \xrightarrow{\partial_{-3}} X \xrightarrow{\partial_{-2}} W \xrightarrow{\partial_{-1}} V \xrightarrow{\Theta} \mathfrak{h} \subset \mathfrak{g}.$$

Observation : this chain complex possesses a canonical differential graded Lie algebra (dgLa) structure containing every physical relevant information (Greitz et al., 2014 ; Bonezzi & Hohm, 2020, 2021).

Objective of the talk : build a canonical assignment (functor)

$$\{\text{embedding tensors}\} \longrightarrow \{\text{dgLa}\}$$

Strategy : building the tower of spaces step by step after making a wise and justified choice for the first step. This construction has been widely influenced by (Greitz et al., 2014 ; Cederwall & Palmkvist, 2015).

Content and main results

An embedding tensor is a **generalization** of a differential (Lie algebra) crossed module $\mathfrak{g}_{-1} \xrightarrow{\partial} \mathfrak{g}_0$.

A differential crossed module is equivalently a dgLa $(\mathfrak{g}_0 \oplus \mathfrak{g}_{-1}, \partial, [\,.\,,.\,])$.

Theorem (sloppy formulation)

There is an injective on objects function :

$$\{\text{embedding tensors}\} \longrightarrow \{\text{dgLa}\}$$

1. it is a faithful functor on a *wide* subcategory of $\{\text{embed. tensors}\}$
2. restricts to the canonical correspondence on the subcategory of differential crossed modules :

$$\mathfrak{g}_{-1} \xrightarrow{\partial} \mathfrak{g}_0 \longrightarrow (\mathfrak{g}_0 \oplus \mathfrak{g}_{-1}, \partial, [\,.\,,.\,])$$

Leibniz algebras

Definition (Loday-Pirashvili)

A (left) Leibniz algebra is a vector space V together with a bilinear operation $\circ : V \otimes V \longrightarrow V$ satisfying the relation :

$$x \circ (y \circ z) = (x \circ y) \circ z + y \circ (x \circ z).$$

A *morphism* of Leibniz algebra $\chi : V \longrightarrow V'$ is a linear map satisfying :

$$\chi(x \circ y) = \chi(x) \circ' \chi(y)$$

Example 1. Lie algebras : the Leibniz product is skew symmetric.

Example 2. Generalized diffeomorphisms in double and exceptional geometry.

What is an embedding tensor ?

For V Leibniz, the adjoint map $\mathrm{ad} : V \longrightarrow \mathrm{End}(V), x \longmapsto x \circ -$ is :

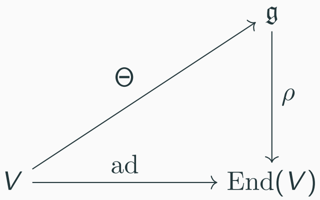
- a derivation of $\circ$:

$$\mathrm{ad}_x(y \circ z) = \mathrm{ad}_x(y) \circ z + y \circ \mathrm{ad}_x(z))$$

- a morphism of Leibniz algebras

$$\mathrm{ad}_{x \circ y} = [\mathrm{ad}_x, \mathrm{ad}_y]$$

Assume that V is a representation of a Lie algebra $\mathfrak{g}$ via $\rho : \mathfrak{g} \to \mathrm{End}(V)$.
An embedding tensor is a **lift** of ad :



What is an embedding tensor ?

Definition

A Lie-Leibniz triple is a triple $(\mathfrak{g}, V, \Theta)$ where :

1. $\mathfrak{g}$ is a Lie algebra,
2. V is a $\mathfrak{g}$-**module**,
3. $\Theta : V \longrightarrow \mathfrak{g}$ is a linear map called the embedding tensor, satisfying the **quadratic constraint** :

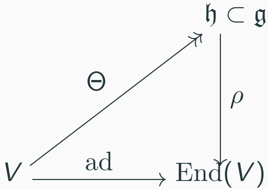
$$\Theta(\Theta(x) \cdot y) = [\Theta(x), \Theta(y)]_{\mathfrak{g}}$$

Consequences :

- V is a Leibniz algebra :

$$x \circ y = \Theta(x) \cdot y$$

- $\mathfrak{h} = \mathrm{Im}(\Theta)$ is a **Lie subalgebra of** $\mathfrak{g}$
- Θ is $\mathfrak{h}$-**equivariant**



Embedding tensors vs differential crossed modules

Example 1 : Let V be a Leibniz algebra, then $\big(\mathrm{End}(V), V, \mathrm{ad}\big)$ and $\big(\mathrm{Der}(V), V, \mathrm{ad}\big)$ are Lie-Leibniz triples. In the latter, Θ is $\mathfrak{g}$-*equivariant* :

$$\delta(\mathrm{ad}_x)(y) = \underbrace{\delta(\mathrm{ad}_x(y))}_{\delta(x)\circ y + x\circ(\delta(y))} - \mathrm{ad}_x(\delta(y)) = \mathrm{ad}_{\delta(x)}(y)$$

Example 2 : A differential crossed module $\mathfrak{g}_{-1} \xrightarrow{\Theta} \mathfrak{g}_0$ consists of the following data :

1. a Lie algebra $\mathfrak{g}_0$,
2. a $\mathfrak{g}_0$-module $\mathfrak{g}_{-1}$, and
3. a $\mathfrak{g}_0$-equivariant linear map $\Theta : \mathfrak{g}_{-1} \longrightarrow \mathfrak{g}_0$ satisfying :

$$\Theta(x) \cdot y = -\Theta(y) \cdot x$$

Proposition

Differential crossed modules are Lie-Leibniz triples $(\mathfrak{g}, V, \Theta)$ for which **a)** the Leibniz algebra V is a **Lie algebra**, and **b)** Θ is $\mathfrak{g}$-**equivariant**.

In categorical language ?

The category of Lie-Leibniz triples contains that of differential crossed modules :

$$\mathbf{diff{\times}mod} \subset \mathbf{LieLeib}$$

A differential crossed module $\mathfrak{g}_{-1} \xrightarrow{\Theta} \mathfrak{g}_0$ is, equivalently, a 2-term differential graded Lie algebra $(\mathfrak{g}_0 \oplus \mathfrak{g}_{-1}, \partial, [\,.\,,.\,])$:

- $[\,.\,,.\,]\big|_{\mathfrak{g}_0 \wedge \mathfrak{g}_0} = [\,.\,,.\,]_{\mathfrak{g}_0}$,
- $[\,.\,,.\,]\big|_{\mathfrak{g}_{-1} \wedge \mathfrak{g}_{-1}} = 0$,
- $[\,.\,,.\,]\big|_{\mathfrak{g}_0 \wedge \mathfrak{g}_{-1}} =$ action of $\mathfrak{g}_0$ on $\mathfrak{g}_{-1}$,
- $\partial = \Theta$

Questions

1) Does the faithful functor $\mathbf{diff{\times}mod} \to \mathbf{DGLie}$ *canonically* extends to a functor $\mathbf{LieLeib} \to \mathbf{DGLie}$?

2) If yes, what is the image of $\mathbf{LieLeib} \to \mathbf{DGLie}$?

Precise statements

Questions

1) Does the faithful functor **diff**$\times$**mod** $\to$ **DGLie** *canonically* extends to a functor **LieLeib** $\to$ **DGLie** ?

2) If yes, what is the image of **LieLeib** $\to$ **DGLie** ?

1) Yes, if one associates to $\mathfrak{g}_{-1} \xrightarrow{\Theta} \mathfrak{g}_0$ the differential graded Lie algebra :

$$\mathfrak{g}_{-1} \xrightarrow{\Theta} \mathfrak{g}_0 \xrightarrow{0} \mathbb{R} \tag{1}$$

where $\mathbb{R}$ is the **trivial $\mathfrak{g}_0$-module generated by Θ in** $\mathrm{Hom}(\mathfrak{g}_{-1}, \mathfrak{g}_0)$.
From a Lie-Leibniz triple $V \xrightarrow{\Theta} \mathfrak{g}$ we can build a dgLa :

$$\ldots \xrightarrow{\partial_{-3}} T_{-3} \xrightarrow{\partial_{-2}} T_{-2} \xrightarrow{\partial_{-1}} V \xrightarrow{\Theta} \mathfrak{g} \longrightarrow R_{\Theta}$$

which reduces to (1) if $V \xrightarrow{\Theta} \mathfrak{g}$ is a differential crossed module.

2) The resulting dgLa is the tensor hierarchy associated to Θ

Lie-Leibniz triples and dgLa

Θ generates a $\mathfrak{g}$-submodule of $\mathrm{Hom}(V,\mathfrak{g})$ by the successive action of $\mathfrak{g}$:

$$R_\Theta = \mathrm{Span}(\Theta, a_1 \cdot (a_2 \cdot (\ldots (a_n \cdot \Theta))) \mid a_1, a_2, \ldots, a_m \in \mathfrak{g})$$

object	corresponding dgLa
Differ. Crossed Module $\mathfrak{g}_{-1} \xrightarrow{\Theta} \mathfrak{g}_0$	$\mathfrak{g}_{-1} \xrightarrow{\Theta} \mathfrak{g}_0 \xrightarrow{0} \mathbb{R}$
$\mathfrak{g}_{-1} \xrightarrow{\Theta} \mathfrak{g}_0$ but Θ not $\mathfrak{g}_0$-equivariant	$\mathfrak{g}_{-1} \xrightarrow{\Theta} \mathfrak{g}_0 \xrightarrow{-\eta(-,\Theta)} \mathrm{R}_\Theta$
Lie-Leibniz triple $V \xrightarrow{\Theta} \mathfrak{g}$	$\ldots \xrightarrow{\partial_{-2}} T_{-2} \xrightarrow{\partial_{-1}} V \xrightarrow{\Theta} \mathfrak{g} \xrightarrow{-\eta(-,\Theta)} \mathrm{R}_\Theta$

Important !

The part of the chain complex of degree ≤ -2 is a direct consequence of the **symmetric part** of the Leibniz product.

Slogan : the tensor hierarchy is the **Lie-fication** of an embedding tensor

Properties of Leibniz algebras

The Leibniz product $\circ$ can be split in two parts :

$$[x,y] = \frac{1}{2}\big(x \circ y - y \circ x\big) \qquad \text{and} \qquad \{x,y\} = \frac{1}{2}\big(x \circ y + y \circ x\big)$$

so that :

$$x \circ y = [x,y] + \{x,y\}$$

The skew-symmetric bracket $[\,.\,,.\,]$ does not satisfy the Jacobi identity :

$$\big[x,[y,z]\big]+\big[y,[z,x]\big]+\big[z,[x,y]\big] = -\frac{1}{3}\Big(\big\{x,[y,z]\big\}+\big\{y,[z,x]\big\}+\big\{z,[x,y]\big\}\Big)$$

Let the ideal of squares be $\mathcal{I} = \mathrm{Span}\big(x \circ x = \{x,x\} \;\big|\; x \in V\big)$

The properties of the embedding tensor implies the inclusion :

$$\mathcal{I} \underbrace{\subset}_{(a)} \mathrm{Ker}(\Theta) \underbrace{\subset}_{(b)} \mathrm{Ker}(\mathrm{ad})$$

(a) $\Theta(x \circ y) = [\Theta(x), \Theta(y)]_{\mathfrak{g}}$ (b) $\mathrm{ad}_x(y) = \Theta(x) \cdot y$

Construction of T_{-2}

In supergravity, one notices that the ideal of squares $\mathcal{I}$ is generated by the 2-forms B. More precisely :

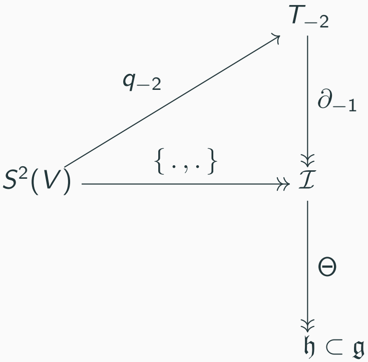
- gauge fields 1-forms A span $T_{-1} = V$
- gauge fields 2-forms B span some $\mathfrak{g}$-**module** T_{-2} such that $\partial_{-1} : T_{-2} \longrightarrow V$ is onto $\mathcal{I}$

♦ if $\mathcal{I}$ is not a $\mathfrak{g}$-module $\Longrightarrow T_{-2} \neq \mathcal{I}$
♦ Idea : factorize $\{\,.\,,.\,\} : S^2 V \twoheadrightarrow \mathcal{I}$
♦ $\mathrm{Ker}\big(\{\,.\,,.\,\}\big) \subset S^2(V)$ is a $\mathfrak{h}$-module but **may *not* be** a $\mathfrak{g}$-module.

First step of the construction

We define K_{-2} to be the **biggest** $\mathfrak{g}$-sub-module of $\mathrm{Ker}\big(\{\,.\,,.\,\}\big)$, and

$$T_{-2} = S^2(V) \big/ K_{-2}$$



Construction of T_{-3}

Let $F_{-1} = V$ (in degree -1) and let $F_\bullet = \bigoplus_{i \geq 1} F_{-i}$ be the free graded Lie algebra generated by F_{-1}.

$$F_{-2} = \wedge^2(F_{-1}) \simeq S^2(V), \quad \wedge^3(F_{-1}) = S^3(V), \quad V \otimes S^2(V) \simeq \wedge^3(F_{-1}) \oplus F_{-3}$$

$$\begin{array}{ccccccc}
0 & \longrightarrow & V \otimes K_{-2} & \xrightarrow{\text{id}} & V \otimes K_{-2} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \\
S^3(V) & \longrightarrow & V \otimes S^2(V) & \longrightarrow & F_{-3} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \text{id} \otimes q_{-2}} & & \downarrow & & \\
S^3(V) & \longrightarrow & V \otimes T_{-2} & \xrightarrow{q_{-3}} & F_{-3}/K_{-3} & \longrightarrow & 0
\end{array}$$

Exactness of the second row implies exactness of the third, so we set

$$T_{-3} = F_{-3}/K_{-3}$$

Construction of $T_{-(n+1)}$

Suppose that all the $T_{-i} = {F_{-i}}\big/{K_{-i}}$ have been built up to order n, where $K_{-i} \subset F_{-i}$ is a $\mathfrak{g}$-submodule.

$$
\begin{array}{ccccccc}
0 & \longrightarrow & \bigoplus_{j=1}^{n-1} F_{-j} \otimes K_{-(n+1)+j} & \xrightarrow{\mathrm{id}} & \bigoplus_{j=1}^{n-1} F_{-j} \otimes K_{-(n+1)+j} & \longrightarrow & 0 \\
 & & \big\downarrow & & \vdots\downarrow & & \\
\Lambda^3(F_\bullet)\big|_{-(n+1)} & \longrightarrow & \Lambda^2(F_\bullet)\big|_{-(n+1)} & \longrightarrow & F_{-(n+1)} & \longrightarrow & 0 \\
 & & \big\downarrow & & \big\downarrow & & \\
\Lambda^3\big(F_\bullet\big)\big|_{-(n+1)} & \longrightarrow & \Lambda^2\big(\bigoplus_{i=1}^{n} T_{-i}\big)\big|_{-(n+1)} & \xrightarrow{q_{-(n+1)}} & {F_{-(n+1)}}\big/{K_{-(n+1)}} & \longrightarrow & 0
\end{array}
$$

We define $T_{-(n+1)} = {F_{-(n+1)}}\big/{K_{-(n+1)}}$.

The end of the hierarchy

Continuing the induction provides us with a (possibly infinite) **graded vector space** $T_\bullet = \bigoplus_{i=1}^{\infty} T_{-i}$, the quotient of $F_\bullet$ by the graded ideal $K_\bullet$.

It has the following properties :

1. Every vector space T_{-i} is a $\mathfrak{g}$-module ;
2. every map $q_{-i} : \wedge^2 T_\bullet|_{-i} \twoheadrightarrow T_{-i}$ is $\mathfrak{g}$-equivariant ;
3. $T_{-1} = V$ (in degree -1) ;
4. $T_{-i} = 0$ for every $i \geq 2$ **if and only if** V is a Lie algebra.

$T_\bullet$ can be equipped with a graded Lie algebra structure with bracket :

$$q = [\![\, . \, , . \,]\!] : \wedge^2 T_\bullet \longrightarrow T_\bullet$$

where $q|_{\wedge^2 T_\bullet|_{-i}}$ is the quotient map $q_{-i} : \wedge^2 T_\bullet|_{-i} \twoheadrightarrow T_{-i}$.

Graded Lie algebra structure

One can *extend* the graded Lie algebra structure to $\mathbb{T} = T_\bullet \oplus \mathfrak{g} \oplus R_\Theta$:

$$[\![a, x]\!] = a \cdot x \quad \text{and} \quad [\![a, b]\!] = [a, b]_\mathfrak{g}$$

for every $a, b \in \mathfrak{g}$ and $x \in T_\bullet$, and (among other brackets) :

$$[\![\Theta, u]\!] = \partial(u) \quad \text{and} \quad [\![R_\Theta, R_\Theta]\!] = 0$$

for every $u \in T_\bullet \oplus \mathfrak{g}$.

Proposition

There exists a *unique* differential ∂ on $\mathbb{T} = T_\bullet \oplus \mathfrak{g} \oplus R_\Theta$

$$\ldots \xrightarrow{\partial_{-3}} T_{-3} \xrightarrow{\partial_{-2}} T_{-2} \xrightarrow{\partial_{-1}} V \xrightarrow{\Theta} \mathfrak{g} \xrightarrow{-\eta(-,\Theta)} R_\Theta$$

such that **a)** $\partial\big([\![x, y]\!]\big) = 2\{x, y\}$ for all $x, y \in V$, and
b) the *graded Leibniz identity* is satisfied :

$$\partial\big([\![u, v]\!]\big) = [\![\partial(u), v]\!] + (-1)^{|u|}[\![u, \partial(v)]\!]$$

The tensor hierarchy of a Lie-Leibniz triple

Definition

We call tensor hierarchy associated to a Lie-Leibniz triple $(\mathfrak{g}, V, \Theta)$ the differential graded Lie algebra $(\mathbb{T}, [\![\,.\,,\,.\,]\!], \partial)$.

Fact : when the Lie-Leibniz triple is a differential crossed module $\mathfrak{g}_{-1} \xrightarrow{\Theta} \mathfrak{g}_0$, its associated *tensor hierarchy* is $\mathfrak{g}_{-1} \xrightarrow{\Theta} \mathfrak{g}_0 \xrightarrow{0} \mathbb{R}$.

Question : is it the same as in (Cederwall & Palmkvist, 2015) ?

Proposition

The function G : **Lie-Leib** $\longrightarrow$ **DGLie** associating to a Lie-Leibniz triple its tensor hierarchy is *injective on objects*.

Final results

A morphism between two Lie-Leibniz triples $(\mathfrak{g}, V, \Theta)$ and $(\mathfrak{g}', V', \Theta')$ is a *pair* (φ, χ) with $\varphi : \mathfrak{g} \to \mathfrak{g}'$ and $\chi : V \to V'$, such that $\Theta'\chi = \varphi\Theta$ and $\chi(a \cdot x) = \varphi(a) \cdot \chi(x)$.

The morphism (φ, χ) induces a morphism of dgLa between the tensor hierarchies $\mathbb{T}$ and $\mathbb{T}'$ **if and only if** $\exists\, \tau$ making the diagram commutative :

$$\begin{array}{ccc} S^2(V) & \xrightarrow{\ \chi \odot \chi\ } & S^2(V') \\ {\scriptstyle q_{-2}} \big\downarrow & & \big\downarrow {\scriptstyle q'_{-2}} \\ T_{-2} & \xrightarrow{\ \tau\ } & T'_{-2} \end{array}$$

The existence of the map $\tau : S^2(V) \big/ K_{-2} \to S^2(V') \big/ K'_{-2}$ is based on the condition that $\chi \odot \chi(K_{-2}) \subset K'_{-2}$.

Proposition

The restriction of the function G : **Lie-Leib** $\longrightarrow$ **DGLie** to the *wide* subcategory where the morphisms are those from $(\mathfrak{g}, V, \Theta)$ to $(\mathfrak{g}', V', \Theta')$ sending K_{-2} to K'_{-2} is a **faithful functor**.

What about physical implications ?

⧫ We have explicitly constructed such a (differential) graded Lie algebra structure on the tensor hierarchy, differently than in (Palmkvist, 2013) and (Palmkvist & Cederwall, 2015). It remains to check if the two constructions *coincide*.

⧫ It is known that the graded Lie bracket of the tensor hierarchy contains *all relevant physical informations* on the field strengths and the gauge transformations (Greitz et al., 2013 ; Bonezzi & Hohm, 2019)

⧫ Beyond it mathematical interest *per se*, the construction has promising applications in giving a better understanding of higher gauge theories in *double and exceptional field theory*, as well as any Leibniz gauge theory.

And after ?

Generalizing the result to the Lie/Leibniz algebroid category.

Slogan : a Lie algebroid is a vector bundle generalizing the tangent bundle

Definition

A Lie algebroid is a vector bundle $A \longrightarrow M$ together with :

1. a Lie bracket $[\,.\,,.\,]_A$ on sections,
2. a vector bundle morphism $\rho : A \longrightarrow TM$ (the *anchor*)

satisfying the compatibility conditions $(\forall\, x, y \in \Gamma(A), f \in \mathcal{C}^\infty(M))$:

Leibniz identity $\quad [x, fy]_A = f[x, y]_A + \rho(x)(f)y$

Lie algebra homomorphism $\quad \rho\big([x, y]_A\big) = \big[\rho(x), \rho(y)\big]_{TM}$

-oidization of embedding tensors

A **Lie algebroid crossed module** is made of :

- a Lie algebroid $A \xrightarrow{\rho} TM$,
- a Lie algebra bundle $L \to M$, on which A acts,
- a morphism of Lie algebroids $\phi : L \to A$, such that $\rho \circ \phi = 0$,

satisfying the usual formulas.

The Atiyah sequence of $P \simeq M \times G$ is a Lie algebroid crossed module :

$$
\begin{array}{ccc}
P \times_G \mathfrak{g} & \longrightarrow & {TP}/{G} \\
 & {\scriptstyle 0} \searrow \quad \swarrow {\scriptstyle \rho} & \\
 & TM &
\end{array}
$$

What if one weakens $P \times_G \mathfrak{g}$ to be a Leibniz algebra bundle instead ?

Other improvements

♦ **Category theory :** does this extend to other operads ? How does the equivalence of 2-categories between **diff**$\times$**mod** and 2-terms dgLas extend to embedding tensors ?

♦ *G*-**algebroids :** there is a formalization of embedding tensors in the Courant/*G*-algebroids setup (Bugden et al. 2021 ; Hulik & Valach 2022). How is it related to the present construction ?

♦ **Singular foliations :** we don't know if *locally* every singular foliation come from a Lie algebroid. Does it come from a Leibniz algebra bundle ?
$\Longrightarrow$ would partially answer the question.

Thank You

with Jim Stasheff, arXiv :2003.07838

Example

$A = \mathcal{M}_{2\times 2}(\mathbb{R})$ is an associative algebra and let $D : A \to A$ be :

$$D : \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \longmapsto \begin{pmatrix} a_{11} & 0 \\ 0 & a_{22} \end{pmatrix}$$

Then, we can define a Leibniz product on A :

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \circ \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} 0 & (a_{11} - a_{22})b_{12} \\ -(a_{11} - a_{22})b_{21} & 0 \end{pmatrix}$$

We have : $\mathfrak{g} = (A, [\,.\,,.\,]) = \mathfrak{gl}_2(\mathbb{R})$, $\mathfrak{h} =$ diagonal matrices, $V = (A, \circ)$, $\Theta = D$.

The symmetric space $S^2 A = S^2(\mathfrak{gl}_2(\mathbb{R}))$ is completely reducible into four irreducible $\mathfrak{gl}_2(\mathbb{R})$-submodules (equivalently : $\mathfrak{sl}_2(\mathbb{R})$-modules) :

$$U_1 = \langle XT - YZ \rangle$$
$$U_2 = \langle X^2 + T^2 + XT + YZ \rangle$$
$$U_3 = \left\langle X^2 - T^2, XY + YT, XZ + ZT \right\rangle$$
$$U_4 = \left\langle Y^2, Z^2, X^2 + T^2 - 2(XT + YZ), XY - YT, XZ - ZT \right\rangle$$

The three first $\mathfrak{g}$-submodules U_1, U_2, U_3 are contained in $\mathrm{Ker}\big(\{\,.\,,.\,\}\big)$:

$$\mathrm{Ker}\big(\{\,.\,,.\,\}\big) = U_1 \oplus U_2 \oplus U_3 \oplus \left\langle Y^2, Z^2, X^2 + T^2 - 2(XT + YZ) \right\rangle$$

Example

We can give an explicit formula of the bracket $[\![\,.\,,.\,]\!] : S^2(A) \to S^2(A)\big/ U_4$:

$$
\left[\!\!\left[\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \right]\!\!\right] =
$$

$$
\begin{aligned}
&\frac{1}{6}\begin{pmatrix} 2a_{11}b_{11} - (a_{11}b_{22} + b_{11}a_{21}) \\ +2a_{22}b_{22} - (a_{12}b_{21} + b_{12}a_{21}) \end{pmatrix}(X^2 + T^2 - 2(XT + YZ)) \\
&\quad + \frac{1}{2}\big((a_{11}b_{12} + b_{11}a_{12}) - (a_{12}b_{22} + b_{12}a_{22})\big)(XY - YT) \\
&\quad\quad + \frac{1}{2}\big((a_{11}b_{21} + b_{11}a_{21}) - (a_{21}b_{22} + b_{21}a_{22})\big)(XZ - ZT) \\
&\quad\quad\quad + 2\,a_{12}b_{12}Y^2 + 2\,a_{21}b_{21}Z^2
\end{aligned}
$$

The linear map $\partial_{-1} : T_{-2} \to T_{-1}$ is then defined as :

$$
\begin{matrix} pY^2 + qZ^2 \\ +r(X^2 + T^2 - 2(XT + YZ)) \\ +s(XY - YT) + t(XZ - ZT) \end{matrix} \xmapsto{\ \partial_{-1}\ } \begin{pmatrix} 0 & 2s \\ -2t & 0 \end{pmatrix}
$$

so that we indeed have $\partial_{-1}([\![a, b]\!]) = 2\{a, b\}$.

Example

The tensor hierarchy associated to the Lie-Leibniz triple $(\mathfrak{g}, V, D)$ then begins with the following sequence :

$$\ldots \longrightarrow U_4[2] \xrightarrow{\partial_{-1}} A[1] \xrightarrow{D} A \xrightarrow{-\eta(-;D)} R_D[-1]$$

where $R_D \subset \mathrm{End}(A)$ is the $\mathfrak{g}$-submodule generated by D.

In order to define T_{-3}, one needs to decompose the following diagram in terms of irreducible $\mathfrak{sl}_2(\mathbb{R})$-modules :

$$\begin{array}{ccccccc}
0 & \longrightarrow & A \otimes (U_1 \oplus U_2 \oplus U_3) & \xrightarrow{\mathrm{id}} & A \otimes (U_1 \oplus U_2 \oplus U_3) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle u_{-3}} & & \downarrow{\scriptstyle k_{-3}} & & \\
S^3(A) & \xrightarrow{-d_3} & A \otimes S^2(A) & \xrightarrow{-d_2} & F_{-3} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \\
S^3(A) & \longrightarrow & A \otimes U_4 & \xrightarrow{q_{-3}} & T_{-3} & \longrightarrow & 0
\end{array}$$

Example

The colors mark the correspondence between domain and range of maps :

$$
\begin{array}{ccccccc}
 & & 0 & \longrightarrow & 3\,V_0 \oplus 4\,V_2 \oplus V_4 & \xrightarrow{\text{id}} & 3\,V_0 \oplus 4\,V_2 \oplus V_4 & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & \begin{matrix} 2\,V_0 \oplus 2\,V_2 \\ \oplus V_4 \oplus V_6 \end{matrix} & \longrightarrow & \begin{matrix} 2\,V_0 \oplus V_0 \\ \oplus V_2 \oplus V_2 \oplus 3\,V_2 \\ \oplus V_4 \oplus 2\,V_4 \\ \oplus V_6 \end{matrix} & \longrightarrow & V_0 \oplus 3\,V_2 \oplus 2\,V_4 & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & \begin{matrix} 2\,V_0 \oplus 2\,V_2 \\ \oplus V_4 \oplus V_6 \end{matrix} & \longrightarrow & V_2 \oplus 2\,V_4 \oplus V_6 & \longrightarrow & 2\,V_4 & \longrightarrow & 0
\end{array}
$$

From this we deduce that $T_{-3} = (V_4 \oplus V_4)[3]$:

$$
\ldots \longrightarrow (V_4 \oplus V_4)[3] \xrightarrow{\partial_{-2}} V_4[2] \xrightarrow{\partial_{-1}} A[1] \xrightarrow{D} A \xrightarrow{-\eta(-;D)} R_D[-1]
$$