(Conformal) geometry as a gauge PDE

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Background

- Batalin-Vilkovisky (BV) formalism.
- BV on jet-bundles, local BRST cohomology *Henneaux et all*
- Alexandrov, Kontsevich, Schwartz, Zaboronsky (AKSZ) construction of BV for Lagrangian topological models
- Unfolded approach in higher spin gauge theories *M.Vasiliev*
- Vinogradov's approach to PDEs *Vinogradov, Krasilshik, . . .*
- FDA approach to SUGRA *d'Auria, Fre, . . .*
- BRST first quantized (also known as L_∞) approach to SFT and gauge fields *Zwiebach; Thorn, Bochicchio, Stern, Ouvry, . . .*
- Fedosov quantization

Linear local BV-BRST system

$\mathcal{H}_0 \to X$ – vector bundle of fields; $\mathcal{H}_1$ – of equations; $\mathcal{H}_1$ – of gauge parameters, etc.

$$\mathcal{H} = \oplus \mathcal{H}_i \qquad \text{vect. bundle. with } \mathbb{Z}\text{-graded fiber}$$

$$\Omega = \sum_i \Omega_{(i)} : \Gamma(\mathcal{H}) \to \gamma(\mathcal{H}), \quad \mathrm{gh}(\Omega) = 1, \quad \Omega^2 = 0$$

a differential operator. This defines linear local BV gauge theory. As $\mathrm{gh}(\Omega) = 1$, $\Omega^2 = 0$ can be seen as a complex:

$$\ldots \xrightarrow{\Omega_{(-2)}} \Gamma(\mathcal{H}_{-1}) \xrightarrow{\Omega_{(-1)}} \Gamma(\mathcal{H}_0) \xrightarrow{\Omega_{(0)}} \Gamma(\mathcal{H}_1) \xrightarrow{\Omega_{(1)}} \ldots$$

If $\Phi_i \in \Gamma(\mathcal{H}_i)$, equations of motion and gauge symmetries:

$$\Omega \Phi_0 = 0\,, \qquad \delta \Phi_0 = \Omega \Phi_{-1}\,, \qquad \ldots$$

+ extra requirements (exact in positive degree on jets).

If in addition $\Gamma(\mathcal{H})$ is equipped with degree -1 inner product $\int d^n x \langle \cdot , \cdot \rangle$ such that Ω is formally symmetric:

$$S[\Phi_0] = \int d^n x \langle \Phi_0, \Omega \Phi_0 \rangle$$

Gauge invariant action familiar from SFT and first quantized BRST approach. The approach can be seen as formal BRST quantum mechanics.

Interactions can be introduced by polydifferential operators

$$l_i : \Gamma(\mathcal{H}) \otimes \ldots \otimes \Gamma(\mathcal{H}) \to \Gamma(\mathcal{H}), \quad i > 1\,, \quad \mathrm{gh}(l_i) = 1$$

Compatibility: L_∞-conditions for $l_0 = \Omega, l_1, l_2, \ldots$ so that $\Gamma(\mathcal{H})$ remains an L_∞ algebra. Gives a standard setup for perturbatove QFT. E.g. perturbatoive S-matrix can be obtained as a minimal model with a proper choice of solution space. *Zwiebach, Lada, Stasheff, Jurco, Arvanitakis, Saemann, Wolf, Hohm,. . .*

BV perspective

Replace vector bundle $\mathcal{H} \to X$ with a bundle of graded manifolds whose fiber $_x$ is $\mathcal{H}_x$ considered as a graded manifold (i.e. associated algebra is $Sym(\mathcal{H}_x^*)$)
$J^\infty(F)$ is naturally equipped with Cartan distribution and BRST differential

$$s\Psi = \Omega\Psi\,, \qquad \Psi = \psi^A e_A$$

$\psi^A e_A$ – known as string field. In the Lagrangian case:

$$S_{BV} = \int d^n x \langle \Psi, \Omega\Psi \rangle\,, \qquad \Psi = \psi^A e_A \quad \text{– string field}$$

(cf. quadratic SFT action) *Bochicchio, Thorn 1986*

Def *[Henneaux,. . . , Barnich,MG; Lyakhovich,Sharapov]* BV (EOM-level) $E \to X$, $J^\infty(E)$ is equipped with vertical evolutionary s, $\mathrm{gh}(s) = 1$, $s^2 = 0$ + technical conditions

L_∞-setup reproduced upon perturb. expansion about a solution. Convenient to pull-back $J^\infty(E)$ to $T[1]X$. Algebra of functions – horizontal local forms on $J^\infty(E)$.

$$s^2 = 0, \quad \mathrm{gh}(s) = 1, \qquad [s, d_{\mathsf{h}}] = 0\,, \quad d_{\mathsf{h}} = \theta^a D_a$$

Cohomology $H(s|d_h)$-local BRST cohomology: deformations, anomalies, global symmetries, conservation laws etc.

In fact invariant information is contained in $Q = s + d_{\mathsf{h}}$ (total degree = form deg + gh). Solutions, gauge symmetries, can be defined in terms of Q and d_X *Barnich, MG*

Towards gauge PDEs

The notion of BV is restricted to jets. Generalization?
Def Q-manifold (M, Q) is a $\mathbb{Z}$-graded supermanifold M equipped with the odd nilpotent vector field of degree 1, i.e.

$$Q^2 = 0\,, \qquad \mathrm{gh}(Q) = 1$$

$\phi : (M_1, Q_1) \to (M_2, Q_2)$ is a Q-map if $\phi^* \circ Q_2 = Q_1 \circ \phi^*$
Example: Odd tangent bundle: $(T[1]X, d_X)$. If θ^a are coordinates on the fibres of $T[1]M$ in the basis $\frac{\partial}{\partial x^a}$:

$$d_X := \theta^a \frac{\partial}{\partial x^a}$$

Example: CE complex $(\mathfrak{g}[1], Q)$. If $\mathfrak{g}$ is a Lie algebra then $\mathfrak{g}[1]$ is equipped with Q structure. If c^α are coordinates on $\mathfrak{g}[1]$ in the basis e_α then

$$Q = \frac{1}{2} c^\alpha c^\beta U^\gamma_{\alpha\beta} \frac{\partial}{\partial c^\gamma}\,, \qquad [e_\alpha, e_\beta] = U^\gamma_{\alpha\beta} e_\gamma$$

Example: $(V[1](M), Q)$ where $V(M)$ Lie algebroid. Indeed generic Q of degree 1 locally reads as:

$$Q = c^\alpha R_\alpha - \frac{1}{2} c^\alpha c^\beta U^\gamma_{\alpha\beta}(z) \frac{\partial}{\partial c^\gamma}$$

R_α gives anchor, $U^\gamma_{\alpha\beta}$ bracket, $Q^2 = 0$ encodes compatibility.

Proposition: [AKSZ] Let (M, Q) a Q-manifol, $p \in M$ and $Q|_p = 0$ then T_pM is an L_∞ algebra.

Equivalence of Q-manifolds:

Idea: restrict to local analysis. Let

$$M = N \times T[1]V\,, \qquad Q = Q_N + d_{T[1]V}$$

with V a graded space. Then (M, Q) and (N, Q_N) are equivalent. Q-manifold $(T[1]V, d_{T[1]V})$ is called contractible. In coordinates:

$$Q = Q_N + v^\alpha \frac{\partial}{\partial w^\alpha}\,, \qquad Q_N = q^i(\phi)\frac{\partial}{\partial \phi^i}\,.$$

Often one finds a "minimal" equivalent Q-man. In the formal setup this gives a minimal model of the respective L_∞ algebra.

Geometric characterization: let w^a be independent functions such that w^a, Qw^a are also independent then the surface $w^a = 0 = Qw^a$ is a Q-submanifold isomorphic to (N, Q_N). Simple geometric picture of the homotopy transfer

In the context of gauge theories: w^α, v^α – are known as "generalized auxiliary fields" *Henneaux, 1990; Barnich, M.G. 2004*.

Def. *[Kotov, Strobl]* Locally trivial bundle $\pi : E \to M$ of Q-manifolds is called Q-bundle if π is a Q-map. Section $\sigma : M \to E$ is called Q-section if it's a Q-map.
In general, $\pi : E \to M$ is not a locally trivial Q-bundle.
Indeed, although locally $E \cong M \times F$ (product of manifolds) in general Q is not a product Q-structure of Q_F and Q_M.

Example: let $\pi_X : E \to X$ be a fiber bundle then
$\pi = d\pi_X : (T[1]E, d_E) \to (T[1]X, d_X)$ is a Q-bundle.

Def. (M, Q) is called an equivalent reduction of (M', Q') if (M', Q') is a locally trivial Q-bundle over (M, Q) with a contractible fiber and (M', Q') admits a global Q-section.

This generates an equivalence relation for Q-manifolds.

Gauge PDEs

Def. Gauge pre-PDE $(\mathcal{E}, T[1]X, Q)$ is a Q-bundle $(\mathcal{E}, Q)$ over $(T[1]X, \mathsf{d}_X)$

Equivalence of Q-manifolds extends to Q-bundles over $T[1]X$, giving the notion of equivalent reduction and equivalence of gauge pre-PDEs. Notion of gauge pre-PDE is too wide:

gauge PDE: equivalent to nonnegatively graded, realizable in term of super-jet bundle in a regular way. In applications we often (but not always!) also want gauge PDE to be proper – i.e. that all the gauge symmetries of the underlying PDE are taken into account by Q.

Equations of motion and gauge symmetries

Solutions: $\sigma : T[1]X \to \mathcal{E}$ is a solution if

$$d_X \circ \sigma^* = \sigma^* \circ Q$$

Gauge transformations:

$$\delta\sigma^* = d_X \circ \epsilon_\sigma^* + \epsilon_\sigma^* \circ Q,$$

Gauge parameter: $\epsilon_\sigma^* : \mathcal{C}^\infty(\mathcal{E}) \to \mathcal{C}^\infty(T[1]X)$,

$$\mathrm{gh}(\epsilon_\sigma^*) = -1, \quad \epsilon_\sigma^*(fg) = \epsilon_\sigma^*(f)\sigma^*(g) + \sigma^*(f)\epsilon_\sigma^*(g)$$

Gauge for gauge symmetries . . .

Example: BV formulation (EOM level)

Take as $\mathcal{E}$ bundle $J^\infty(E)$ pulled back to $T[1]X$ (horizontal forms on $J^\infty(E)$) and $Q = d_{\mathsf{h}} + s$. Locally, gauge system determined by $(\mathcal{E}, T[1]X, Q)$ is equivalent to the one encoded in the BV formulation $(J^\infty(\mathcal{E}), s)$. *Barnich, MG 2010*

The notion of gauge PDE includes BV as a particular case and hence all reasonable gauge theories. Justifies definition.

Example: zero-curvature equation

Take $\mathcal{E} = (T[1]X, d_X) \times (\mathfrak{g}[1], Q)$, where $\mathfrak{g}$ is a Lie algebra and Q is a CE differential seen as a vector field. If C^α denote coordinates on $\mathfrak{g}[1]$ then $QC^\alpha = -\frac{1}{2}U^\alpha_{\beta\gamma}C^\beta C^\gamma$. Denoting $\sigma^*(C^\alpha) = A^\alpha_a(x)\theta^a$ we get

$$d_X \circ \sigma^* = \sigma^* \circ Q \quad \Longrightarrow \quad dA + \frac{1}{2}[A, A] = 0$$

Gauge transformations:

$$\delta A = d\epsilon + [A, \epsilon]$$

Topological PDE. $\mathcal{E}$ can be thought of as a finite-dimensional BV analog of the *Vinogradov's* diffiety. Example known from *AKSZ*

Example: PDE

Let $\mathcal{E}_0 \to X$ be a bundle equipped with Cartan distribution. Extend to a bundle $\mathcal{E} \to T[1]X$, the Cartan distribution defines d_{h} on $\mathcal{E}$:

$$d_{\mathsf{h}} = \theta^a D_a\,, \qquad (\theta^a \equiv dx^a)$$

We arrive at Q-bundle $(\mathcal{E}, T[1]X, d_{\mathsf{h}})$.
Seen as a section of $\mathcal{E} \to T[1]X$, a solution is a Q-section. If ψ^A are local fiber coordinates the section is parametrized by $\sigma^A(x) = \sigma^*(\psi^A)$
Q-map condition $d_X \circ \sigma^* = \sigma^* \circ d_{\mathsf{h}}$ gives:

$$\frac{\partial}{\partial x^a}\sigma^A(x) = \Gamma^A_a(\sigma(x), x)\,, \qquad d_{\mathsf{h}} = \theta^a D_a = \theta^a\left(\frac{\partial}{\partial x^a} + \Gamma^A_a(\psi, x)\frac{\partial}{\partial \psi^A}\right)$$

also known as "unfolded" representation *M.Vasiliev*.
Usual PDEs are gauge PDEs with horizontal degree.

Riemannian geometry as a gauge PDE

Take $E = S^2(T^*X) \oplus T[1]X$. Consider $J^\infty(E)$ pulled back to $T[1]X$. Local trivialization:

$$x^a, \theta^a, \qquad g_{ab}, g_{ab|c}, \dots, \quad \xi^a, {\xi^a}_{|c} \cdots$$

In a suitable trivialization (cf. AKSZ):

$$Q = d_x + \gamma\,, \quad \gamma g_{ab} = \xi^c g_{ab|c} + {\xi^c}_{|a} g_{cb} + {\xi^c}_{|b} g_{ac}\,, \quad \gamma \xi^a = \xi^c {\xi^a}_{|c}\,, \dots$$

E.g. Lagrangians: $H^n(Q, local functions)$, $n = \dim X$. Applies to generic off-shell (equivalent to jets) gauge PDEs. *MG, 2010*

Locally, $\mathcal{E} = (T[1]X, d_X) \times (\mathcal{F}, q)$, i.e. Locally-trivial Q-bundle.

Minimal model

Restrict to local analysis. $\Gamma^a_{(bc|d\ldots)}$ form contractible pairs with $\xi^a_{bcd\ldots}$ and g_{ab} with symmetric part of ξ^a_b. Resulting minimal model *Stora; Barnich, Brandt, Henneaux; Vasiliev ...*:

Coordinates: $x^\mu, \theta^\mu\,,\qquad \xi^a, \rho^a{}_b,\qquad R_{ab}{}^c{}_d, R_{a(b}{}^c{}_{de)}, \ldots, R_{a(b}{}^c{}_{de\ldots)}, \ldots$

$$Qx^\mu = \theta^\mu\,,\quad Q\xi^a = \rho^a{}_c\,\xi^c\,,\qquad q\rho^{ab} = \rho^a{}_c\,\rho^{cb} + \lambda\xi^a\xi^b + \xi^c\xi^d R^{ab}_{cd}\,,$$
$$QR^{R_{ab}{}^c{}_d} = \xi^e R_{a(b}{}^c{}_{de)} + \rho_a{}^f R_{fb}{}^c{}_d + \ldots\,,\qquad \ldots$$

For instance $H^0(Q)$ immediately gives Riemannian invariants. On-shell version: R are totally traceless (only Weyl tensors).

Section:

$$\sigma^*(\xi^a) = e^a_\mu(x)\theta^\mu, \quad \sigma^*(\rho^{ab}) = \omega^{ab}_\mu(x)\theta^\mu, \quad \sigma^*(R_{ab}{}^c{}_d) = \mathsf{R}_{ab}{}^c{}_d(x), \ldots$$

Equations of motion:

$$d_X e + \omega e = 0\,, \quad d_X\omega + \omega\omega = \mathsf{R}\,, \quad \ldots$$

Cartan structure equations. Taking a total degree "gh+form degree" is crucial. Frame-like formulations.

On shell version – equivalent form of Einstein equations.
What about Lagrangians in the on-shell version?

Presymplectic structure and Cartan-Weyl action

Presymplectic structure on the fiber F of the minimal model:

Alkalaev, M.G. 2013

$$\omega = \epsilon_{abcd}\xi^a d\xi^b d\rho^{cd}\,, \quad \omega = d\chi$$

$$L_Q\omega = 0\,, \qquad d\omega = 0 \quad \Rightarrow \quad dH = i_Q\omega$$

AKSZ-like action

$$S[\sigma] = \int_{T[1]X} \sigma^*(\chi)(d_X) - \sigma^*(H) = \int_{T[1]X} (d_X\omega^{ab} + \omega^a{}_c\omega^{cb})\epsilon_{abcd}e^c e^d$$

Familiar Cartan-Weyl action for GR. Generalization for general $n > 4$ and $\Lambda \neq 0$ is straightforward.

What about remaining components of section? What about full-scale BV formulation available in usual AKSZ?

Idea: assume ω regular and take the symplectic quotient. But ω is not regular for $n > 3$!

Restrict to local analysis. Refined idea: locally, sections are fiber-valued functions, take:

$$Smaps(T[1]X, F) = Smaps(X, \bar{F})\,, \qquad \bar{F} = Smaps(\mathbb{R}^n[1], F))$$

$\bar{F}$ is finite-dimensional provided F is. Natural lift of ω to $\bar{F}$

$$\bar{\omega} = \int d^n\theta\; \omega_{AB}(\psi(\theta))d\psi^A(\theta) \wedge d\psi^B(\theta)\,, \qquad \mathrm{gh}(\bar{\omega}) = -1$$

Now assume that $\bar{\omega}$ regular and take a symplectic quotient. We have arrived at BV formulation! With BV symplectic structure determined by $\bar{\omega}$!

State of the art: for a Lagrangian system such a representation always exists but not necessarily in the minimal model. Counter-example: massive spin-2 field.

Regularity

$Smaps(\mathbb{R}^n, F)$ explicitly:

$$\hat{\sigma}^*(\xi^a) = \overset{0}{\xi}{}^a(x) + e^a_\mu \theta^\mu + \overset{2}{\xi}{}^a_{\mu\nu}\theta^\mu\theta^\nu + \dots\,,$$
$$\hat{\sigma}^*(\rho^{ab}) = \overset{0}{\rho}{}^{ab} + \omega^{ab}_\mu \theta^\mu + \overset{2}{\rho}{}^{ab}_{\mu\nu}\theta^\mu\theta^\nu + \dots\,,$$

form-degree k components carry ghost degree $1-k$.

Prop. *[Kotov, MG 2020]* $\bar{\omega}$ is regular provided e^a_μ is invertible.

$$S[\hat{\sigma}] = \int \hat{\sigma}^*(\chi)(d_X) - \hat{\sigma}^*(H)$$

induces a proper BV action on the symp. quotient.

Formal path integral: $\qquad Z = \int_L exp(\frac{i}{\hbar} S_{BV})$

L comprise gauge condition and gauge condition for zero modes of $\bar{\omega}$. No need to take symplectic quotient explicitly! AKSZ-like

Conformal geometry as a gauge PDE

Take $E = S^2(T^*X) \oplus T[1]X \oplus \mathcal{C}^\infty(X)[1]$. Consider as $\mathcal{E}$ the $J^\infty(E)$ pulled back to $T[1]X$. Local trivialization:

$$x^a, \theta^a\,, \qquad g_{ab|c\ldots}, \quad \xi^a{}_{|c\ldots}, \quad \lambda_{|c\ldots}$$

In a suitable trivialization:

$$Q = d_x + \gamma\,, \quad \gamma g_{ab} = \xi^c g_{ab|c} + \xi^c{}_{|a} g_{cb} + \xi^c{}_{|b} g_{ac} - 2\lambda g_{ab}$$
$$\gamma \xi^a = \xi^c \xi^a{}_{|c}\,, \quad \gamma\lambda = \xi^a \lambda_{|a} \cdots$$

Minimal model $T[1]X \times F$ (locally):
Degree 1 variables:

$$\xi^a, \;\rho^{ab}\; \kappa_a, \;\lambda$$

Degree 0 variables:

$$W_{ab}{}^c{}_d\,, \quad W_{a(b}{}^c{}_{de)}\,, \quad W_{a(b}{}^c{}_{d\ldots)}\,, \quad \ldots$$

The Q structure (a version of that obtained by Boulanger,2004)

$$Q = \rho^a{}_c \xi^c + \xi^a \lambda\,,$$
$$Q\rho^a{}_b = \rho^a{}_c \rho^c{}_b + (\xi^a \kappa_b - \xi_b \kappa^a) + \frac{1}{2}\xi^c \xi^d W^a{}_{bcd}\,,$$
$$Q\kappa_b = \kappa_c \rho^c{}_b + \lambda \kappa_b + \frac{1}{2}\xi^c \xi^d C_{bcd}\,,$$
$$Q\lambda = \kappa_c \xi^c\,.$$

Here $C_{abc} = W_{ab}{}^d{}_{cd}$ – Cotton tensor.

$$QW^a{}_{bcd} = \xi^k W^a{}_{bcd|k} - \rho_k{}^a W^k{}_{bcd} + \ldots\,,$$
$$QC_{abc} = \xi^k C_{abc|k} + \rho_a{}^k C_{kbc} + \ldots\,.$$

Resulting equations of motion (Cartan structure equations):

$$d_X A + \frac{1}{2}[A, A] = e^a e^b (W^{cd}_{ab} J_{cd} + C_{ab}{}^c K_c)\,, \quad A = e^a T_a + \omega^{ab} J_{ab} + f^a K_a + vD$$

(Bach-flat version – all $W_{\ldots}$ are $o(n-1,1)$-irreducible)

Restrict to conformal gravity in $n=4$ (i.e. Bach-flat metrics). The compatible presymplectic structure: *Dneprov, MG 2022*

$$\omega = \omega_W - 2\omega_C \,,$$
$$\omega_W = d(\rho_{ab})d(W^{abnm}\epsilon_{nmpk}\xi^p\xi^k) \,, \qquad \omega_C = d(\xi_a)d(C^a{}_{bc}\epsilon^{bcpk}\xi_p\xi_k)$$

$$d\omega = 0, \qquad L_Q\omega = 0, \qquad \mathrm{gh}(\omega) = n-1$$

Defines presymplectic AKSZ system. The action:

$$S[e,\omega,W,C] = \int_X \Big[(d\omega_{ab} + \omega_{ac}\omega^c{}_b)W^{abnm}\epsilon_{nmpk}e^pe^k +$$
$$+ W_{abcd}e^ce^dW^{abnm}\epsilon_{nmpk}e^pe^k - 2(de_a + \omega_{ad}e^d)C^a{}_{bc}\epsilon^{bcpk}e_pe_k\Big] \,,$$

Equivalent to CGR Lagrangian $\sqrt{g}W^2$ upon elimination of auxiliary fields and passing to the symplectic quotient. First principle frame-like action *(cf. Kaku et all)*

Presymplectic structures: general setup

Def. Compatible presymplectic structure on gauge PDE $(E, T[1]X, Q)$ is a vertical 2-form ω on E satisfying:

$$d\omega = 0\,, \qquad L_Q\omega = 0\,, \qquad \mathrm{gh}(\omega) = n-1$$

Here $n = \dim X$ and vertical forms are understood as equivalence classes + techincal assumptions.

Defines "Hamiltonian" (or, better, covariant BRST charge) via

$$i_Q\omega = d\mathcal{H}\,, \qquad \mathrm{gh}(\mathcal{H}) = n$$

ω is directly related to the BV symplecic structure $\overset{n}{\omega}$ extended as $\omega = \overset{n}{\omega} + \overset{n-1}{\omega} + \ldots \overset{0}{\omega}$ to be a cocycle of $d_\mathsf{h} + s$, i.e. $L_{d_\mathsf{h}+s}\omega = 0$.

Intrinsic BV action

Action functional on the space of section of $(E, T[1]X, Q, \omega)$

$$S[\sigma] = \int_{T[1]X} (\sigma^*(\chi)(d_X) - \sigma^*(\mathcal{H}))$$

where χ is a presymplectic potential, i.e. $\omega = d\chi$. $\chi \to \chi + d\rho$ adds boundray term.

BV extension (AKSZ-type). Supersection $\hat{\sigma}$:

$$S^{BV}[\hat{\sigma}] = \int_{T[1]X} (\hat{\sigma}^*(\chi)(d_X) - \hat{\sigma}^*(\mathcal{H}))$$

R gh$(C) = 1$ then $\sigma^*(C) = A_a(x)\theta^a$ while $\hat{\sigma}^*(C) = \overset{0}{C}{}^a + A_a\theta^a + \overset{2}{\xi}_{ab}\theta^a\theta^b + \dots$,

As before: interpretation through the symplectic quotient on $Smaps(\mathbb{R}^n, F)$

Example: Maxwell

Recall: $E = T[1]X \times F$, Fiber coordinates:

$$C, \quad \text{gh}(C) = 1\,, \quad F^{a|b}, \quad F^{a|b_1 b_2}, \quad \ldots \quad F^{a|b_1 \ldots b_l} \quad \ldots \quad \text{gh}(F^{\cdots}) = 0$$

$$Qx^a = \theta^a, \quad Q\theta^a = 0, \quad QC = \frac{1}{2} F^{a|b} \theta_a \theta_b, \quad QF^{a|b} = \theta_c F^{a|bc}, \quad \ldots$$

indexes rised/lowered with Minkowski metric. $F^{a|b_1 \ldots b_l}$ – irreducible tensors (totally traceless + Young condition)

Presymplectic structure: *Alkalaev, M.G. 2013 (also A. Sharapov 2017)*

$$\omega = (\theta)^{(n-2)}_{ab} dF^{a|b} dC\,,$$

indexes rised/lowered with Minkowski metric
Intrinsic action ($\sigma^*(C) = A_a(x)\theta^a, \sigma^*(F^{a|b}) = F^{a|b}(x)$):

$$S[\sigma] = \int d^n x (\partial_a A_b - \partial_b A_a) F^{a|b} - \frac{1}{2} (F^{a|b})^2$$

Presymplectic structure on supermaps gives correct BV form!

$$\bar{\omega} = d\overset{0}{C} \wedge \overset{2}{F}{}^{a|b}_{ab} + dA_a \wedge \overset{1}{F}{}^{a|b}_{b} + d\overset{0}{F}{}^{a|b} \wedge \overset{2}{C}_{ab}$$

Here:

$$\hat{\sigma}^*(C) = \overset{0}{C}(x) + A_a(x)\theta^a + \frac{1}{2}\overset{2}{C}_{ab}(x)\theta^a\theta^b \ldots$$

$$\hat{\sigma}^*(F^{a|b}) = \overset{0}{F}{}^{a|b}(x) + \overset{1}{F}{}^{a|b}_{c}(x)\theta^c + \frac{1}{2}\overset{2}{F}{}^{a|b}_{cd}(x)\theta^c\theta^d + \ldots$$

All the fields are in the kernel except for:

$$C = \overset{0}{C}, \quad C^* = \overset{2}{F}{}^{a|b}_{ab}, \quad A_a, \quad A^a_* = \overset{1}{F}{}_b{}^{a|b}, \quad F^{a|b}, \quad F^*_{ab} = \overset{2}{C}_{ab}$$

BV master action (standrad BV for Maxwell in first order formalism. Extension to YM is straitforward.):

$$S_{BV} = S + \int d^n x F^{a|b}_b \partial_a \overset{0}{C}$$

Conclusions

- Gauge PDEs as geometric objects. Well suited to work with diffeomorphims-invariant and topological models. Notion of equivalence.
- Determines a "canonical" first-order realization in terms of a jet-bundle associated to the equation manifold
- Comprise "frame-like" formulation of the system. The respective FDA arise from BRST differential. E.g. the Cartan-Weyl form of gravity arises from a minimal model of the respective BRST complex.
- Full scale BV and its BV symplectic structure are encoded in the graded presympletic structure on the gauge PDE.

- In the case of variational systems unifies Lagrangian and Hamiltonian BRST formalism, cf. BV/BFV approach of *Cattaneo et all.*

- Gives an invariant approach to study boundary values of gauge fields. In particular in the AdS/CFT correspondence context. *Bekaert, M.G. 2012*. In particular, Fefferman-Graham construction (and tractor calculus) can be seen as a certain gauge PDE. *M.G. 2012, M.G. Waldron 2011, Bekaert, M.G. Skvortsov 2017*

- Succesful applications in constructing new models of HS theory, e.g. Type-B theory (AdS holographic dual to conformal spinor on the boundary) *M.G. Skvortsov 2018*

- Recent construction of Lagrangians for AdS_4 higher spin gravity in terms of presymplectic AKSZ. *Sharapov, Skvortsov 2020*

Presymplectic structures and intrinsic actions

Lagrangian induces presymplectic structure $\omega \in \Omega^{(n-1,2)}(\mathcal{E})$ on the equation manifold.

Kijowski, Tulczyjew 1979, Crnkovic, Witten, 1987, Hydon 2005, Khavkine 2012, Alkalaev M.G. 2013, Sharapov 2016

Given a Lagrangian $\mathcal{L} \in \bigwedge^{n,0}(J^\infty(\mathcal{F}))$ define $\widehat{\chi} \in \bigwedge^{n-1,1}(J^\infty(\mathcal{F}))$:

$$d_{\mathsf{v}}\mathcal{L} = d_{\mathsf{v}}\phi^i \frac{\delta^{EL}\mathcal{L}}{\delta\phi^i} - d_{\mathsf{h}}\widehat{\chi}$$

Define $\widehat{\omega} = d_{\mathsf{v}}\widehat{\chi}$

$$d_{\mathsf{v}}\omega = d_{\mathsf{h}}\omega = 0\,, \qquad \omega = \widehat{\omega}|_{\mathcal{E}}$$

More generally, let a generic $\omega \bigwedge^{n-1,2}(\mathcal{E})$ satisfies the above. It follows $\omega = d(\chi + l)$ for some $\chi \in \bigwedge^{n-1,1}(\mathcal{E}), l \in \bigwedge^{n,0}(\mathcal{E})$. These define a natural action functional on section of $\mathcal{E}$ called intrinsic action: *MG, 2016*

$$S^c[\sigma] = \int_X \sigma^*(\chi + l)$$

What this has to do with the PDE in question?

S^c doesn't depend on fields in the vertical kernel of ω. Assuming regularity take a symplectic quotient. The resulting Lagrangian system is weaker, $\mathcal{E} \subset \mathcal{E}^c$. For a class of systems containing YM, Gravity etc. there exists ω such that S^c is equivalent to the standard Lagrangian. Counterexample: systems with degree zero differential consequences, e.g. massive spin-2 system. *M.G. Gritzaenko 2021*

Example: scalar field

Lagrangian:

$$L = \frac{1}{2}\eta^{ab}\phi_a\phi_b - V(\phi)$$

$\mathcal{E}$ is coordinatized by $x^a, \phi, \phi_a, \phi_{ab}, \ldots$ with $\phi_{abc\ldots}$ traceless.

$$d_{\mathsf{h}}x^a = dx^a\,, \quad d_{\mathsf{h}}\phi = dx^a\phi_a,\,, \quad d_{\mathsf{h}}\phi_a = dx^b(\phi_{ab} - \frac{1}{n}\eta_{ab}\frac{\partial V}{\partial\phi})\,, \quad \ldots$$

The presymplectic potential and 2-form:

$$\chi = (dx)^{n-1}_a\phi^a d_{\mathsf{v}}\phi\,, \quad \omega = (dx)^{n-1}_a d_{\mathsf{v}}\phi^a d_{\mathsf{v}}\phi$$

The Hamiltonian:

$$\mathcal{H} = (dx)^n(\phi_a\phi^a - L|_{\mathcal{E}}) = \frac{1}{2}\phi^a\phi_a + V(\phi)$$

The intrinsic Larangian: *Schwinger*

$$\mathcal{L}^c = (dx)^n\left(\phi^a(\partial_a\phi - \frac{1}{2}\phi_a) - V(\phi)\right)$$