Cohomologies of Lie Superalgebras

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Some motivations

Supergravity backgrounds based on supergroups

1. Geometric backgrounds solving Einstein's equations plus non-trivial fluxes
2. Chern-Simons gauge theories on supergroups (3D supergravity)
3. JT supergravity

Structure of supergravity theories

1. Self-dual form backgrounds
2. Free Differential Algebras
3. Superspace formulations
4. Auxiliary fields

Superstring theories

1. Supermoduli spaces (integration of vertex operators)
2. Pure spinor formulation and target space PCO
3. WZW -models with supergroups as target space
4. A_∞ /L_∞-algebra and integral forms

SUMMARY

1. Review of Lie Algebra Chevalley-Eilenberg (CE) Cohomology — Example SL(3)
2. Review of Equivariant CE Cohomology — Example SL(3)/GL(2)
3. Review of Poincaré Polynomials — Greub-Halperin-Vanstone Theorem
4. Lie Superalgebra CE Cohomology — Example OSp(1|2)
5. Lie Superalgebra CE Cohomology — New Results for OSp(2|2)

CE cohomology for Lie algebras

We denote by $\mathfrak{g}$ a simple Lie algebra. Its generators are given by the vector fields

$$X_a \in \mathfrak{g}, \quad a = 1, \dots, \dim \mathfrak{g}, \qquad [X_a, X_b] = f_{ab}{}^{c} X_c$$

where $f_{ab}{}^{c}$ are the structure constants.

We introduce linear functionals (forms) of the dual space $\mathfrak{g}^\star$ denoted by

$$V^a, \qquad V^a \wedge V^b = -V^b \wedge V^a, \quad \langle V^a, X_b \rangle = \delta^a{}_b$$

the forms V^a have a grading: **FORM NUMBER**

The exterior algebra $\Omega(\mathfrak{g}^\star) = \oplus_{p=0}^{dim\mathfrak{g}} \Omega^p(\mathfrak{g}^\star)$ is filtered with the form number and it is endowed with a differential $d_{CE} : \Omega^p(\mathfrak{g}^\star) \to \Omega^{p+1}(\mathfrak{g}^\star)$

$$dV^a = f_{bc}{}^{a} V^b \wedge V^c, \qquad d^2 = 0$$

The Chevalley-Eilenberg coholomogy (with trivial coefficients) is defined as follows

$$H(d, \Omega(\mathfrak{g}^\star), \mathbb{R}) = \oplus_{p=0}^{\dim\mathfrak{g}} H^{(p)}(d, \Omega(\mathfrak{g}^\star), \mathbb{R})$$

CE Cohomology of Lie algebra $\mathfrak{sl}(3)$

Its CE cohomology (in the trivial module $\mathbb{R}$) is represented by the ring of forms

$$\widetilde{\omega}^{(0)} = 1\,, \qquad \widetilde{\omega}^{(3)} = \eta_{ab} V^a \wedge dV^b = f_{abc} V^a \wedge V^b \wedge V^c\,,$$

$$\widetilde{\omega}^{(5)} = d_{abc} V^a \wedge dV^b \wedge dV^c = d_{ars} f^r{}_{bc} f^s{}_{de} V^a \wedge V^b \wedge V^c \wedge V^d \wedge V^e\,,$$

$$\widetilde{\omega}^{(8)} = \widetilde{\omega}^{(3)} \wedge \widetilde{\omega}^{(5)} = \epsilon_{abcdefgh} V^a \wedge \ldots \wedge V^h\,,$$

where η_{ab}, d_{abc} are the invariant tensors corresponding to the Casimirs (in the enveloping algebra)

$$C_2 = \eta^{ab} X_a \otimes X_b\,, \qquad C_3 = d^{abc} X_a \otimes X_b \otimes X_c$$

The "Volume" form $\widetilde{\omega}^{(8)}$ represents the top form and it transforms under automorphisms as the determinant, we will rename it as $\det \mathfrak{g}^\star$. In the case of compact real forms, this represents the Haar measure of the corresponding Lie group.

The Lie algebra $\mathfrak{sl}(3)$ has several subalgebras $\mathfrak{h}$, in terms of which we can construct some homogenous spaces. One can build the CE cohomologies of $\mathfrak{sl}(3)$ starting from the CE cohomologies of subalgebras $\mathfrak{h}$ and of its cosets spaces. Since this turns out a very powerful methods for Lie superlgebras, I will briefly review it.

Equivariant CE Cohomology

For homogeneous spaces, the correct definition is not the Chevalley-Eilenberg cohomology, but rather the equivariant CE cohomology (the cohomology of d on basic and invariant - under the subgroup - forms)

For example in the case of SL(3)/GL(2) one has to study the action of the differential d up to gauge transformations of the subgroup GL(2).

$$\widetilde{\omega}^{(0)}_{SL(3)} = \omega^{(0)}_{GL(2)} \wedge \omega^{(0)}_{SL(3)/GL(2)}$$

$$\widetilde{\omega}^{(3)}_{SL(3)} = \omega^{(3)}_{GL(2)} \wedge \omega^{(0)}_{SL(3)/GL(2)} + \omega^{(1)}_{GL(2)} \wedge \omega^{(2)}_{SL(3)/GL(2)}$$

$$\widetilde{\omega}^{(5)}_{SL(3)} = \omega^{(3)}_{GL(2)} \wedge \omega^{(2)}_{SL(3)/GL(2)} + \omega^{(1)}_{GL(2)} \wedge \omega^{(2)}_{SL(3)/GL(2)} \wedge \omega^{(2)}_{SL(3)/GL(2)}$$

$$\widetilde{\omega}^{(8)}_{SL(3)} = \omega^{(3)}_{GL(2)} \wedge \omega^{(2)}_{SL(3)/GL(2)} \wedge \omega^{(2)}_{SL(3)/GL(2)}$$

where $\omega^{(0)}_{GL(2)}, \omega^{(1)}_{GL(2)}, \omega^{(3)}_{GL(2)}, \omega^{(1)}_{GL(2)} \wedge \omega^{(3)}_{GL(2)}$ are the cohomology classes of $\mathfrak{gl}(2)$

and $\omega^{(0)}_{SL(3)/GL(2)}, \omega^{(2)}_{SL(3)/GL(2)}$ are the cohomologies of the coset SL(3)/GL(2)

The complex

$$0 \to \Omega^0(\mathfrak{g}^\star) \to \ldots \to \Omega^{\dim \mathfrak{g}}(\mathfrak{g}^\star) \to 0$$

is finite dimensional and it displays the duality between $\Omega^p(\mathfrak{g}^\star) \longleftrightarrow \Omega^{\dim \mathfrak{g}-p}(\mathfrak{g}^\star)$

This Poincaré duality allows us to split the complex in two isomorphic parts (for even dimensional algebras)

$$\Omega(\mathfrak{g}^\star) = \left(\oplus_{p=0}^{dim\,\mathfrak{g}/2} \Omega_S^{(p)} \right) \oplus \left(\oplus_{p=dim\,\mathfrak{g}/2}^{dim\,\mathfrak{g}} \Omega_I^{(p)} \right)$$

and the elements of the first complex $\Omega_S^{(p)}(\mathfrak{g}^\star)$ are related to those of the complement complex by the relations

$$\omega_S^{(p)} \wedge \omega_I^{(dim\mathfrak{g}-p)} = \det(\mathfrak{g}^\star)$$

where $\det(\mathfrak{g}^\star)$ is the top form of the complex.

The part $\Omega_I^{(p)}(\mathfrak{g}^\star)$ can also be identified with the chains

$$\hat{\Omega}_I^{(p)} = \iota_{X_1}\ldots\iota_{X_p}\det(\mathfrak{g}^\star)$$

It follows that

$$H_{CE}(d, \Omega_S) \sim H(d_K, \hat{\Omega}_I)$$

where d_K is the Koszul differential acting on chains $\hat{\Omega}_I$.

Poincaré Polynomials

This is a very important tool to study the cohomology and they are defined as follows

$$\mathbb{P}_{\mathfrak{g}}(t) = \sum_{p=0}^{dim\,\mathfrak{g}} \dim H^{(p)}(d,\Omega)(-1)^p t^p = \prod_{q=0}^{\infty}(1-t^q)^{N(q)}$$

1. $\dim H^{(p)}(d,\Omega)$ is the dimension of the cohomology at a given form number,

2. t^p encodes the form number (or, physics wise, the scaling of the Maurer-Cartan forms),

3. $(-1)^p$ encodes the parity of the corresponding cohomologies

For example for $\mathfrak{sl}(3)$, we have

$$\mathbb{P}_{\mathfrak{sl}(3)}(t) = (1-t^3)(1-t^5) = 1 - t^3 - t^5 + t^8 = (1-t^3) - t^8(t^{-3} - 1)$$

The 3 different expressions call for different interpretations.

1. The first is the usual Hopf decomposition of a Lie algebra in terms of sphere cohomologies. Each binomial $(1-t^x)$ takes into account a rank of the group, the power x takes care of the dimension of the sphere.

2. The second equality describes the 4 cohomologies classes:
   1 denotes the cosntat cohomology,
   $-t^3$ denotes the 3-cocycle (always presents for a Lie algebra) $\omega^{(3)}$,
   $-t^5$ corresponds to the 5-cocycle $\omega^{(5)}$ related to the 3-Casimir,
   t^8 corresponds to the top form $\omega^{(8)}$.

3. The last equality separates the polynomials into two parts, corresponding to Ω_S and Ω_I. The isomorphism is evident, notice that even though it seems trivial, the two parts of the polynomial represent two different cohomologies.

Poincaré Polynomials for Cosets

Following Grueb-Halperin-Vanstone theorem considering the biggest subalgebra $\mathfrak{h}$ with the same rank as $\mathfrak{g}$ we have

$$\mathbb{P}_{\mathfrak{g}/\mathfrak{h}}(t) = \frac{\prod_{q=0}^{\infty}(1 - t^{q+1})^{N_g(q)}}{\prod_{q=0}^{\infty}(1 - t^{q+1})^{N_h(q)}}$$

where $\mathbb{P}_{\mathfrak{g}/\mathfrak{h}}(t)$ is the Poincaré polynomial for the equivariant cohomology of the coset space G/H. The exponents $N_g(q)$ and $N_h(q)$ depend upon the group G and its subgroup H.

In the case of $\mathfrak{sl}(3)$, if we choose $\mathfrak{gl}(2)$, we have $\mathbb{P}^2$ as the coset space (of the real forms SU(3)/U(2)) and therefore we have

$$\mathbb{P}_{\mathbb{P}^2}(t) = \frac{(1-t^4)(1-t^6)}{(1-t^2)(1-t^4)} = 1 + t^2 + t^4$$

The cohomology of $\mathbb{P}^2$ is entirely described in terms of its Kähler form K_2

$$H(\mathbb{P}^2) = \{1, K_2, K_2 \wedge K_2\}$$

corresponding to the three terms of the Poincaré polynomial.

Lie Superalgebras

We use the same symbol $\mathfrak{g}$, generated by even vectors X_a with $a = 1,\ldots,\dim\mathfrak{g}_0$ where $\dim\mathfrak{g}_0$ is the bosonic dimension, and by odd vectors $\hat{X}_\alpha$ with $\alpha = 1,\ldots,\dim\mathfrak{g}_1$ where $\dim\mathfrak{g}_1$ is the fermionic dimension.

$$[X_a, X_b] = f_{ab}{}^c X_c\,, \qquad [X_a, \hat{X}_\alpha] = f_{a\alpha}{}^\beta \hat{X}_\beta\,, \qquad [\hat{X}_\alpha, \hat{X}_\beta]_+ = f_{\alpha\beta}{}^a X_a\,.$$

where $f_{ab}{}^c, f_{a\alpha}{}^\beta\ f_{\alpha\beta}{}^a$ are the structure constants. The Lie algebra respects the parity of the generators.

We introduce the linear functionals V^a, ψ^α defined as

$$\langle V^a, X_b\rangle = \delta^a_b\,, \qquad \langle V^a, \hat{X}_\beta\rangle = 0\,, \qquad \langle \psi^\alpha, X_b\rangle = 0\,, \qquad \langle \psi^\alpha, \hat{X}_\beta\rangle = \delta^\alpha_\beta\,,$$

$$V^a \wedge V^b = -\,V^b \wedge V^a\,, \qquad V^a \wedge \psi^\alpha = \psi^\alpha \wedge V^a\,, \qquad \psi^\alpha \wedge \psi^\beta = \psi^\beta \wedge \psi^\alpha\,,$$

The Chevalley-Eilenberg differential is defined

$$dV^a = f_{\alpha\beta}{}^a \psi^\alpha \wedge \psi^\beta + f_{bc}{}^a V^b \wedge V^c\,, \qquad d\psi^\alpha = f_{a\beta}{}^\alpha V^a \wedge \psi^\beta\,,$$

the differential is nilpotent because of the super Jacobi identities.

FUKS' THEOREM

$$H^{\bullet}_{CE}(\mathfrak{osp}(n\,|\,m)) = \begin{cases} H^{\bullet}_{CE}(\mathfrak{so}(n)), & \text{if} \quad n \geq 2m\,, \\ H^{\bullet}_{CE}(\mathfrak{sp}(m)), & \text{if} \quad n < 2m \end{cases}$$

This theorem states that the cohomology of the Lie Superalgebra (in the present example of the orthosymplectic one) is isomorphic only to one sector of the bosonic subalgebra. So, we wondered what happened at the rest of the cohomology, if any.

In terms of those generators, we can build the space $\Omega(\mathfrak{g}^\star)$ which locally can be written as

$$\omega^{(n)} = \omega_{[a_1\dots a_p],(\alpha_1\dots\alpha_q)} V^{a_1}\wedge\dots\wedge V^{a_p}\wedge\psi^{\alpha_1}\wedge\dots\wedge\psi^{\alpha_q} \quad \in \quad \Omega(\mathfrak{g}^\star)$$

where the coefficients are constant. The total form number is n=p+q.
All elements of the Chevalley-Eilenberg cohomology are scalars with respect to automorphisms.

1. There is no-top form in $\Omega(\mathfrak{g}^\star)$ (where is the Berezinian?)

2. The space $\Omega(\mathfrak{g}^\star)$ is unbounded from above, the form number can be $n \geq 0$

3. There is no Poincaré duality (this can be easily checked by counting the dimensions of each space $\Omega^{(p)}(\mathfrak{g}^\star)$)

4. The introduction of new ingredient is motivated also from supermanifold point of view: integration theory for supermanifold is not equivalent to normal manifold and new ingredients have to be introduced.

We have to introduce a new generator $\delta(\psi^\alpha)$ with the (distribution like) properties

$$\psi^\alpha \delta(\psi^\alpha) = 0\,, \qquad \delta(\psi^\alpha) \wedge \delta(\psi^\beta) = -\,\delta(\psi^\beta) \wedge \delta(\psi^\alpha)\,,$$

$$d\delta(\psi^\alpha) = d\psi^\alpha \delta'(\psi^\alpha) = -\,V^{\alpha\alpha}\delta(\psi^\alpha) + \sum_{\beta\neq\alpha} V^{\alpha\beta}\psi_\beta \delta'(\psi^\alpha)$$

$$\psi^\alpha \delta'(\psi^\alpha) = -\,\delta(\psi^\alpha) \qquad\qquad \delta(\psi^\alpha) \wedge V^a = -\,V^a \wedge \delta(\psi^\alpha)$$

In this enlarged space a generic form has the following generic structure

$$\omega^{(t|r)} = \omega_{[a_1 \ldots a_p],(\alpha_1 \ldots \alpha_q)[\beta_1 \ldots \beta_r]} V^{a_1} \wedge \ldots \wedge V^{a_p} \wedge \psi^{\alpha_1} \wedge \ldots \wedge \psi^{\alpha_q} \wedge \delta^{(g_1)}(\psi^{\beta_1}) \wedge \ldots \wedge \delta^{(g_r)}(\psi^{\beta_r})$$

where $\delta^{(g)}(\psi^\alpha) = \frac{\partial^g}{\partial^g \psi^\alpha}\delta(\psi^\alpha)$ is the g-the derivative of $\delta(\psi^\alpha)$,

The form number of $t = p + q - \sum_{i=1}^{m} g_i$ and there is another grading (PICTURE) = r

$$r = 0\,, \quad t \geq 0\,, \qquad r = m\,, \quad t \leq n\,, \qquad r \neq 0,m\,, \quad t \in \mathbb{Z}$$

Top form (Berezinian)

$$\omega^{(n|m)} = \omega_{[a_1 \ldots a_n],[\beta_1 \ldots \beta_m]} V^{a_1} \wedge \ldots \wedge V^{a_n} \wedge \delta(\psi^{\beta_1}) \wedge \ldots \wedge \delta(\psi^{\beta_m})$$

which transforms as $\omega^{(n|m)} \to \mathrm{Sdet}(J)\, \omega^{(n|m)}$ where J is the Jacobian of the transformation.

1. It plays the role of the “volume” form
2. On supermanifolds $\omega^{(n|m)}$ defines a meaningful geometrical integration theory. For supergroups this corresponds to the Maurer-Cartan forms and $\omega^{(n|m)}$ is the super Haar measure,

The space of form $\Omega^{(p|q)}(\mathfrak{g}^\star)$ can be decomposed according to the two numbers

$$\Omega(\mathfrak{g}^\star) = \oplus_{q=0}^{dim\mathfrak{g}_1} \oplus_p \Omega^{(p|q)}(\mathfrak{g}^\star)$$

$$q = 0\,, \quad p \geq 0\,, \qquad q = m\,, \quad p \leq n\,, \qquad q \neq 0,m\,, \quad p \in \mathbb{Z}$$

The differential d preserves the filtration

$$d : \Omega^{(p|q)}(\mathfrak{g}^\star) \longrightarrow \Omega^{(p+1|q)}(\mathfrak{g}^\star)$$

At the moment is it not clear if the complex $\Omega(\mathfrak{g}^\star)$ is a double complex. Nobody was able to construct a covariant vertical differential (new progresses with Noja, Cremonini and Aschieri)

There are also some new interesting operators, a.k.a. Picture Changing Operators $\mathbb{Y}^{(0|1)}$ and $\mathbb{Z}^{(0|-1)}$

$$\mathbb{Y}^{(0|1)} : \Omega^{(p|q)}(\mathfrak{g}^\star) \longrightarrow \Omega^{(p|q+1)}(\mathfrak{g}^\star)$$

$$\mathbb{Z}^{(0|-1)} : \Omega^{(p|q)}(\mathfrak{g}^\star) \longrightarrow \Omega^{(p|q-1)}(\mathfrak{g}^\star)$$

SPECTRUM OF FORMS FOR A SUPERMANIFOLD

$$
\begin{array}{ccc|ccccccc|ccc}
 & 0 & \xrightarrow{\mathrm{d}} & \Omega^{(0|0)} & \xrightarrow{\mathrm{d}} & \dots & \Omega^{(r|0)} & \dots & \xrightarrow{\mathrm{d}} & \Omega^{(n|0)} & \xrightarrow{\mathrm{d}} & \Omega^{(n+1|0)} & \dots \\
 & {\scriptstyle Z}\uparrow & & {\scriptstyle Z}\uparrow\downarrow{\scriptstyle Y} & & & {\scriptstyle Z}\uparrow\downarrow{\scriptstyle Y} & & & {\scriptstyle Z}\uparrow\downarrow{\scriptstyle Y} & & \downarrow{\scriptstyle Y} & \\
 & \vdots & & \vdots & & & \vdots & & & \vdots & & \vdots & \\
\dots & \Omega^{(-1|s)} & \xrightarrow{\mathrm{d}} & \Omega^{(0|s)} & \xrightarrow{\mathrm{d}} & \dots & \Omega^{(r|s)} & \dots & \xrightarrow{\mathrm{d}} & \Omega^{(n|s)} & \xrightarrow{\mathrm{d}} & \Omega^{(n+1|s)} & \dots \\
 & \vdots & & \vdots & & & \vdots & & & \vdots & & \vdots & \\
 & {\scriptstyle Z}\uparrow & & {\scriptstyle Z}\uparrow\downarrow{\scriptstyle Y} & & & {\scriptstyle Z}\uparrow\downarrow{\scriptstyle Y} & & & {\scriptstyle Z}\uparrow\downarrow{\scriptstyle Y} & & \downarrow{\scriptstyle Y} & \\
\dots & \Omega^{(-1|m)} & \xrightarrow{\mathrm{d}} & \Omega^{(0|m)} & \xrightarrow{\mathrm{d}} & \dots & \Omega^{(r|m)} & \dots & \xrightarrow{\mathrm{d}} & \Omega^{(n|m)} & \xrightarrow{\mathrm{d}} & 0 &
\end{array}
$$

Poincaré Polynomials for Lie Superalgebras

As for Lie algebras, we can construct the Poincaré polynomials/series for Lie superalgebra. Due to the presence of two distinct gradings (form number and picture number) we add to variables t and $\tilde{t}$

$$\mathbb{P}_{\mathfrak{g}}(t, \tilde{t}) = \sum_{q=0}^{dim\, \mathfrak{g}_1} \sum_{p} \dim H^{(p|q)}(d, \Omega)(-1)^{p+q} t^p \tilde{t}^q$$

where the second sum is over the ranges

$$q = 0\,, \quad p \geq 0\,, \qquad q = m\,, \quad p \leq n\,, \qquad q \neq 0, m\,, \qquad p \in \mathbb{Z}$$

Contrary to Poincaré polynomials for Lie algebra, where the Weyl integration formula (Molien-Weyl formula) provides a direct computation tool to derive the corresponding the above expression, in the case of Lie superalgebras, this has never been built. It would be very interesting to construct such a formula. (see recent developments 2106.09353 [hep-th])

The case OSp(1|2)

The generators of the Lie superalgebra are $X_{(\alpha\beta)}, \hat{X}_\alpha$ and they commutation relations are

$$[X_{\alpha\beta}, X_{\gamma\delta}] = \epsilon_{\alpha\gamma}X_{\delta\beta} + \epsilon_{\beta\gamma}X_{\delta\alpha} + \epsilon_{\alpha\delta}X_{\gamma\beta} + \epsilon_{\beta\delta}X_{\gamma\alpha}$$

$$[X_{\alpha\beta}, \hat{X}_\gamma] = \epsilon_{\gamma\alpha}\hat{X}_\beta + \epsilon_{\gamma\beta}\hat{X}_\alpha$$

$$[\hat{X}_\alpha, \hat{X}_\beta]_+ = X_{\alpha\beta}$$

$\alpha, \beta, \gamma = 1{,}2$

The generators of the dual space $\mathfrak{g}^\star$ are $V^{(\alpha\beta)} = V^{(\beta\alpha)}, \psi^\alpha$

$$dV^{(\alpha\beta)} = \psi^\alpha\psi^\beta + (V \wedge V)^{(\alpha\beta)}, \qquad d\psi^\alpha = V^{\alpha\beta}\epsilon_{\beta\gamma}\psi^\gamma,$$

where $(V \wedge V)^{(\alpha\beta)} = \epsilon_{\rho\sigma}V^{(\alpha\rho)} \wedge V^{(\sigma\beta)}$ and $\epsilon_{\alpha\beta} = -\epsilon_{\beta\alpha}, \epsilon_{12} = 1$

1. It preserves the algebraic curve $x_0^2 - x_1^2 - x_2^2 + \theta_1\theta_2 = R^2$
2. The dimension is (3|2), three bosonic dimensions and two fermionic dimensions
3. Rank = 1

Cohomology

$$\mathbb{P}_{\mathfrak{osp}(1|2)}(t,\tilde{t}) = (1-t^3)(1+\tilde{t}^2) = 1 - t^3 + \tilde{t}^2 - t^3\tilde{t}^2 = (1-t^3) + \tilde{t}^2(1-t^3)$$

from which we read

$$H^{(p|q)}(d,\Omega) = \{\omega^{(0|0)}, \omega^{(3|0)}, \omega^{(0|2)}, \omega^{(3|2)}\}$$

$$\omega^{(0|0)} = 1$$

$$\omega^{(3|0)} = \psi^\alpha\psi^\beta\epsilon_{\alpha\alpha'}\epsilon_{\beta\beta'}V^{\alpha'\beta'} + \frac{1}{3!}(V\wedge V\wedge V)$$

$$\omega^{(0|2)} = \delta^2(\psi) + (V\wedge V)^{\alpha\beta}\frac{\partial}{\partial\psi^\alpha}\frac{\partial}{\partial\psi^\beta}\delta^2(\psi)$$

$$\omega^{(3|2)} = V\wedge V\wedge V\wedge\delta^2(\psi)$$

Interpretation of the Poincaré polynomial

1. The second equality gives the different classes of the cohomology. Parity and gradings are indicated
2. The third equality shows the duality between to different type of generators leading again to

$$\Omega(\mathfrak{g}^\star) = \Omega_S \otimes \Omega_I$$

where we denote Ω_S the space of superforms (zero picture) and Ω_I the space of integral forms (picture =2)

and we also notice the duality $$H(d,\Omega_S) \sim H(d,\Omega_I)$$

Cohomology of the Coset

Let us consider the supercoset OSp(1|2)/Sp(2). This is a purely fermionic coset whose geometry is easily described using the corresponding Maurer-Cartan equations as follows

$$R^{\alpha\beta} \equiv dV^{\alpha\beta} - (V \wedge V)^{\alpha\beta} = \psi^\alpha \psi^\beta \,, \qquad \nabla \psi^\alpha \equiv d\psi^\alpha - V^{\alpha\beta} \epsilon_{\beta\gamma} \psi^\gamma = 0$$

where $R^{\alpha\beta}$ is the curvature of Sp(2) and ∇ is the covariant derivative w.r.t. to the connection $V^{\alpha\beta}$

This implies that ψ^α are covariantly closed and therefore any function of them is automatically closed. Then, it is easy to compute the cohomology

$$H(d, \Omega) = \{\omega^{(0|0)}, \omega^{(0|2)}\}$$

where

$$\omega^{(0|0)} = 1 \,, \qquad \omega^{(0|2)} = \epsilon^{\alpha\beta} \delta(\psi^\alpha) \delta(\psi^\beta)$$

This is in accord with the Poincaré polynomial (computed using GHV theorem)

$$\mathbb{P}_{\mathfrak{g}/\mathfrak{h}}(t, \tilde{t}) = 1 + \tilde{t}^2$$

Again we have the duality between $\Omega_S = \{\omega^{(0,0)}\}$ and $\Omega_I = \{\omega^{(0|2)}\}$.

Using the ring structure we can reconstruct the cohomology of $\mathfrak{osp}(1\,|\,2)$ from the cohomology of its subalgebra $\mathfrak{sl}(2)$ and its coset

$$\widetilde{\omega}^{(0|0)} = \omega^{(0|0)}_{sl(2)} \wedge \omega^{(0|0)}_{osp/sp}\,, \qquad \widetilde{\omega}^{(3|0)} = \omega^{(3|0)}_{sl(2)} \wedge \omega^{(0|0)}_{osp/sp}$$

$$\widetilde{\omega}^{(0|2)} = \omega^{(0|0)}_{sl(2)} \wedge \omega^{(0|2)}_{osp/sp}\,, \qquad \widetilde{\omega}^{(3|2)} = \omega^{(3|0)}_{sl(2)} \wedge \omega^{(0|2)}_{osp/sp}$$

up to d-exact terms.

So, we are able to reconstruct completely the cohomology of the original algebra in terms of the cosets and of its subgroup. This can be formalised into the usual procedure of spectral sequence. Indeed, the result can be computed also in that way.

There is another way to interpret the result by considering the two complexes

$$0 \to \Omega^{(0|0)} \to \Omega^{(1|0)} \to \Omega^{(2|0)} \to \ldots \to \Omega^{(n|0)} \to \Omega^{(n+1|0)} \to \ldots$$

$$\ldots \to \Omega^{(-1|2)} \to \Omega^{(0|2)} \to \Omega^{(1|2)} \to \Omega^{(2|2)} \to \Omega^{(3|2)} \to 0$$

The first complex is of the superforms Ω_S and the second one of Ω_I of the integral forms.
However, this point of view is misleading since one is erroneously brought to think that there is double complex, but nobody has found the vertical differential.

The cohomology class $\widetilde{\omega}^{(0|2)}$ can be used to map vertically $H^{(p|0)} \to H^{(p|2)}$

The crucial case OSp(2|2)

Let us come to the crucial case, this is a new result that requires a deeper understanding.

1. It is rank =2 example
2. The bosonic subgroup SO(2)xSp(2)
3. It preserves the algebraic curve $x_0^2 - x_1^2 - x_2^2 + y^2 + \theta_1\theta_2 + \theta_1'\theta_2' = R^2$
4. The Lie superalgebra is generated by the (4|4) vectors $X_{\alpha\beta}, X_0, \hat{X}^I_\alpha$ (with $I = 1{,}2$)

$$[X_{\alpha\beta}, X_{\gamma\delta}] = \epsilon_{\alpha\gamma}X_{\delta\beta} + \epsilon_{\beta\gamma}X_{\delta\alpha} + \epsilon_{\alpha\delta}X_{\gamma\beta} + \epsilon_{\beta\delta}X_{\gamma\alpha}$$

$$[X_{\alpha\beta}, X_0] = 0\,, \quad [X_0, X_0] = 0\,, \quad [X_{\alpha\beta}, \hat{X}^I_\gamma] = \epsilon_{\gamma\alpha}\hat{X}^I_\beta + \epsilon_{\gamma\beta}\hat{X}^I_\alpha$$

$$[X_0, \hat{X}^I_\alpha] = \epsilon^{IJ}\eta_{IJ}\hat{X}^J_\alpha \qquad [\hat{X}^I_\alpha, \hat{X}^J_\beta]_+ = X_{\alpha\beta}\eta^{IJ} + X_0\epsilon^{IJ}\epsilon_{\alpha\beta}$$

The linear functionals are generated by $V^{\alpha\beta}, V_0, \psi^\alpha_I$ with the differential

$$dV^{\alpha\beta} = \psi^\alpha_I \wedge \psi^\beta_J \eta^{IJ} + (V \wedge V)^{\alpha\beta}\,, \qquad dV_0 = \psi^\alpha_I \wedge \psi^\beta_J \epsilon^{IJ}\epsilon_{\alpha\beta}$$

$$d\psi^\alpha_I = V^{\alpha\beta}\epsilon_{\beta\gamma} \wedge \psi^\gamma_I + V_0\epsilon_{IJ}\eta^{JK} \wedge \psi^\alpha_K$$

η_{IJ} is the euclidean metric.

using the spectral sequences (using the decomposition under the subgroup OSp(1|2)) we obtain the result

$$\mathbb{P}_{OSp(2|2)}(t,\tilde{t}) = (1-t^3)(1-t\tilde{t}^2)(1+\tilde{t}^2) = (1-t^3)(1+\tilde{t}^2-t\tilde{t}^2-t\tilde{t}^4)$$
$$= (1-t^3)+(1-t^3)\tilde{t}^2-(1-t^3)t\tilde{t}^2+(1-t^3)t\tilde{t}^4$$

where we have selected four sectors corresponding to following decomposition

$$\Omega(\mathfrak{g}^\star) = \Omega_S(\mathfrak{g}^\star)\oplus\Omega_P(\mathfrak{g}^\star)\oplus\Omega_I(\mathfrak{g}^\star)$$

where we added the sector of PSEUDOFORMS $\Omega_P(\mathfrak{g}^\star)$ (corresponding to the two central pieces of the equation). Locally, they can be written as

$$\omega^{(t|2)} = \omega_{[a_1\dots a_p],(\alpha_1\dots\alpha_q)[\beta_1\beta_2]}V^{a_1}\wedge\dots\wedge V^{a_p}\wedge\psi^{\alpha_1}\wedge\dots\wedge\psi^{\alpha_q}\wedge\delta^{(g_1)}(\psi^{\beta_1})\wedge\delta^{(g_2)}(\psi^{\beta_2})$$

The expressions $\delta^{(g)}(\psi^\alpha)$ are not tensorial under automorphisms

$$\delta(\psi'^{\alpha}) = \delta(\psi^\alpha+\Lambda^\alpha{}_\beta\psi^\beta) = \delta(\psi^\alpha)+\Lambda^\alpha{}_\beta\psi^\beta\frac{\partial}{\partial\psi^\alpha}\delta(\psi^\alpha)+\mathcal{O}(\Lambda)$$

Notice that increasing the power of ψ^α we can increase the number of derivatives of $\delta^{(g)}(\psi^\alpha)$, and the form number does not change. This means that the subspaces $\Omega^{(p|2)}(\mathfrak{g}^\star)$ are infinitely generated.

$$\dots \to \Omega^{(-n|2)} \to \dots \to \Omega^{(-1|2)} \to \Omega^{(0|2)} \to \Omega^{(1|2)} \to \dots \to \Omega^{(n|2)} \to \dots$$

is unbounded from below and from above.

The geometry of pseudoforms is not yet studied in detail.

In works 1907.07152 and 1912.10807 with C.A. Cremonini we show how the computations are consistent and some physical applications.

Cohomology of the Cosets of OSp(2|2)

In order to compute the cohomology it is convenient as shown above to compute the cohomology of the coset OSp(2|2)/SO(2)xSp(2). The geometry is described by

$$R^{\alpha\beta} \equiv dV^{\alpha\beta} - (V \wedge V)^{\alpha\beta} = \psi^\alpha_I \psi^\beta_J \eta^{IJ} , \qquad R \equiv dV_0 = \psi^\alpha_I \psi^\beta_J \epsilon^{IJ} \epsilon_{\alpha\beta}$$

$$\nabla \psi^\alpha_I \equiv d\psi^\alpha_I - V^{\alpha\beta} \epsilon_{\beta\gamma} \psi^\gamma_I - V_0 \epsilon_{IJ} \eta^{JK} \psi^\alpha_K = 0$$

where $R^{\alpha\beta}, R$ are the curvature of Sp(2) and of SO(2).

The cohomology of the coset (which is purely fermionic) is described entirely by the generators ψ^α_I. Any invariant expression is clearly closed.

The computation can be done using again the GHV theorem

$$\mathbb{P}_{\mathfrak{g}/\mathfrak{h}}(t, \tilde{t}) = \frac{(1-t^4)(1-t^2\tilde{t}^2)(1+\tilde{t}^2)}{(1-t^2)(1-t^4)} = \frac{(1-t^2\tilde{t}^2)(1+\tilde{t}^2)}{(1-t^2)} = \frac{1}{(1-t^2)} + \tilde{t}^2 + \frac{1}{(1-\frac{1}{t^2})}\tilde{t}^4$$

Let us discuss the three different pieces

The first piece $\frac{1}{(1-t^2)} \quad \Longrightarrow \quad \Omega_S(\mathfrak{g}/\mathfrak{h})$

corresponds to superforms. Therefore the explicit expressions should be written in terms of ψ_I^α. The only invariant combination is

$$K_2 = \psi_I^\alpha \psi_J^\beta \epsilon^{IJ} \epsilon_{\alpha\beta}$$

which is a 2-form, commuting and closed $\nabla K_2 = 0$ and not exact. Therefore any power of it represents a cohomology class $K_2, K_2 \wedge K_2, K_2 \wedge K_2 \wedge K_2, \ldots$ The counting of them reproduces exactly the Poincaré series above.

The second piece $\frac{1}{(1-\frac{1}{t^2})} \tilde{t}^4 \quad \Longrightarrow \quad \Omega_I(\mathfrak{g}/\mathfrak{h})$

The factor $\tilde{t}^4$ corresponds to the factor $\delta^4(\psi) \equiv \delta(\psi_1^1) \wedge \ldots \wedge \delta(\psi_2^2)$

The first factor $\frac{1}{(1-\frac{1}{t^2})}$ is relate to the derivative of the delta's $\iota_2 \delta^4(\psi) \equiv \epsilon_{\alpha\beta} \epsilon^{IJ} \frac{\partial}{\partial \psi_I^\alpha} \frac{\partial}{\partial \psi_J^\beta} \delta^4(\psi)$

Again we can reiterate the differential operator ι_2 at wish, and since it counts -2 as form degree, resuming all contributions we get the interesting factor.

$$\iota_2^n \delta^4(\psi) \Longrightarrow t^{-2n}\tilde{t}^4$$

Let us come to the last piece: $\tilde{t}^2$. This is due to pseudoforms at picture 2.

It is rather difficult to build them. The strategy however was to first notice that the combinations

$$F(\psi^1) = \epsilon^{\alpha\beta}\delta(\psi^1_\alpha)\delta(\psi^1_\alpha)\,, \qquad F(\psi^2) = \epsilon^{\alpha\beta}\delta(\psi^2_\alpha)\delta(\psi^2_\alpha)\,,$$

are invariant under Sp(2) transformations. Therefore we need to impose the invariance under SO(2) symmetry. This is achieved by requiring if it exists a combination of them such that satisfy the differential equation

$$TG(F(\psi_1), F(\psi_2), \psi^I_\alpha) = \left(\psi_1^\alpha \frac{\partial}{\partial \psi_2^\alpha} - \psi_2^\alpha \frac{\partial}{\partial \psi_1^\alpha}\right) G(F(\psi_1), F(\psi_2), \psi^I_\alpha) = 0$$

this equation can be solve and it has a unique solution which can be easily expressed as the Taylor expansion of a modified Bessel function of first kind I_0

$$G(\psi^1, \psi^2) = \epsilon^{\alpha\beta}\delta(\psi^1_\alpha)\delta(\psi^1_\alpha) - \frac{1}{4}\psi^2_\alpha\psi^2_\beta \frac{\partial}{\partial\psi^1_\alpha}\frac{\partial}{\partial\psi^1_\beta}\epsilon^{\alpha\beta}\delta(\psi^1_\alpha)\delta(\psi^1_\alpha) + \dots\,,$$

Now reconstructing the full cohomology from the equivariant cohomology is rather easy, and this confirms the result obtained by spectral sequences

$$\begin{aligned}\mathbb{P}_{OSp(2|2)}(t,\tilde{t}) &= (1-t^3)(1-t\tilde{t}^2)(1+\tilde{t}^2) = (1-t^3)(1+\tilde{t}^2-t\tilde{t}^2-t\tilde{t}^4) \\ &= (1-t^3)+(1-t^3)\tilde{t}^2-(1-t^3)t\tilde{t}^2+(1-t^3)t\tilde{t}^4 \\ &= (1-t^3)+(1-t^3)(1-t)\tilde{t}^2+(1-t^3)t\tilde{t}^4\end{aligned}$$

where we have

$$H^{(p|0)}(d,\Omega_S) \longrightarrow (1-t^3)$$

$$H^{(p|2)}(d,\Omega_P) \longrightarrow (1-t^3)(1-t)\,\tilde{t}^2$$

$$H^{(p|4)}(d,\Omega_I) \longrightarrow (1-t^3)\tilde{t}^4$$

all cohomology groups are isomorphic (using again the duality) and they have been checked using spectral sequences, Poincaré polynomials and explicit computations.

Conclusions

In general we have

$$\mathbb{P}_{\mathfrak{osp}(2|2n)}(t, \tilde{t}) = \mathbb{P}_{\mathfrak{sp}(2n)}(t)(1 - t\tilde{t}^{2n})(1 + \tilde{t}^{2n})$$

which can be analysed along the same lines. One can establish the spectral sequence argument using all possible subgroups of OSp(2|2n).

1. We have found unexpected and non trivial results in the cohomology of Lie superalgebra in the trivial module.
2. We have found a way to compute the cohomologies using the Poincaré polynomials (or series) extending the theorem by Grueb-Halperin-Vanstone.
3. We have completed the Fuks theorem to the complete set of forms for a Lie superalgebra.
4. We have found invariant pseudoforms in the spectrum of those cohomologies.

Thank you for your attention