# Kaluza-Klein reductions of non-standard $D = 5$ maximally supersymmetric backgrounds

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# Killing spinors

- A supergravity bosonic background is a solution of the equations of motion with fermionic fields set to zero. Bosonic fields are then invariant under supersymmetry transformations.
- Not so fermionic fields. The variation of the fermionic field $\Psi$ associated to the metric, the gravitino, takes the form

$$\delta_\epsilon \Psi = \mathcal{D}\epsilon, \qquad \mathcal{D} = \nabla - \beta, \quad \beta \in \Gamma(T^*M \otimes \mathrm{End}(\Sigma)),$$

  where $\nabla$ is the LC connection on $M$, $\Sigma$ the spinor bundle.
- Killing spinors are solutions of $\mathcal{D}\epsilon = 0$ (+ algebraic constraints from the variations of other fermionic fields if present).
- The fraction of preserved susy is $\nu = \dim(\mathrm{Ker}\,\mathcal{D})/\,\mathrm{rk}(\Sigma)$. If $\nu = 1$ the background is maximally supersymmetric.

# The Killing superalgebra

- Extend these definitions to any supergravity-like theory defining Killing spinors to be parallel section with respect to some connection on the spinor bundle.
- Let $\mathfrak{h}_{\bar{0}}$ be the space of Killing vector fields preserving any additional bosonic field, $\mathfrak{h}_{\bar{1}}$ the space of Killing spinors. The Killing superalgebra (KSA) is $\mathfrak{h} = \mathfrak{h}_{\bar{0}} \oplus \mathfrak{h}_{\bar{1}}$.
  - The bracket on $\mathfrak{h}_{\bar{0}}$ is the usual bracket of vector fields;
  - The bracket $[\mathfrak{h}_{\bar{0}}, \mathfrak{h}_{\bar{1}}]$ is the spinorial Lie derivative;
  - The bracket $[\mathfrak{h}_{\bar{1}}, \mathfrak{h}_{\bar{1}}]$ is the Dirac current.
- In many cases the "supergravity equations of motion" can be obtained by taking the $\Gamma$-trace of the associated curvature $R_{\mathcal{D}}$.
- For simply connected manifolds, vanishing of $R_{\mathcal{D}}$ is equivalent to maximal supersymmetry.

# The Killing superalgebra (2)

The Poincaré superalgebra $\mathfrak{g}$ of $V$ has a $\mathbb{Z}$-grading $\mathfrak{g}_0 \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_{-2}$ with $\mathfrak{g}_0 = V$, $\mathfrak{g}_{-2} = \mathfrak{so}(V)$, $\mathfrak{g}_{-1}$ the spinor representation.

- Homogeneity theorem (Figueroa-O'Farrill, Hustler): if $\dim(\mathfrak{h}_{\bar{1}})/\dim(\mathfrak{g}_{-1}) > 1/2$ then $[\mathfrak{h}_{\bar{1}}, \mathfrak{h}_{\bar{1}}] = \mathfrak{g}_{-2}$ and the background is determined by the KSA up to local isometry.
- (Figueroa-O'Farrill, Santi) The KSA $\mathfrak{k}$ of a bosonic background is a filtered subdeformation of $\mathfrak{g}$, i.e. $\mathfrak{k}$ is a filtered superalgebra and its associated graded superalgebra is isomorphic (as a graded superalgebra) to some graded subalgebra $\mathfrak{a} \subset \mathfrak{g}$. Concretely, $\mathfrak{k} \simeq \mathfrak{a}$ as a vector space, but the brackets can differ by terms of positive degree.

# The Killing superalgebra (3)

- Study/classification of supersymmetric backgrounds: take the KSA as the starting point and classify filtered subdeformations of the Poincaré superalgebra.
- Filtered subdeformations are classified via Spencer cohomology. From the cohomological data it is possible to reconstruct the gravitino connection $\mathcal{D}$.
- In some cases, e.g. $D = 11$ and minimal $D = 4$, the $\mathcal{D}$ obtained is precisely the supergravity one. Other cases, e.g. $D = 5, 6$, allow for both standard and non-standard supergravity theories.

## The case $D = 5$

Results by Beckett, Figueroa-O'Farrill.

- The Spencer cohomology data corresponds to a 2-form $C$ and an $\mathfrak{sp}(1)$-valued 1-form $F$.
- The corresponding gravitino connection is $\mathcal{D} = \nabla - \beta$, $\beta$ a 1-form with values in the endomorphisms of the spinor bundle,

$$\beta_X s = \frac{1}{4} X^\flat \cdot C \cdot s - \frac{3}{4} C \cdot X^\flat \cdot s - \frac{1}{8} X^\flat \cdot F \cdot s - \frac{3}{8} F \cdot X^\flat \cdot s.$$

- $F = 0 \Rightarrow \mathcal{D}$ connection of standard minimal $D = 5$ SUGRA.
- Maximally supersymmetric backgrounds have either $C = 0$ or $F = 0$. Let us focus on the non-standard case $C = 0 \neq F$.
- Then $F = \varphi \otimes r$ with $\varphi$ parallel, $r$ a fixed element of $\mathfrak{sp}(1)$.

# Non-standard maximally supersymmetric backgrounds

For $C = 0 \neq F = \varphi \otimes r$ there are 3 classes of maximally supersymmetric backgrounds depending on the causal character of $\varphi$. They all are locally symmetric spaces.

- If $\varphi$ is timelike then $M = -\mathbb{R} \times S^4_R$ and $R^2|\varphi|^2 = -2$.
- If $\varphi$ is spacelike then $M = \mathbb{R} \times \mathrm{Ads}^4_R$ and $R^2|\varphi|^2 = 2$.
- if $\varphi$ is null then $M$ is a Cahen-Wallach space.

We (J. Figueroa-O'Farrill, G.F.) studied the Kaluza-Klein reduction of these backgrounds.

- We found a number of non-homogeneous quotients whose KSA is not a filtered subdeformation of $N = 1$, $D = 4$ super-Poincaré. These quotients admit $N = 1$ supersymmetry and could be used to construct new rigidly supersymmetry field theories.

# The Kaluza-Klein reduction process in a nutshell

Let $M$ be one of the non-standard maximally supersymmetric backgrounds, $\Gamma$ a 1-parameter subgroup of $\mathrm{Iso}(M)$. The Kaluza-Klein reduction of $M$ along $\Gamma$ is the quotient $M/\Gamma$.

- Classify 1-parameter subgroups of $\mathrm{Iso}(M)$ and identify those subgroups leading to a smooth quotients admitting a Riemannian or Lorentzian metric.
- Determine the amount of preserved supersymmetry of the quotient and study its geometry.

# 1-parameter subgroups

- Let $G = \mathrm{Iso}(M)$, $g \in G$, $\Gamma$ a 1-parameter subgroup. We need to identify the isometric quotients

$$M/\Gamma \simeq (g \cdot M)/(g\Gamma g^{-1}).$$

- At the Lie algebra level

$$\Gamma \subset G \;\leftrightarrow\; \text{KVF } \xi_X|_p = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mathrm{e}^{-tX} \cdot p, X \in \mathfrak{g},$$
$$\Gamma \sim g\Gamma g^{-1} \;\leftrightarrow\; \xi_X \sim \xi_{\lambda g X g^{-1}}, \lambda \in \mathbb{R}^\times,$$

hence the problem is equivalent to classifying 1-dimensional subalgebra of $\mathfrak{g}$ under the equivalence relation

$$X \sim \lambda g X g^{-1}, \text{ for any } g \in G,\ \lambda \in \mathbb{R}^\times.$$

- Well-defined pseudo-Riemannian quotient $\Rightarrow |\xi_X| \neq 0$ everywhere so $\xi_X$ is either spacelike or timelike.

# Supersymmetry preserved by the quotient

- (Figueroa-O'Farrill, Simon) Let $M$ be simply connected spin, $\xi$ a KVF. Then $M/\Gamma_\xi$ is spin if and only if the $\Gamma_\xi$ action on $M$ lifts to the spin bundle in a $\Gamma_\xi$-equivariant way.
- This always happens if $\Gamma_\xi$ is a line, which is the case for all the examples we consider. Then a Killing spinor (KS) on $M$ which is invariant under the $\Gamma_\xi$ action descends to a spinor on $M/\Gamma_\xi$.
- We do not specify a supergravity theory on the quotient and define KS on $M/\Gamma_\xi$ to be invariant KS on $M$.
- $\epsilon$ is invariant under the $\xi$ action if $L_\xi \epsilon = \nabla_\xi \epsilon + \frac{1}{4} \mathrm{d}\xi^\flat \cdot \epsilon = 0$.
- $\epsilon$ KS $\Rightarrow \mathcal{D}_\xi \epsilon = \nabla_\xi \epsilon - \beta_\xi \epsilon = 0$ hence the condition is algebraic

$$\beta_\xi \epsilon + \frac{1}{4} \mathrm{d}\xi^\flat \cdot \epsilon = 0.$$

# Geometry of the quotient

- KK geometry: $\pi : M \to M/\Gamma_\xi$ is a principal $\Gamma$-bundle. The metrics $g$ on $M$ and $h$ on $M/\Gamma_\xi$ are related by

$$g = \pi^* h + \frac{\xi^\flat \otimes \xi^\flat}{g(\xi, \xi)}.$$

- In adapted local coordinates with $\xi = \partial_z$,

$$g = \pi^* h \pm \mathrm{e}^{2\pi^* \phi} (\mathrm{d}z + A)^2$$

  with $\phi \in C^\infty(M/\Gamma_\xi)$, $A$ basic.
- Let $G = \mathrm{Iso}(M)$. Any subgroup $H \subset N_\Gamma(G)$ of the normaliser of $\Gamma$ in $G$ acts on $M/\Gamma$ by isometries. Therefore

$$N_\Gamma(G)/\Gamma \subset \mathrm{Iso}(M/\Gamma).$$

- The quotient may have additional "accidental" isometries.

# The Cahen-Wallach background

- The background $M$ is an indecomposable Cahen-Wallach (CW) pp-wave with metric, $|x|^2 = x_1^2 + x_2^2 + x_3^2$,

$$g = 2\mathrm{d}x^+\mathrm{d}x^- + |x|^2(\mathrm{d}x^-)^2 + |\mathrm{d}x|^2.$$

  It has a parallel null vector field $\varphi = \sqrt{2}\partial_+$ and $W = 0$.

- It has isometry algebra $\mathfrak{g} \rtimes \mathfrak{so}(3)$, where $\mathfrak{g}$ has generators $(e_i, e_i^*, e_+, e_-)$, $i = 1, 2, 3$, and non-trivial brackets

$$[e_-, e_i] = e_i^*, \quad [e_-, e_i^*] = e_i, \quad [e_i^*, e_j] = \delta_{ij} e_+.$$

- Taking generators $(V_i)$ of $\mathfrak{so}(3)$, $[V_i, V_j] = -\epsilon_{ijk} V_k$ we have

$$[V_i, e_j] = -\epsilon_{ijk} e_k, \quad [V_i, e_j^*] = -\epsilon_{ijk} e_k^*, \quad [V_i, e_\pm] = 0.$$

- The corresponding KVFs are, for $R_{ij} = x_i\partial_j - x_j\partial_i$,

$$\xi_{e_\pm} = \partial_\pm, \quad \xi_{V_1} = R_{23}, \ \xi_{V_2} = R_{31}, \ \xi_{V_3} = R_{12},$$

$$\xi_{e_i} = \cosh x^- \partial_i - x^i \sinh x^- \partial_+, \ \xi_{e_i^*} = x^i \cosh x^- \partial_+ - \sinh x^- \partial_i.$$

# 1-parameter subgroups

- Let $\xi$ be a KVF of the CW background and assume that $|\xi|$ never vanishes. Then there are coordinates such that $\xi$ is the KVF associated to one of the following elements in $\mathfrak{g} \rtimes \mathfrak{so}(3)$:

$$\begin{aligned} X_1 &= e_- + bV_3 + \gamma e_+, \quad \gamma > 0, \\ X_2 &= V_3 + ce_3, \; c \neq 0, \\ X_4^{\pm} &= V_3 + c(e_3 \pm e_3^*), \quad c \neq 0, \\ X_6 &= e_3, \\ X_8^{\pm} &= e_3 \pm e_3^*, \\ X_9 &= e_3 + d_3 e_3^* + d_1 e_1^*, \quad d_1 \neq 0. \end{aligned}$$

- There are 3 further inequivalent KVFs but the quotient does not admit a pseudo-riemannian metric.
- The corresponding KVFs are all spacelike and the 1-parameter groups they generate all have the topology of a line.

# Fraction of preserved supersymmetry

The KVFs preserving a fraction $\nu > 0$ of supersymmetry are:

| $X \in \mathfrak{g} \rtimes \mathfrak{so}(3)$ | Condition | $\nu$ |
|---|---|---|
| $X_9$ | - | $1/2$ |
| $X_6$ | - | $1/2$ |
| $X_8^{\pm}$ | - | $1/2$ |
| $X_1$ | $4b^2 = 9$ or $4b^2 = 1$ | $1/4$ |

# Geometry of the quotient

- All the quotients $M/\Gamma_{\xi_{X_i}}$ are lorentzian 4-manifolds with the topology of $\mathbb{R}^4$.
- The Lie algebra $\mathfrak{l}$ of the normaliser of $\Gamma_{\xi_{X_i}}$ is given by

| $X \in \mathfrak{g} \rtimes \mathfrak{so}(3)$ | $\mathfrak{l}$ | generators |
|---|---|---|
| $X_1,\ b = 0$ | $\mathbb{R} \oplus \mathfrak{so}(3)$ | $e_+;\ V_1, V_2, V_3$ |
| $X_1,\ b \neq 0$ | $\mathbb{R} \oplus \mathfrak{so}(2)$ | $e_+;\ V_3$ |
| $X_2$ | $\mathbb{R} \oplus \mathfrak{so}(2)$ | $e_+;\ V_3$ |
| $X_4^{\pm}$ | $\mathbb{R} \oplus \mathfrak{so}(2)$ | $e_+;\ V_3$ |
| $X_6$ | $\mathfrak{h} \rtimes \mathfrak{so}(2)$ | $e_+, e_2, e_2^*, e_1, e_1^*;\ V_3$ |
| $X_8^{\pm}$ | $\mathfrak{h} \rtimes (\mathbb{R} \oplus \mathfrak{so}(2))$ | $e_+, e_2, e_2^*, e_1, e_1^*;\ e_-, V_3$ |
| $X_9$ | $\mathfrak{h}$ | $e_+, e_2, e_2^*, e_1 + d_1 e_3^*, e_1^*$ |

where $\mathfrak{h}$ is the 5-dimensional Heisenberg algebra.

# The $\nu = \frac{1}{2}$ quotients

- Let us look at the $\nu = \frac{1}{2}$ cases $X_6$, $X_8^{\pm}$, $X_9$ in more detail.
- Let $\xi$ be the KVF generating $\Gamma$, $(\chi_1, \chi_2, \chi_3, \chi_4)$ a basis for $\xi^{\perp}$.

$$g = \frac{\xi^\flat \otimes \xi^\flat}{g(\xi,\xi)} + \sum_{i=1}^{4} C_{ij} \chi_i^\flat \odot \chi_j^\flat,$$

where the coefficients $C_{ij} = C_{ji}$ need to be determined.
- The quotient metric $h$ on $M/\Gamma$ is $h = \sum_{i=1}^{4} C_{ij} \chi_i^\flat \odot \chi_j^\flat$ which we need to rewrite in terms of coordinates $\tilde{x}_i$ well-defined on the quotient, i.e. $\xi(\tilde{x}_i) = 0$.

# The $\nu = \frac{1}{2}$ quotients: $X_6$ and $X_8^{\pm}$

- $X_6 = e_3$, $\xi_{X_6} = \cosh x^- \partial_3 - x_3 \sinh x^- \partial_+$,
  $\xi^\perp = \mathrm{Span}\left(\partial_+, \partial_1, \partial_2, \chi = x_3 \sinh x^- \partial_3 + \cosh x^- \partial_-\right)$.
- We find $h = \mathrm{d}x_1^2 + \mathrm{d}x_2^2 + \mathrm{d}x^- \left(C_1 \mathrm{d}x^- + C_2 \chi^\flat\right)$, with
  $C_1 = -|x|^2 - x_3^2 \tanh^2(x^-), \quad C_2 = 2/\cosh(x^-)$.
- In terms of coordinates $(x_1, x_2, x^-, \tilde{x}^+ = x^+ + \frac{x_3^2}{2}\tanh(x^-))$ well-defined on the quotient we have

  $$h = 2\mathrm{d}x^- \mathrm{d}\tilde{x}^+ + (x_1^2 + x_2^2)(\mathrm{d}x^-)^2 + \mathrm{d}x_1^2 + \mathrm{d}x_2^2.$$

  which we recognise as the metric of a $D = 4$ CW space.
- Comparing with the isometries inherited from above, we see that in this case the quotient has an "accidental" isometry.
- $X_8^{\pm}$ gives the same quotient as the $X_6$, in agreement with the fact that $X_6$ can be viewed as an asymptotic limit of $X_8^{\pm}$. There are no "accidental" isometries.

# The $\nu = \frac{1}{2}$ quotients: $X_9$

- $X_9 = e_3 + d_3 e_3^* + d_1 e_1^*$. For simplicity we take $d_3 = 0$.
  $\xi_{X_9} = \cosh x^- \partial_3 - d_1 \sinh x^- \partial_1 + (d_1 x_1 \cosh x^- - x_3 \sinh x^-)\partial_+$,
  $\xi^\perp = \mathrm{Span}(\partial_+, \partial_2, \chi_1, \chi_2)$ with

$$\chi_1 = d_1 \sinh x^- \partial_3 + \cosh x^- \partial_1,$$
$$\chi_2 = d_1 \sinh x^- \partial_- + (d_1 x_1 \cosh x^- - x_3 \sinh x^-)\partial_1.$$

- In terms of well-defined coordinates $(\tilde{x}^+, x^-, u, x^2)$,

$$\tilde{x}^+ = x^+ + \frac{x_1^2}{2} \coth x^- + \frac{x_3^2}{2} \tanh x^-, u = d_1 x^3 \sinh x^- + x^1 \cosh x^-,$$

  the quotient metric is

$$h = 2\mathrm{d}\tilde{x}^+ \mathrm{d}x^- + x_2^2 (\mathrm{d}x^-)^2 + \frac{(\mathrm{d}u - 2u \coth(2x^-)\mathrm{d}x^-)^2}{\cosh^2 x^- + d_1^2 \sinh^2 x^-} + \mathrm{d}x_2^2.$$

- The metric $h$ is not conformally flat.

# Other reductions

- Reductions of $-\mathbb{R} \times S^4$ lead to riemannian metrics on $S^4$. We get a 2-parameter family of $\nu = \frac{1}{4}$ reductions which becomes $\nu = \frac{1}{2}$ if one of the parameters vanishes. The $\nu = \frac{1}{2}$ case is a cohomogeneity one metric with isometry group $O(3) \times O(2)$.
- For $\mathbb{R} \times \mathrm{AdS}^4$ we find 6 inequivalent 1-parameter subgroups leading to lorentzian or riemannian supersymmetric quotients. In four cases the quotient has $\nu = \frac{1}{4}$.
- The other two cases have $\nu = \frac{1}{2}$. The quotient is diffeomorphic to $\mathbb{R}^4$ with a lorentzian (riemannian) cohomogeneity one action of $O(2,1) \times O(2)$ (resp. $O(3) \times O(2)$). In both cases the geometry is similar to the $\nu = \frac{1}{2}$ quotient of $-\mathbb{R} \times S^4$ and is not conformally flat.

# So what? Conclusions and outlook

- We have classified the Lorentzian and Riemannian KK reductions of the maximally supersymmetric backgrounds of a non-standard $D = 5$ supergravity theory, determined the fraction of supersymmetry preserved by the reduction and studied the geometry of the quotient.
- The most interesting quotients are the three distinct $\nu = \frac{1}{2}$ lorentzian ones arising from the reduction of CW (two cases) and $\mathrm{AdS}^4$ (one case). Are they new?
- Maximally supersymmetric Killing superalgebras arising as filtered subdeformations of the $N = 1$ Poincaré superalgebra in $D = 4$ have been classified by De Medeiros, Figueroa-O'Farrill, Santi. They corresponding geometries are all conformally flat.

- The 4-dimensional CW reduction is isometric to the metric on the Nappi-Witten group and is contained in their list.
- The remaining two quotients are not conformally flat hence the associated Killing superalgebras are not filtered subdeformations of the $N = 1$ 4-dimensional Poincaré superalgebra. They provide novel examples of spacetimes admitting $N = 1$ rigid supersymmetry.
- It would be interested to explicitly determining the associated Killing superalgebras, which would also be the first step in the construction of rigidly supersymmetric theories on these reductions.
- It would be interesting to consider reductions by null KVFs which could lead to novel four-dimensional supersymmetric Newton–Cartan geometries.

Thank you very much for your attention!