Super Geometric Quantization

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References

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- C. Carmeli, R. Fioresi, V. S. Varadarajan, *Highest weight Harish-Chandra supermodules and their geometric realizations*, Transformation Groups, 2019.
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Plan of the Talk

- Linear Supergeometry
- The functor of points: supermanifolds and superschemes
- Supergroups, SHCP, representations
- Unitary Representations: infinitesimal and global
- Super Quantization and Super Moment map

Linear Supergeometry

- **Super vector space**: $V = V_0 \oplus V_1$.
  Example: $k^{m|n} := k^m \oplus k^n$, superspace of dimension $m|n$, $k = \mathbb{R}, \mathbb{C}$.
- **Superalgebra**: super vector space with product respecting parity, p.
- **Commutative Superalgebra**: $ab = (-1)^{p(a)p(b)} ba$, (or $p(a) = |a|$)
  Prototype of commutative superalgebra: polynomial superalgebra.

$$k[x_1 \dots x_m, \xi_1 \dots \xi_n] = Sym(x_1 \dots x_m) \otimes \wedge(\xi_1 \dots \xi_n)$$

- **Supermodules**: modules over commutative superalgebras.
  $A^{m|n} := A \otimes k^{m|n}$ free A-module of dimension $m|n$.
  $A_0^{m|n} := A_0 \otimes k^m \oplus A_1 \otimes k^n = \{(a_1, \dots, a_m, \alpha_1, \dots, \alpha_n)\}$
  $A_1^{m|n} := A_0 \otimes k^n \oplus A_1 \otimes k^m = \{(\alpha_1, \dots, \alpha_m, a_1, \dots, a_n)\}$

EVEN: latin letters, ODD: greek letters.

The functor of points approach

The A points of the supervector space $V = k^{m|n}$ are:

$$V(A) := (A \otimes V)_0 = A_0^{m|n} = A_0 \otimes k^m \oplus A_1 \otimes k^n = \{(a_1, \dots a_m, \alpha_1, \dots, \alpha_n)\}$$

where A is a **commutative** (always!) superalgebra.

We may interpret them as:

$$\begin{aligned} V(A) &= \{f : k[x_1 \dots x_m, \xi_1 \dots \xi_n] \longrightarrow A\} = \\ &= \mathrm{Hom}_{(\mathrm{salg})}(k[x_1 \dots x_m, \xi_1 \dots \xi_n], A) \end{aligned}$$

In fact:

$$\begin{array}{cccc} f: & k[x_1 \dots x_m, \xi_1 \dots \xi_n] & \longrightarrow & A \\ & x_i & \mapsto & a_i \\ & \xi_j & \mapsto & \alpha_j \end{array}$$

The Categories of Supermanifolds and Superschemes

- **Supermanifold**: $(M_0, \mathcal{O}_M)$
  M_0 topological space
  $\mathcal{O}_M|_U \cong C^\infty_{M_0}|_U \otimes \wedge(\xi_1, \dots, \xi_n)$ sheaf of comm. superalgebras on M_0
- **Superscheme**: $(M_0, \mathcal{O}_M)$
  M_0 topological space
  $\mathcal{O}_M|_U \cong \underline{Spec} A$ sheaf of commutative superalgebras on M_0
- **Functor of points** of M supermanifold or (affine) superschemes:

$$M : (\text{smflds})^o \longrightarrow (\text{sets}), \qquad T \mapsto \text{Hom}(T, M) = \text{Hom}(\mathcal{O}(M), \mathcal{O}(T)),$$

$$M(\phi)(f) = f \circ \phi$$

The functor of points of $\mathrm{GL}(m|n)$

The A points of $\mathrm{GL}(m|n)$ are:

$$\mathrm{GL}(m|n)(A) = \mathrm{Hom}_{(\mathrm{salg})}(C^\infty(x_{ij}) \otimes \wedge(\xi_{kl})[u^{-1}, v^{-1}], A)$$

$$u = det(x_{ij})_{1 \leq i,j \leq m}, \qquad v = det(x_{kl})_{m+1 \leq k,l \leq n}$$

We can write the A points of $\mathrm{GL}(m|n)$ as matrices:

$$\mathrm{GL}(m|n)(A) = \left\{ \begin{pmatrix} (t_{ij}) & (\theta_{il}) \\ (\theta_{kj}) & (t_{kl}) \end{pmatrix} \right\}$$

(t_{ij}), (t_{kl}) invertible matrices with entries in A_0,
(θ_{il}), (θ_{kj}) matrices with entries in A_1.
In fact, we can write a morphism as (both supermanifolds, superschemes):

$$\begin{array}{cccc} f: & \mathcal{O}(\mathrm{GL}(m|n)) & \longrightarrow & A = \mathcal{O}(T) \\ & x_{ij},\, x_{kl} & \mapsto & t_{ij},\, t_{kl} \\ & \xi_{il},\, \xi_{kj} & \mapsto & \theta_{il},\, \theta_{kj} \end{array}$$

The functor of points of $\mathrm{GL}(m|n)$ and actions

The functor of points is very useful for the action language!

We naturally recover the action of $\mathrm{GL}(m|n)$ on $k^{m|n}$:

$$\begin{array}{ccc} \mathrm{GL}(m|n)(A) \times k^{m|n}(A) & \longrightarrow & k^{m|n}(A) \\ \begin{pmatrix} (t_{ij}) & (\theta_{il}) \\ (\theta_{kj}) & (t_{kl}) \end{pmatrix}, \begin{pmatrix} a_1 \\ \vdots \\ \alpha_n \end{pmatrix} & \mapsto & \begin{pmatrix} (t_{ij}) & (\theta_{il}) \\ (\theta_{kj}) & (t_{kl}) \end{pmatrix} \begin{pmatrix} a_1 \\ \vdots \\ \alpha_n \end{pmatrix} \end{array}$$

Super Harish-Chandra pairs (SHCP)

Supergroup G as a **Super Harish-Chandra pair**: $G \sim (G_0, \mathfrak{g})$,

- G_0: Lie group;
- $\mathfrak{g}$: Lie superalgebra, $\mathfrak{g}_0 = \mathrm{Lie}(G_0)$;
- G_0 acts on $\mathfrak{g}$, on $\mathfrak{g}_0$ as Ad ($Ad(g) : X \mapsto gXg^{-1}$)

Example

$\mathrm{GL}(m|n) = (\mathrm{GL}(m) \times \mathrm{GL}(n), \mathfrak{gl}(m|n))$, where $\mathfrak{gl}(m|n)$ is the vector superspace of $m+n$ matrices with entries in $\mathbb{R}$ or $\mathbb{C}$ and parity ($u_{ij}, v_{kl} \in k$):

$$\begin{pmatrix} (u_{ij}) & 0 \\ 0 & (u_{kl}) \end{pmatrix} \quad \text{even}, \qquad \begin{pmatrix} 0 & (v_{il}) \\ (v_{kj}) & 0 \end{pmatrix} \quad \text{odd}$$

The bracket:

$$[X, Y] = XY - (-1)^{p(X)p(Y)} YX$$

Antisymmetry and Jacobi identity come with signs.

A **Representation** of G in $V = V_0 \oplus V_1$ is equivalently as follows:

1. **Action** of G on V via functor of points: $G(A) \times V(A) \to V(A)$
2. **Natural transformation**: $G(A) \to \mathrm{GL}(V)(A)$
3. **SHCP representation**,

$$\pi_0 : G_0 \to \mathrm{GL}(V_0) \times \mathrm{GL}(V_1) \qquad \rho^\pi : \mathfrak{g} \to \mathrm{End}(V)$$

with compatibility conditions

$$\pi_0(g)\rho^\pi(X)\pi_0(g)^{-1} = \rho^\pi(Ad(g)X), \qquad \rho^\pi\big|_{\mathfrak{g}_0} \simeq \mathrm{d}\pi_0$$

Example of SHCP representations: $\mathrm{Osp}(1|2)$

$$\mathrm{Osp}(m|2n)(A) := \{X \in \mathrm{GL}(m|2n)(A) \,|\, X^t J X = J\}, \quad J = \begin{pmatrix} I_m & 0 & 0 \\ 0 & 0 & I_n \\ 0 & -I_n & 0 \end{pmatrix}$$

In SHCP notation: $\mathrm{Osp}(1|2) = (\mathrm{SL}_2(k), \mathrm{osp}(1|2))$ where:

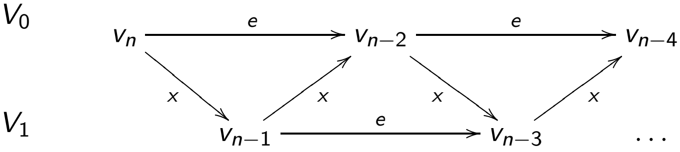
$$\mathrm{osp}(1|2) = \left\{ \begin{pmatrix} 0 & \alpha & \beta \\ \beta & & \\ -\alpha & & B \end{pmatrix} |B \in \mathrm{sl}_2 \right\} \subset \mathfrak{gl}(1|2), \text{ with basis}$$

$$\mathrm{sl}_2(k): \quad e = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, f = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, h = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

$$x = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, y = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad [x,x] = e, \ [y,y] = -f$$

Classification of highest weight $\mathrm{Osp}(1|2)$ representations

Theorem. *Let $V = V_0 \oplus V_1$ be the highest weight representation of $\mathrm{Osp}(1|2) = (\mathrm{SL}_2(k), \mathrm{osp}(1|2))$ with highest weight $\lambda \in \mathbb{C}$, v_λ highest weight vector. V is finite dimensional if and only if $\lambda = n \in \mathbb{Z}_{\geq 0}$.*
V_0: highest weight module for sl_2 with highest weight λ.
V_1: highest weight module for sl_2 with highest weight $\lambda - 1$.



Note: $\mathrm{span}\{e, f, h\} \cong \mathrm{sl}_2$;
y and f reverse the arrows of x and e.

Hermitian and Super Hermitian products

V: complex super vector space

Definition. $(,)$ is a *super hermitian product* on V if:

$$(u, v) = (-1)^{|u||v|}\overline{(v, u)}, \quad \forall u, v \in V$$

$$(u, v) = 0 \quad \text{for} \quad |u| \neq |v|$$

If X is an endomorphism of V, its *adjoint* X^* is

$$(Xu, v) = (-1)^{|u||X|}(u, X^* v),$$

Observation.hermitian product $\langle \, , \, \rangle \iff (\, , \,)$ super hermitian product:

$$\langle u, v \rangle = \begin{cases} i(u, v) & |u| = |v| = 1 \\ (u, v) & \text{otherwise} \end{cases}$$

(assuming V_0 and V_1 are orthogonal).

Unitary representations of Lie superalgebras

$\mathfrak{g}$: real Lie superalgebra; $\gamma : \mathfrak{g} \longrightarrow \mathrm{End}(V)$ representation.

Definition. $\gamma : \mathfrak{g} \longrightarrow \mathrm{End}(V)$ is **unitary** if and only if

$$\gamma(X)^* = \begin{cases} -\gamma(X), & |X| = 0 \\ +\gamma(X), & |X| = 1 \end{cases} \quad \text{or} \quad \gamma(X)^\dagger = \begin{cases} -\gamma(X), & |X| = 0 \\ -i\gamma(X), & |X| = 1 \end{cases}$$

$\gamma(X)^*$ adjoint for $(,)$, $\gamma(X)^\dagger$ adjoint for $\langle , \rangle$.

Equivalently (Carmeli, Cassinelli, Toigo, Varadarajan):
Proposition. γ is *unitary* if and only if V is equipped with an ordinary hermitian product $\langle \, , \, \rangle$ in which V_0 and V_1 are orthogonal, and
(U1) For all $X \in \mathfrak{g}_0$, $i\gamma(X)$ is symmetric on V.
(U2) For all $Z \in \mathfrak{g}_1$, $\gamma(Z) := e^{-i\pi/4}\gamma(Z)$ is symmetric on V.

Definition. Let $G = (G_0, \mathfrak{g})$ be a real supergroup. A supergroup morphism $\pi : G \longrightarrow \mathrm{Aut}(\mathcal{H})$ is a **unitary representation** for G in the Hilbert superspace $\mathcal{H}$ if:

1. π_0 is an even unitary representation of G_0 in $\mathcal{H}$;
2. $\gamma = d\pi : \mathfrak{g} \longrightarrow End(V) \subset End(C^\omega_{\pi_0})$ (analytic vectors)
   γ is a unitary representation of $\mathfrak{g} = \mathrm{Lie}(G)$;
3. $\mathcal{H}$ is the completion of V.

Harish-Chandra representations

V: complex super vector space (infinite dimensional)
$\mathfrak{g}_{\mathbb{C}}$: complex contragredient Lie superalgebra
($\mathfrak{g}_{\mathbb{C}}$ is $A(m,n)$, $B(m,n)$, $C(n)$, $D(m,n)$, $F(4)$, $G(3)$)
$\mathfrak{g}_{\mathbb{C}} = \mathfrak{k}_{\mathbb{C}} \oplus \mathfrak{p}_{\mathbb{C}}$: *Cartan decomposition* (complexified)
E.g. $\mathfrak{su}(p,q|r,s) = \mathfrak{su}(p) \oplus \mathfrak{su}(q) \oplus \mathfrak{su}(r) \oplus \mathfrak{su}(s) \oplus i\mathbb{R} \oplus i\mathbb{R} \oplus \mathfrak{p}$
$\pi : \mathfrak{g}_{\mathbb{C}} \longrightarrow \mathrm{End}(V)$ a representation.

Definition. V is a *Harish-Chandra module* for $\mathfrak{g}_{\mathbb{C}}$ if
$\exists\ \pi_{K_{\mathbb{R}}} : K_{\mathbb{R}} \longrightarrow \mathrm{Aut}(V)$, $\pi : \mathfrak{g}_{\mathbb{C}} \longrightarrow \mathrm{End}(V)$:

1. $\pi(Ad(k)X) = \pi_{K_{\mathbb{R}}}(k)\pi(X)\pi_{K_{\mathbb{R}}}(k)^{-1}$
2. $V = \sum_{\tau \in \Theta} V(\tau)$
   $V(\tau)$: span of all the finite dimensional subspaces with τ as character
3. $V(\tau)$ is finite dimensional.

Infinitesimal Harish-Chandra highest weight unitary representations

Definition. Harish-Chandra homomorphism β:

$$\begin{array}{rccl} \beta: & Z(\mathfrak{g}_\mathbb{C}) & \longrightarrow & S(\mathfrak{h}_\mathbb{C})^W \\ & u & \mapsto & \beta(u) \cong u \quad mod(\mathcal{P}), \quad \mathcal{P} = \sum_{\alpha>0} \mathcal{U}(\mathfrak{g}_\mathbb{C})\mathfrak{g}_{\mathbb{C}\alpha} \end{array}$$

Theorem. (Kac). The Harish-Chandra homomorphism identifies the center $Z(\mathfrak{g}_\mathbb{C})$ of the universal enveloping superalgebra with the subalgebra $I(\mathfrak{h}_\mathbb{C})$ of $S(\mathfrak{h}_\mathbb{C})^W$:

$$I(\mathfrak{h}_\mathbb{C}) = \{\phi \in S(\mathfrak{h}_\mathbb{C})^W \,|\, \phi(\lambda+t\alpha) = \phi(\lambda), \, \forall\, \lambda \in \langle\alpha\rangle^\perp, \, \alpha \, \text{isotropic}, \, \forall\, t \in \mathbb{C}\}$$

This is a new super phenomenon!

Infinitesimal Harish-Chandra highest weight unitary representations

Theorem (Carmeli-F.-Varadarajan, 2021). Let $\lambda \in \mathfrak{h}^*_{\mathbb{C}}$ and let π_λ be the irreducible highest weight representation of highest weight λ. Then:

$$\pi_\lambda \text{ is unitary if and only if } (-i)^{|a|}\beta(a^*a)(\lambda) > 0 \text{ for all } a \in \mathcal{U}(\mathfrak{g}_{\mathbb{C}}).$$

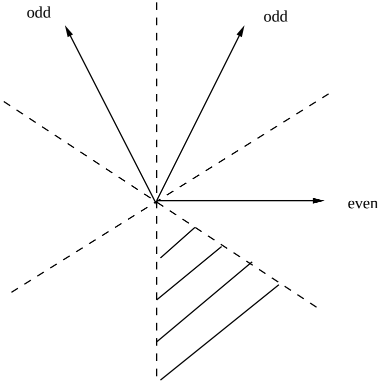
In particular it is necessary to be in the **Harish-Chandra cone**:

1. $\lambda(H_\alpha) \geq 0$ for α even compact
2. $\lambda(H_\alpha) < 0$ for α non compact.

Furthermore the weight spaces are orthogonal.

Note: $* : \mathcal{U}(\mathfrak{g}_{\mathbb{C}}) \longrightarrow \mathcal{U}(\mathfrak{g}_{\mathbb{C}})$ antiautomorphism extending the adjoint.

The Harish-Chandra cone in $\mathfrak{sl}(2|1)$



Example

Example: $\mathfrak{g}_{\mathbb{C}} = \mathrm{osp}_{\mathbb{C}}(1|2)$ and its real form $\mathrm{osp}_{\mathbb{R}}(1|2)$.
Theorem. Let V_t be the universal (Verma) $\mathrm{osp}_{\mathbb{C}}(1|2)$ module of highest weight $t \in \mathbb{C}$. (**Note:** $t = \lambda(H_\alpha)$).

1. V_t is a unitary module for $\mathrm{osp}_{\mathbb{R}}(1|2)$ if and only if $t \in \mathbb{R}$, $t < 0$
2. V_t unitary is irreducible.
3. All unitary representation of the real Lie supergroup $\mathrm{Osp}_{\mathbb{R}}(1|2) = (\mathrm{SL}_2(\mathbb{R}), \mathrm{osp}(1|2))$ are given on the completion $\mathcal{H}$ of V_t:
    - $V_t \subset C^\omega(\pi_0)$ analytic vectors
    - π_0: unitary representation of $\mathrm{SL}_2(\mathbb{R})$ in $\mathcal{H}$ integrating $(V_t)_0$.

Remark. $t = \lambda(H_\alpha) < 0$, α the only (non compact) root.
[$\mathrm{osp}(m|2n)$ does not admit compact forms]

Geometric realizations of HC Unitary Representations

Assume:

- G: real Lie supergroup (e.g. $G = \mathrm{SU}(r, s|p, q)$)
- T: maximal compact torus (e.g. T purely imaginary diagonal)
- A: maximal non compact torus (e.g. A real diagonal matrices)
- $\mathfrak{g} = \mathrm{Lie}(G)$: contragredient superalgebra
- $$\mathrm{rk}\,\mathfrak{k} = \mathrm{rk}\,\mathfrak{h}, \qquad \mathfrak{h} = \mathfrak{t} \oplus \mathfrak{a}, \qquad \mathfrak{a} = i\mathfrak{t}$$

  where:

  $$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}, \qquad \text{Super Cartan decomposition}$$

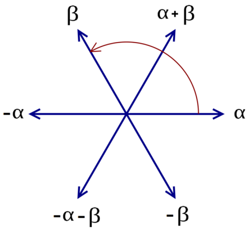
  (e.g. $\mathfrak{k} = \mathfrak{su}(r) \oplus \mathfrak{su}(s) \oplus \mathfrak{su}(p) \oplus \mathfrak{su}(q) \oplus i\mathbb{R} \oplus i\mathbb{R}$)

Admissible Positive Systems

Fix a positive system Δ^+ *admissible* for $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$:

$$\Delta_{\mathfrak{k}} + \Delta_{\mathfrak{p}}^+ \subset \Delta_{\mathfrak{p}}^+, \qquad \Delta_{\mathfrak{p}}^+ + \Delta_{\mathfrak{p}}^+ \subset \Delta_{\mathfrak{p}}^+$$

Example: admissible systems for A_2, $\Delta_{\mathfrak{k}} = \{\pm\alpha\}$, $\Delta_{\mathfrak{p}} = \{\pm\beta, \pm(\alpha+\beta)\}$



$\{\alpha, \beta, \alpha+\beta\}$: admissible
$\{\alpha+\beta, -\beta, \alpha\}$: **not** admissible

Key Point: P^+ subsupergroup of superalgebra $\mathfrak{p}^+$.

Hermitian Symmetric Super Spaces

Fix an admissible positive system for $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}^- \oplus \mathfrak{p}^+$.
Let $P^{\pm}$ the complex supergroup associated with the subalgebras $\mathfrak{p}^{\pm}$.

Theorem (Harish-Chandra decomposition). *We have an isomorphism onto the image:*

$$\begin{array}{ccc} P^- \times K \times P^+ & \longrightarrow & G \\ (p^-, k, p^+) & \mapsto & p^- k p^+ \end{array}$$

Corollary.

$$G_{\mathbb{R}}/K_{\mathbb{R}} \subset G_{\mathbb{R}} K P^+ / K P^+$$

hence $G_{\mathbb{R}}/K_{\mathbb{R}}$ acquires naturally a $G_{\mathbb{R}}$ invariant complex structure.

Example: The upper half plane $\mathrm{SL}_2(\mathbb{R})/\mathrm{SO}(2)$

Define the **super Lagrangian** $\mathcal{L}$ as:

$$\mathcal{L} = \mathrm{Osp}(m|2n)/P,$$

where:

$$P(A) = \left\{ \begin{pmatrix} a & 0 & \alpha_2 \\ \beta_1 & b_{11} & b_{12} \\ 0 & 0 & b_{22} \end{pmatrix} \right\} \subset \mathrm{Osp}(m|2n)(A).$$

$$\mathrm{Osp}(m|2n)(A) := \{X \in \mathrm{GL}(m|2n)(A) \,|\, X^t J X = J\}$$

$$J = \begin{pmatrix} I_m & 0 & 0 \\ 0 & 0 & I_n \\ 0 & -I_n & 0 \end{pmatrix}$$

The Siegel superspace

Define the **Siegel superspace**:

$$\mathcal{S}(A) = \{(z, \zeta) \,|\, \zeta^t \zeta + z^t - z = 0,\ z = x + iy,\ |y| > 0\} \subset \mathbb{C}^{n^2|mn}(A),$$

It is an analytic supermanifold of dimension $n^2 - n(n-1)/2|mn$.

Theorem

1. $\mathrm{Osp}_{\mathbb{R}}(m|2n)$ acts transitively on the super Siegel space.
2. The stabilizer of the topological point $(iI, 0) \in \mathcal{S}$ is the subgroup:

$$K_{\mathbb{R}} = Stab(iI, 0) \quad = \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & b_{11} & b_{12} \\ 0 & -b_{12} & b_{22} \end{pmatrix} \in \mathrm{Osp}_{\mathbb{R}}(m|2n) \right\}$$

3. We have:

$$\mathcal{S} \cong \mathrm{Osp}_{\mathbb{R}}(m|2n)/K_{\mathbb{R}}$$

hence $\mathrm{Osp}_{\mathbb{R}}(m|2n)/K_{\mathbb{R}}$ acquires naturally a complex structure.

Geometric realizations of HC Unitary Representations

Define super sections of a super line bundle on G/B:

$$L_\chi(U) = \{f \in \mathcal{H}(U) | f(ga) = \chi^{-1}(a) f(g)\}, \qquad \chi = e^\lambda, \lambda \in \imath\mathfrak{t}^* \subset \mathfrak{h}^*$$

Theorem (Main Result 1, Carmeli-F.-Varadarajan, 2019).
Assume $L_\chi(G_\mathbb{R} B^+/B^+) \neq 0$ modulo J the odd ideal. Then:

1. $L_\chi(G_\mathbb{R} B^+/B^+)$ contains ψ analytic continuation of the const. poly. 1;
2. $\overline{\ell(\mathcal{U}(\mathfrak{g}))\psi} \subset L_\chi(G_\mathbb{R} B^+/B^+)$ is a Fréchet $G_\mathbb{R}$-module, $K_\mathbb{R}$-finite
3. $\lambda(H_\alpha) \in \mathbb{Z}_{\geq 0}$ for all compact positive roots α.

Hence:

- $V = \langle \ell(\mathcal{U}(\mathfrak{g}))\psi \rangle$ HC highest weight module for $\mathfrak{g}$
- $\mathcal{H} = \overline{\ell(\mathcal{U}(\mathfrak{g}))\psi}$ HC module for $G_\mathbb{R}$.

Note. $(\lambda + \rho)(H_\alpha) < 0$, α non compact $\implies$ V irreducible.

Iwasawa decomposition: $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \oplus \mathfrak{a} \oplus \mathfrak{n}_{\mathbb{C}}^{-}$
Hence $GAN_{\mathbb{C}}$ is *open* in $G_{\mathbb{C}}$.

Examples

1. $G = \mathrm{SU}(1|1)$, $G_{\mathbb{C}} = \mathrm{SL}(1|1)$:

$$N_{\mathbb{C}}(A) = \left\{ \begin{pmatrix} 1 & 0 \\ \nu & 1 \end{pmatrix} \right\} \subset G_{\mathbb{C}} = \left\{ \begin{pmatrix} a & \alpha \\ \beta & b \end{pmatrix} \right\}$$

   Hence:

$$X = G_{\mathbb{C}}/N_{\mathbb{C}} \cong S^{1|2} \longrightarrow G_{\mathbb{C}}/B_{\mathbb{C}} \cong \mathbb{P}^{0|1}$$

2. $G = \mathrm{Osp}(1|2)_{\mathbb{R}}$, $G_{\mathbb{C}} = \mathrm{Osp}(1|2)_{\mathbb{C}}$

$$X = G_{\mathbb{C}}/N_{\mathbb{C}} = (\mathbb{C}^{1|1})^{\times} \longrightarrow G_{\mathbb{C}}/B_{\mathbb{C}} = \mathbb{P}^{1|1}$$

Super Geometric Quantization

Question. Determine "all" *irreducible unitary representations* of a real supergroup G **at once** in the superspace of holomorphic sections of a super line bundle on the supermanifold $X = GA \subset G_{\mathbb{C}}/N_{\mathbb{C}}$ (open).

Facts in ordinary geometry (Guillemin-Chuah 2000):

- There is a unique (up to scalar) $G_0 \times T$ invariant Kahler form ω_0 on $X_0 = G_0 A$ ($T = T_0$, $A = A_0$, $H = TA$).
- $\omega_0 = i\partial\overline{\partial} F_0$ *globally*
- There is an hermitian line bundle $\mathcal{L}_0$ on X_0; ω_0 is its Chern class (trivial as C^∞ line bundle, ω_0 exact)
- There exist a unique $G_0 \times T$ invariant *non vanishing holomorphic* section s_0 of $\mathcal{L}_0$
- The hermitian product on the fiber $\mathcal{L}_z$ is determined by:

$$\langle s_0, s_0 \rangle_z = e^{-F_0(z)}, \qquad z \in X_0$$

Key fact. s_0 realizes the identification:

$$\mathcal{H}(\mathcal{L}_0) \cong \mathcal{H}(X_0)$$

The hermitian structure on $\mathcal{L}_0$ gives an hermitian product on $\mathcal{H}^2(\mathcal{L}_0)$:

$$\langle f, g\rangle = \int_{X_0} \langle fs_0, gs_0\rangle_z = \int_{X_0} f(z)\overline{g(z)}e^{-F_0(z)}dz$$

$\mathcal{H}^2(\mathcal{L}_0)$: section for which the integral exists and it is finite.

$$\mathcal{H}(\mathcal{L}_0)_\lambda = \{f \in \mathcal{H}(\mathcal{L}_0) | f(ga) = \chi^{-1}(a)f(g)\} \subset \mathcal{H}(\mathcal{L}_0)$$

Theorem. (Guillemin, Chuah 2000). *Let $\mathcal{H}(\mathcal{L}_0)_\lambda$ be the (infinite dimensional) representation of G with character $\chi = e^\lambda$, $\lambda \in i\mathfrak{t}^*$. Then $\mathcal{H}(\mathcal{L}_0)_\lambda \subset \mathcal{H}^2(\mathcal{L}_0)$ if and only if λ is in the image of the moment map $\Phi_0 : X_0 \longrightarrow \mathfrak{g}_0$ for the hamiltonian action of G_0 on $X_0 = G_0A$.*

Super Quantization/1

Ingredients:

- The super 1|0 hermitian bundle: $\mathcal{L} = \mathcal{L}_0 \otimes \wedge(\xi)$
- There exist a unique $G \times T$ invariant ω super Kahler form.
- $\omega = i\partial\overline{\partial}F$ globally, with F super kahler potential
- G left acts on $X = GA$ open in $G_{\mathbb{C}}/N_{\mathbb{C}}$
- T right acts on X with a *Hamiltonian action*.
- The super moment map $\Phi : X \longrightarrow \mathfrak{t}^*$ is well defined
- We can define the super representations:

$$\mathcal{H}(\mathcal{L})_\lambda = \{f \in \mathcal{H}(\mathcal{L}) | f(ga) = \chi^{-1}(a)f(g)\}, \quad \chi = e^\lambda, \lambda \in \mathfrak{h}^* \quad \text{integral}$$

Let $\lambda \in \mathfrak{h}^*$ be typical, integral and in the Harish-Chandra cone:

$$C = \{\lambda \in \mathfrak{h} | (\alpha, \lambda) \geq 0 \text{ for all } \alpha \in \Delta_c^+, \, (\beta, \lambda) < 0 \text{ for all } \beta \in \Delta_n^+\}$$

$\Phi : X \longrightarrow \mathfrak{g}^*$ the (super) moment map.

Theorem (Main Result 2, Chuah-F. 2022). The irreducible **unitary** highest weight G-representation with HC super character $\chi_{\lambda+\rho}$ occurs in $\mathcal{H}(\mathcal{L})$ if and only if $\lambda \in \mathrm{Im}(\Phi)$, and it is given by

$$\mathcal{H}(\mathcal{L})_\lambda = \{f \in \mathcal{H}(\mathcal{L}) | f(ga) = \chi^{-1}(a) f(g)\}$$