(Almost) everything you ever wanted to know about four-dimensional supersymmetry*

(*but were afraid to ask)

Geometric structures and supersymmetry
Universitetet i Tromsø
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Veronica Stanciu in memoriam

(1925-2022)

Based on several collaborations

- *“Killing superalgebras for lorentzian four-manifolds”* (**arXiv:1605.00881**) with **Paul de Medeiros** and **Andrea Santi**
- *“Kinematical superspaces”* (**arXiv:1908.11278**) with **Ross Grassie**
- *“Killing superalgebras for lorentzian five-manifolds”* (**arXiv:2105.05775**) with **Andrew Beckett**
- *“Kaluza–Klein reductions of maximally supersymmetric five-dimensional lorentzian spacetimes”* (**arXiv:2207.07430**) with **Guido Franchetti**

Which 4-dimensional geometries can support N=1supersymmetry?

(What are “physically interesting” (4|4)-dimensional supermanifolds?)

Two approaches

- **Lorentzian supersymmetry**

  The “body” of the supermanifold is a four-dimensional lorentzian manifold

- **Non-lorentzian supersymmetry**

  The “body” of the supermanifold is a four-dimensional kinematical spacetime

Lorentzian d=4 supersymmetry

Poincaré supersymmetry

Golfand+Likhtman ’71
Haag+Lopuszanski+Sohnius ’74

(V, η) lorentzian 4-dimensional vector space

$\mathfrak{so}(V)$ Lie algebra of skew-symmetric endomorphisms

Poincaré Lie algebra $\mathfrak{p} = V \oplus \mathfrak{so}(V)$

$$[A, B] = AB - BA$$
$$[A, v] = A(v)$$
$$[v, w] = 0$$

$v, w \in V \quad A, B \in \mathfrak{so}(V)$

$\mathfrak{p} \cong$ Lie algebra of isometries of Minkowski spacetime
(also symmetries of asymptotic geometries of Minkowski spacetime)

$C\ell(V,\eta) \cong \mathrm{Mat}(4,\mathbb{R})$ Clifford algebra

$S \cong \mathbb{R}^4$ symplectic Clifford module

$\langle -,- \rangle : S \times S \to \mathbb{R}$ $\langle v \cdot s_1, s_2 \rangle = -\langle s_1, v \cdot s_2 \rangle = \langle v \cdot s_2, s_1 \rangle$

Dirac current $\kappa : S \times S \to V$ $\eta(\kappa(s_1,s_2),v) = \langle s_1, v \cdot s_2 \rangle$

N=1 Poincaré superalgebra $\mathfrak{s} = V \oplus S \oplus \mathfrak{so}(V)$

$\mathfrak{s}_{\bar{0}} = \mathfrak{p}$ $\mathfrak{s}_{\bar{1}} = S$ $\mathfrak{s} = \mathfrak{s}_{-2} \oplus \mathfrak{s}_{-1} \oplus \mathfrak{s}_0$ $\mathbb{Z}$-graded

$$[A,s] = \sigma(A)s$$
$$[s,s] = \kappa(s)$$
$$[v,s] = 0$$

$A \in \mathfrak{so}(V), s \in S, v \in V$

$\sigma : \mathfrak{so}(V) \to \mathrm{End}(S)$ spin representation

AdS supersymmetry

Zumino ’77

$\mathfrak{so}(3,2) \cong$ Lie algebra of isometries of AdS

κ Killing form

$\mathfrak{so}(3,2) \cong \mathfrak{sp}(4,\mathbb{R})$ “sporadic isogeny” (spin representation)

$(S \cong \mathbb{R}^4, \langle -, - \rangle)$ (symplectic) spinor representation

$\lambda : S \times S \to \mathfrak{sp}(4,\mathbb{R}) \qquad \kappa(\lambda(s_1, s_2), X) = \langle s_1, X \cdot s_2 \rangle = -\langle X \cdot s_1, s_2 \rangle = \langle s_2, X \cdot s_1 \rangle$

$\mathfrak{s}_{\bar{0}} = \mathfrak{sp}(4,\mathbb{R}) \qquad \mathfrak{s}_{\bar{1}} = S \qquad \mathfrak{s} = \mathfrak{s}_{\bar{0}} \oplus \mathfrak{s}_{\bar{1}} \cong \mathfrak{osp}(1|4)$

Parallel spinors

Both Poincaré and AdS supersymmetry are examples of the following construction:

(M, g) lorentzian spin 4-dimensional manifold

$\$$ spinor bundle (real, rank 4)

$D = \nabla - \beta$ $\beta \in \Omega^1(M, \mathrm{End}\$)$ connection on $

Clifford action

Poincaré $\beta = 0$ AdS $\beta_X s = \mu X \cdot s$

"**Killing superalgebra**"

$\mathfrak{s} = \mathfrak{s}_{\bar{0}} \oplus \mathfrak{s}_{\bar{1}}$

$\mathfrak{s}_{\bar{0}} =$ Killing vector fields

$\mathfrak{s}_{\bar{1}} = \{s \in \Gamma(\$) \mid Ds = 0\}$ "**Killing spinors**"

Killing (super)algebras

(M^n, g) n-dimensional (pseudo)riemannian spin manifold

$\$$ spinor bundle (actually, bundle of Clifford modules)

$D = \nabla + \cdots$ connection on $

Killing spinors

$$\mathfrak{s}_{\bar{1}} = \{s \in \Gamma(\$) \mid Ds = 0\}$$

$$\mathfrak{s}_{\bar{0}} = \{X \in \mathscr{X}(M) \mid \mathscr{L}_X D = 0\}$$

assumes X is a (conformal) Killing vector field

$\mathfrak{s} = \mathfrak{s}_{\bar{0}} \oplus \mathfrak{s}_{\bar{1}}$ **Question**: is $\mathfrak{s}$ a Lie (super)algebra?

Three possibilities

1) $\mathfrak{s}$ is not a Lie (super) algebra

2) $\mathfrak{s}$ is a 2-graded Lie algebra e.g., $M = S^7$ $\mathfrak{s} \cong \mathfrak{so}(9)$

$M = S^8$ $\mathfrak{s} \cong \mathfrak{f}_4$

$M = S^{15}$ $\mathfrak{s} \cong \mathfrak{e}_8$

Adams '96
JMF '07

3) $\mathfrak{s}$ is a Lie superalgebra e.g., Poincaré and AdS

Filtered deformations

Fact: Killing superalgebras are **filtered**

Fact: the associated graded superalgebra is a graded subalgebra of the pseudo-euclidean Lie superalgebra associated with $(V, \eta) = (T_p M, g_p)$

JMF+Santi '16
Beckett '22

Fact: filtered deformations are classified by **generalised Spencer cohomology**

Cheng+Kac '98

$$D = \nabla - \beta$$

component of a Spencer cocycle

Geometries with D flat are **maximally supersymmetric**
and Killing superalgebra acts locally transitively!

JMF+Hustler '13

Killing superalgebras for lorentzian 4-manifolds

Festuccia+Seiberg ’11
de Medeiros+JMF+Santi ’16

- Poincaré superalgebra (Minkowski)
- $\mathfrak{osp}(1|4)$ (anti de Sitter)
- filtered deformations of $V \oplus S \oplus \mathfrak{h} \subset V \oplus S \oplus \mathfrak{so}(V)$
  - $\mathfrak{h} \cong \mathfrak{so}(3)$ $(-\mathbb{R} \times S^3)$
  - $\mathfrak{h} \cong \mathfrak{so}(2,1)$ $(\mathbb{R} \times \mathrm{AdS}_3)$
  - $\mathfrak{h} \cong \mathfrak{nw}$ (CW)

Nappi+Witten algebra — Nappi+Witten ’93

Cahen+Wallach lorentzian symmetric space — Cahen+Wallach ’70

Other dimensions

- d=11 JMF+Santi ’15
- d=6 de Medeiros+JMF+Santi ’18
- d=5 Beckett+JMF ‘21
- See **Guido Franchetti**’s talk for new cohomogeneity-one lorentzian 4-manifolds admitting Killing superalgebras obtained as quotients of d=5 maximally supersymmetric geometries
- Other calculations in d=10, d=3,5,6 (extended)

de Medeiros+JMF (wip)
Beckett (wip)

Superspaces

Both Poincaré and AdS supersymmetry are geometrically realised on (4|4)-dimensional (homogeneous) supermanifolds:

$(\mathfrak{s}, \mathfrak{h})$ **super Klein pair**

Minkowski superspace $(V \oplus S \oplus \mathfrak{so}(V), \mathfrak{so}(V))$

AdS superspace $(\mathfrak{osp}(1|4), \mathfrak{so}(3,1))$

$\mathfrak{so}(3,1) \subset \mathfrak{so}(3,2)$ stabiliser of time-like vector in $\mathbb{R}^{3,2}$

Similarly, all other d=4 maximally supersymmetric Killing superalgebras are realised geometrically on homogeneous supermanifolds which are **superisations** of homogeneous lorentzian four-dimensional manifolds:

Santi '10

$$-\mathbb{R} \times S^3 \qquad \mathbb{R} \times \mathrm{AdS}_3 \qquad \mathrm{CW}$$

Non-lorentzian d=4 supersymmetry

Not all spacetimes are lorentzian!

Kinematical spacetimes

Minkowski and AdS are examples of (spatially isotropic) **homogeneous kinematical spacetimes**.

These are homogeneous spaces of (spatially isotropic) **kinematical Lie groups**.

Kinematical Lie algebra $\mathfrak{k}$

$$\mathfrak{k} \supset \mathfrak{h} \cong \mathfrak{so}(d-1)$$

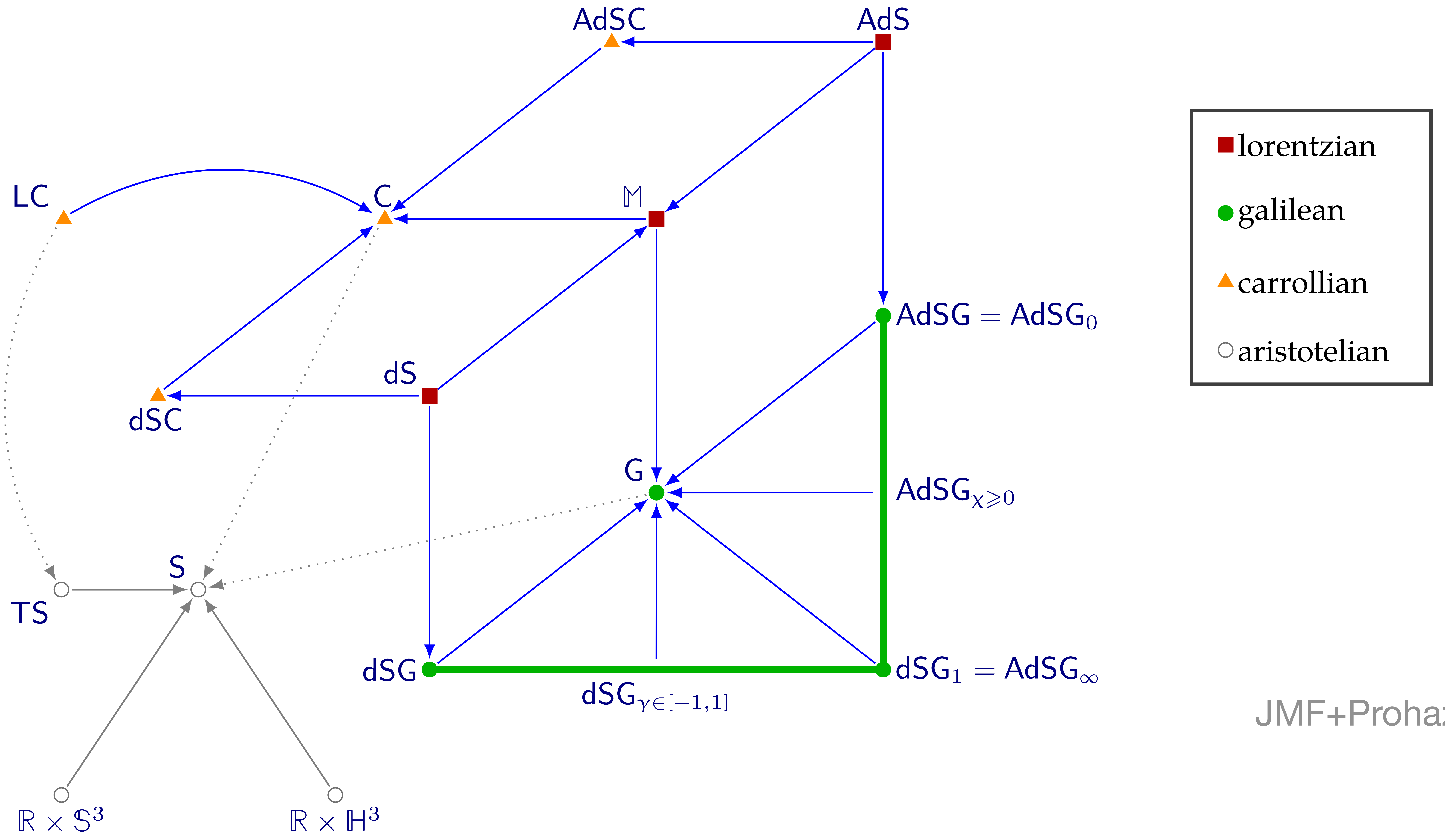
"rotations, boosts and translations"

$$\mathfrak{k} \cong \mathrm{ad} \oplus 2V \oplus \mathbb{R}$$

as $\mathfrak{h}$-modules

$V \cong \mathbb{R}^{d-1}$ vector representation

d=4 spatially isotropic homogeneous kinematical spacetimes



JMF+Prohazka '18

d=4 N=1 kinematical superalgebras

$\mathfrak{s}_{\bar{0}} = \mathfrak{k}$ d=4 kinematical Lie algebra

rotational subalgebra $\mathfrak{h} \cong \mathfrak{so}(3) \cong \mathfrak{sp}(1)$

$\mathfrak{s}_{\bar{1}} = S$ 4-dimensional $\mathfrak{k}$-module and spinor $\mathfrak{h}$-module

$S \cong \mathbb{H}$ 1-dimensional right quaternionic vector space

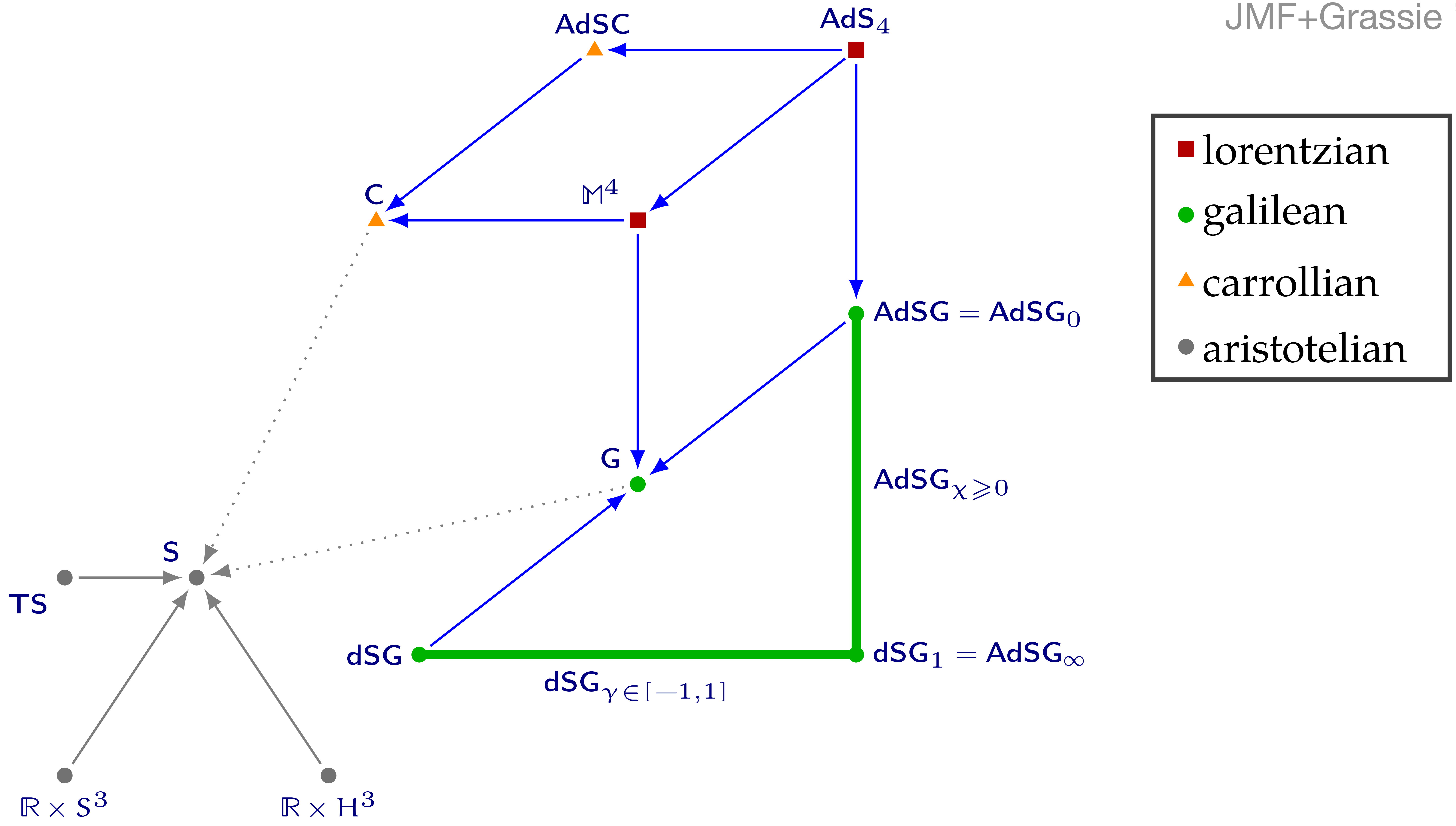
$\mathfrak{h}$ acting by left quaternion multiplication

There is a nice quaternionic calculus which allows for painless classification.

JMF+Grassie '19

Superisable kinematical spacetimes

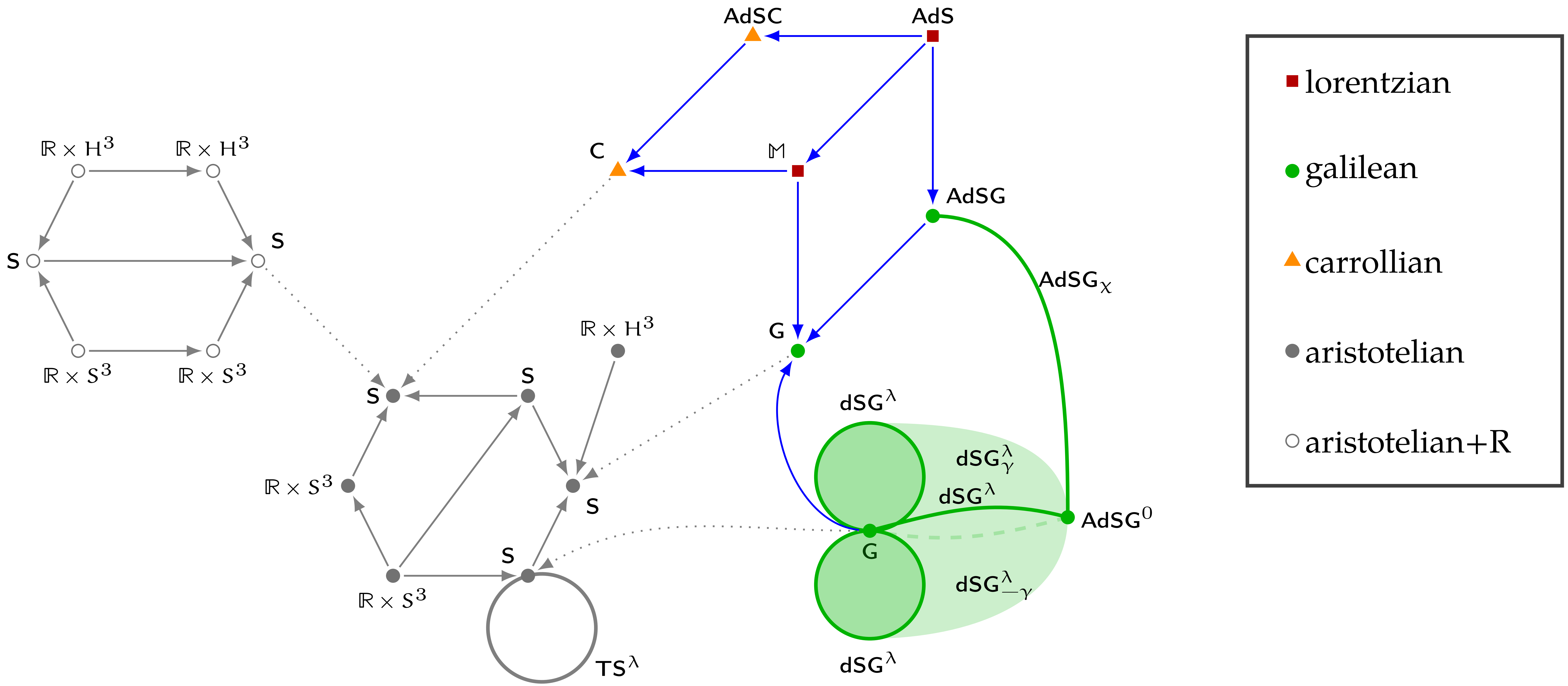
JMF+Grassie '19



Kinematical superspaces

(Impressionistically)

JMF+Grassie ’19



Conclusion

- There are a plethora of lorentzian and non-lorentzian **(4|4)-dimensional supermanifolds**, each one laying claim to being an **arena** for **d=4 N=1 supersymmetric field theories**.
- These are **supersymmetric Klein geometries** and their associated **gaugings**/**Cartan geometries** might result in **novel supergravity theories**.
- The **supersymmetry algebras** are of two types:
  - **filtered deformations** of graded subalgebras of (possibly extended) Poincaré superalgebras (in the lorentzian case)
  - **supersymmetric extensions** of kinematical Lie algebras.
- Their **representation theory** is largely unexplored.