Special Vinberg Cones of rank 3 and their Application to Supergravity

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A convex cone is an open convex $\mathbb{R}^+$ invariant domain $\mathcal{V} \subset W = \mathbb{R}^n$ without straight lines.
It is called homogeneous , if the linear group

$$\mathrm{Aut}(\mathcal{V}) = \{\mathrm{A} \in \mathrm{GL(W)}, \mathrm{A}\mathcal{V} = \mathcal{V}\}$$

acts transitively in $\mathcal{V}$.
Then there exists a solvable subgroup $\mathrm{G} \subset \mathrm{Aut}(\mathcal{V})$ (the Vinber group) which acts simply transitively in $\mathcal{V}$.
The adjoint (dual) cone is the convex cone

$$\mathcal{V}^* = \{\xi \in \mathrm{W}^*, \, \xi(\mathrm{X}) > 0 \, \forall \mathrm{X} \in \mathcal{V}\}$$

If $\mathcal{V}$ is homogeneous, then $\mathcal{V}^*$ is homogeneous.
A homogeneous convex (HC) cone $\mathcal{V}$ is self-adjoint if there is a metric $\mathrm{g} : \mathrm{V} \to \mathrm{V}^*$ s.t. $\mathrm{g} \circ \mathcal{V} = \mathcal{V}^*$.

Characteristic function and canonical metric of a HC cone

The function

$$\chi(\mathrm{x}) = \int_{\mathcal{V}^*} \mathrm{e}^{<\xi,\mathrm{x}>} \mathrm{d}\xi, \ \mathrm{d}\xi := \mathrm{d}\xi^1 \wedge \cdots \wedge \mathrm{d}\xi^{\mathrm{n}}$$

(Vinberg-Koszul characteristic function) is the density of the finite invariant measure $\mu = \chi(\mathrm{x})\mathrm{dx}$ in $\mathcal{V}$ where $\mathrm{dx} = \mathrm{dx}^1 \wedge \cdots \wedge \mathrm{dx}^{\mathrm{n}}$. Points $\mathrm{x} \in \mathcal{V}$ correspond to probability measures in $\mathcal{V}^*$ ("the exponantial family")

$$\mathrm{p_x}(\xi) := \frac{\mathrm{e}^{-<\xi,\mathrm{x}>}}{\chi(\mathrm{x})}.$$

The Hessian metric $\mathrm{g}_{\mathcal{V}} = -\mathrm{Hess}(\ln \chi(\mathrm{x}))$ is $\mathrm{Aut}(\mathcal{V})$-invariant complete Riemannian metric (the Koszul -Vinberg metric).
The manifold $(\mathcal{V}, \mathrm{g}_{\mathcal{V}})$ is a symmetric manifold iff $\mathcal{V}$ is selfdual.

Remark on applications of HC cones

Theory of homogeneous convex cones has many applications to different parts of physics (quantum physics, supergravity, quantum field theory and renormalization), differential geometry (special Kähler and quaternionic Kähler geometry, Frobenious manifolds), harmonic analysis, information geometry, multivariate statistics, Souriau thermodynamics on Lie groups, convex optimization, combinatorics, numerical integration of differential equations etc.

Examples of homogeneous convex (HC) cones

i). $\mathcal{V} = \mathbb{R}^+ \subset \mathbb{R}$.
ii). A direct product $\mathcal{V}_1 \times \mathcal{V}_2 \subset \mathrm{W}_1 \times \mathrm{W}_2$, $\mathcal{V}_\mathrm{i} \subset \mathrm{W}_\mathrm{i}$, $\mathrm{i} = 1, 2$.
In particular, the polyhedral cone $(\mathbb{R}^+)^\mathrm{n} \subset \mathbb{R}^\mathrm{n}$.
(see Noemie Combe,Yuri I. Manin, Matilde Marcolli,Moufang Patterns and Geometry of Information, 2021, for relations of the polyhedral cone $(\mathbb{R}^+)^\mathrm{n}$ with Moufang loops, F-manifolds, 3-webs, Latin squares, perfect tensors, quantum information theory, quantum error connection codes etc.)
iii). $\mathrm{Herm}^+_\mathrm{n}(\mathbb{K}) \subset \mathrm{Herm}_\mathrm{n}(\mathbb{K})$, $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ and for $\mathrm{n} = 3$ $\mathbb{O}$.
iv).The Lorentz cone

$$\mathbb{R}^{1,\mathrm{n}+1}_\mathrm{t} = \mathrm{Herm}_2(\mathbb{R}^\mathrm{n}) =$$

$$\{\mathrm{X} = \begin{pmatrix} \lambda & \mathrm{v} \\ \mathrm{v}^* & \mu \end{pmatrix}, -\det \mathrm{X} =< \mathrm{v}^*, \mathrm{v} > -\lambda\mu > 0, \lambda, \mu > 0\}$$

(see Jh. Baez, Jh. Huerta Division Algebras and Supersymmetry,(for quantum mechanical interpretation of $\mathrm{Herm}^+_2(\mathbb{K})$). It was extended to special rank 3 Vinberg cones (A-Cortes, 2021).

Köcher-Vinberg theorem

There are 1-1 between sefsimilar HC cones $\mathcal{V} = \mathrm{Int J^2}$ and the Euclidean Jordan algebras J. In particular, indecomposable HC cones are cones iii), iv).
(Due to Jordan-von Neumann-Wigner classification of EJA (1933)).
The cone $\mathrm{Herm_n(\mathbb{K}) = G/K := GL_n(\mathbb{K})/K}$ is a symmetric space where the group G acts by

$$\mathrm{G \ni A : X \mapsto A(X) := AXA^*, \; X \in Herm_n(\mathbb{K})}$$

and the simply transitive Vinberg group $\mathrm{G = T_n(\mathbb{K})}$ is the group of upper triangular matrices with positive diagonal elements.

Vinberg construction of HC cones as the orbit $Herm_n^+ = G(Id) = \{X = A \cdot A^*\} \subset Herm_n$ of the upper trangular group G of matrices in $Herm_n$

1. The algebra $\mathcal{T}$ of rank n upper triangular matrices is an associative algebra of upper triangular matrices of the form

$$B = \begin{pmatrix} b_{11} & b_{12} & b_{13} & \cdots b_{1n} \\ 0 & b_{22} & b_{23} & \cdots b_{2n} \\ 0 & 0 & b_{33} & \cdots b_{3n} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots b_{nn} \end{pmatrix}$$

where $b_{ii} \in \mathbb{R}$ and off-diagonal elements $b_{ij} \in V_{ij}$ belong to Euclidean vector spaces V_{ij} and the matrix multiplication is defined by a system of isometric maps

$$V_{ij} \times V_{jk} \to V_{ik}, \ (b_{ij}, b_{jk}) \mapsto b_{ij} \cdot b_{jk}$$

$$|b_{ij} \cdot b_{jk}| = |b_{ij}| \cdot |b_{jk}| \ i < j < k, .$$

Space $Herm_n$ of Hermitian matrices

2. A triangular matrtix $B = ||b_{ij}|| \in \mathcal{T}$ is extended to the Hermitian matrix

$$X = X(B) = ||x_{ij}|| = \begin{pmatrix} b_{11} & b_{12} & b_{13} & \cdots b_{1n} \\ b_{12}^* & b_{22} & b_{23} & \cdots b_{2n} \\ b_{13}^* & b_{23}^* & b_{33} & \cdots b_{3n} \\ \cdots & \cdots & \cdots & \cdots \\ b_{1n}^* & b_{2n}^* & b_{3n}^* & \cdots b_{nn} \end{pmatrix}$$

where $b_{ji}^* = g \circ b_{ij} \in V_{ij}^*$.

3. The (connected) Vinberg group

$$G = \{A = ||a_{ij}||, a_{ii} > 0\} \subset \mathfrak{T}$$

of invertible upper triangular matrices has a naturtal linear reprersentation $\rho : G \to GL(Herm_n)$ in the space $Herm_n = \{X\}$ of Hermitian matices.
The action is generated by the representgation

$$\rho : A \to \rho(A),\ \rho(A)X := AX + XA^* = AX + (AX)^*,\ A \in \mathfrak{g}, X \in Herm_n.$$

of the Lie algebra $\mathfrak{g} := \mathfrak{T}$.
The new products $V_{ij} \times V_{jk} \to V_{ik} \forall i, j, k$ are defined by the relations

$$(X_{ij} \cdot X_{jk})^* := X_{jk}^* \cdot X_{ij}^*,$$

$$< X_{ij}^* \cdot X_{ik}, X_{jk} >=< X_{ik}, X_{ij} \cdot X_{jk} >,\ < X_{ik} \cdot X_{jk}^*, X_{ij} >=< X_{ik}, X_{ij} \cdot X_{jk} >)$$

Vinberg's theorem

Theorem

(Vinberg) The orbit

$$Herm_n^+ := G(Id\ ) = \{X = AA^*,\, A \in G\}$$

of the identity matrix is the homogeneous convex cone (the cone of positive matrices of $Herm_n$*).*
Conversely, any homogeneous convex cone has the form $Herm_n^+$ *for some (associative) upper triangular matrix algebra* $\mathcal{T}$*.*

Isometric maps and Clifford modules

The Vinberg theory partially reduces explicit description of HC cones to description of isometric maps $\varphi : U \otimes V \to W$ between Euclidean vector spaces.We consider two cases.

i) $\dim U = \dim V = \dim W$. Then identifying $V = U = W$, we get a structure of the division algebra in V. The division algebras and associated isometric maps had been classified by A. Hurwitz 1898.

ii) dim U = dim W

Then the isometric map

$$\mu : V \times U \to W,\ (v,u) \mapsto \mu_v u = \cdot u$$

between (pseudo)Euclidean vector spaces, s.t.

$$g(v \cdot u, v \cdot u) = g(v,v) \cdot g(u,u)$$

defines in the space $S = S_0 + S_1 := U + W$ the structure of $\mathbb{Z}_2$ -graded Clifford module over $Cl(V)$. Then the problem of description of the isometric maps μ reduces to
a) classification of $\mathbb{Z}_2$-graded Clifford $CL(V)$- modules (obtained by M. F. Atiyah, R. Bott and A. S. Shapiro (1964)) and
b) classification of admissible metrics in S (such that the Clifford multiplication is skew-symmetric). It was done by D. A. and V. Cortes(1997).
These lead to description on the class of rank 3 homogeneous convex cones and its generalization to indefinite case. (A-Cortes, TG, 2021).

Special (rank 3) triangular algebra $\mathcal{T}$, the Vinberg group G and the dual group G^*

We associate to a metric $\mathbb{Z}_2$-graded Clifford $Cl(V)$ module $(S = S_0 + S_1, g_S)$ a rank 3 associative upper triangular matrix algebra $\mathcal{T}$ where $V_{12} = V$, $V_{23} = S_0$, $V_{13} = S_1$. It consists of the matrix of the form

$$A = \begin{pmatrix} b_1 & b_{12} & b_{13} \\ 0 & b_2 & b_{23} \\ 0 & 0 & b_3 \end{pmatrix} b_i \in \mathbb{R},\ b_{12} \in V, b_{23} \in S_0, b_{13} \in S_1 \qquad (1)$$

The Vinberg group G and , respectively its dual G^* consists of the matrices of the form

$$A = \begin{pmatrix} \alpha_1 & a_{12} & a_{13} \\ 0 & \alpha_2 & a_{23} \\ 0 & 0 & \alpha_3 \end{pmatrix}, A^* = \begin{pmatrix} \alpha_1 & 0 & 0 \\ a_{12}^* & \alpha_2 & 0 \\ a_{13}^* & a_{23}^* & \alpha_3 \end{pmatrix}, \alpha_1, \alpha_2, \alpha_3 > 0. \qquad (2)$$

Special (rank 3) Vinberg cone $\mathcal{V} = Herm_3^+(S)$ and the dual cone $\mathcal{V}' = Herm_3^-(S)$

Theorem

(Vinberg, A-Cortes) The orbit $\mathcal{V} = G(Id\ ) = \{AA^*,\ A \in G\}$, (resp.,$\mathcal{V}' = G^*(Id\ ) = \{A^* \cdot A,\ A^* \in G^*\}$ of the identity matrix are homogeneous cones with the simply transitive action of the Vinberg group G and, respectively, its dual G^*. If g_V and g_S are Euclidean metrics, then the cones $\mathcal{V}, \mathcal{V}'$ are convex.

The cone $\mathcal{V}$ is called the (indefinite) special Vinberg cone and the cone $\mathcal{V}'$ the dual cone.
The metric $< X, Y >= tr\ XY$ in $Herm_3(S)$ transforms the dual cone $\mathcal{V}' = Herm_3^-(S)$ into the adjoint cone $\mathcal{V}^* = g \circ \mathcal{V}'$.

Two types of natural coordinates in Vinberg cone - matrix and group coordinates

The coordinates of the matrix elements $x_i, X_i, i = 1, 2, 3$ of a matrix $X \in \mathcal{V}$ are called matrix coordinates. Since G acts simply transitively in $\mathcal{V}$, the coordinates $\alpha_i, i = 1, 2, 3, a_{12}, a_{23}, a_{13}$ of an element $A \in G$ define coordinates in $X = A \cdot A^*$, called the group coordinates of the cone $\mathcal{V}$. They are closed, but different form de Witt and Van Proeyen coordinates, used in modern physical literature (when S.Cecotti left physics for politics).

Fundamental invariants and description of the cone by inequalities

Following Vinberg, consider the real-valued homogeneous polynomials of degree 1,2, and 4, which value at $X = A \cdot A^* \in \mathcal{V}$ is given by

$$\begin{aligned}
p_3(X) = &\quad x_3 = \alpha_3^2, \\
p_2(X) = &\quad x_3 x_2 - |X_{23}|^2 = (\alpha_2\alpha_3)^2, \\
p_1(X) = &\quad (x_3 x_1 - |X_{13}|^2)(x_3 x_{22} - |X_{23}|^2) \\
- &\quad (X_{33}X_{12} - X_{13}X_{32})(x_3 X_{21} - X_{23}X_{31}) \\
= &\quad x_3[x_1 x_2 x_3 - x_1|X_1|^2 - x_2|X_2|^2 - x_3|X_3|^2 + 2(X_2 \cdot X_1^*) \cdot X_3^*] \\
= &\quad \alpha_1^2 a_2^2 \alpha_3^4.
\end{aligned}$$

The cone $\mathcal{V}$ is defined by inequalities

$$p_1(X) > 0,\ p_2(X) > 0,\ p_3(X) > 0.$$

Vinberg determinant and the Vinberg-Koszul characteristic function

The quotient

$$\begin{aligned} d(X) := & \ \tfrac{p_1(X)}{p_3(X)} = \\ = & \ x_1 x_2 x_3 - x_1|X_1|^2 - x_2|X_2|^2 - x_3|X_3|^2 + 2(X_1 \cdot X_2^*) \cdot X_3 \\ = & \ (\alpha_1\alpha_2\alpha_3)^2 = (\det A)^2 \end{aligned}$$

is a $G^0 = \{A \in G,\ \alpha_1\alpha_2\alpha_3 = 1\}$-invariant cubic, which we call the Vinberg determinant of the matrix $X = AA^* \in \mathcal{V}$.
The Vinberg-Koszul characteristic function χ is given by

$$\chi^{-1}(X) = \det \rho(A)^2 = \alpha_1^{2+n+N} \alpha_2^{2+n+N} \alpha_3^{2+2N}$$

$$= d(X)^{1+\frac{1}{2}(n+N)} p_3(X)^{\frac{1}{2}(N-n)}.$$

where $X = AA^*,\ n = \dim V,\ N = \dim S = 2\dim S_0$.

The fundamental invariants of the dual cone $\mathcal{V}' = Herm_3^-(S)$

The fundamental invariant of the dual cone $\mathcal{V}'$ is obtained from the fundamental invariants of the cone $\mathcal{V}$ by interchanging indexes $1 \Leftrightarrow 3$. More precisely, the Vinberg determinants coincide

$$d'(X) = d(X),\ X \in Herm_3(S),$$

$$p_3'(X) = x_1 = \alpha_1^2,$$
$$p_2'(X) = x_1 x_2 - |X_3|^2 = (\alpha_1\alpha_2)^2$$
$$p_1'(X) = x_1 d(X) = \alpha_1^4\alpha_2^2\alpha_3^2.$$
$$(\chi')^{-1}(X) = d(X)^{1+1/2(n+N)} p_1(X)^{1/2(n+N)}.$$

The dual cubic d^* and the quadratic maps h, h^*

We denote by $\mathcal{H}^* = Hom(\mathcal{H}, \mathbb{R})$ the dual space to the space $\mathcal{H} := Herm_3(S)$ and by $X^\flat =< X, \cdot >\in \mathcal{H}^*$ the 1-form metrically dual to $X \in \mathcal{H}$. The cubic $d(X)$ defines the dual cubic $d^*(X^\flat) := d(X)$, $X \in \mathcal{H}$. Considering $d(X), d^*(X^\flat)$ as symmetric tensors, we define the associated quadratic maps

$$h : \mathcal{H} \to \mathcal{H}^*,\ h^* : \mathcal{H}^* \to \mathcal{H}$$

$$h(X) := d(X, X, \cdot),\ h^*(X) := d^*(X^\flat, X^\flat, \cdot).$$

We set

$$\mathcal{H}_+ = \{d > 0\}, \quad \mathcal{H}_- = \{d < 0\} \subset \mathcal{H},$$

$$\mathcal{H}^*_+ = \{d^* > 0\}, \quad \mathcal{H}^*_- = \{d^* < 0\} \subset \mathcal{H}^*.$$

Recall that G^0 is the unimodular subgroup $(\alpha_1\alpha_2\alpha_3 = 1)$ of G.

The formula for the inversion of the quadratic map $h : \mathcal{H}_+ \to \mathcal{H}_+^*$

Theorem

(A-Marrani-Spiro '21)

(i) *The quadratic maps* $h : \mathcal{H} \to \mathcal{H}^*, h^* : \mathcal{H}^* \to \mathcal{H}$ *are* G^0*-equivariant and, say, the map* h *induces a diffeomorphism of each* G*-orbit in* $\mathcal{H}_+ \cup \mathcal{H}_-$ *onto a* G*-orbit in* $\mathcal{H}_+^*$*.*

(ii) *for any* $X \in \mathcal{H}$

$$(h^* \circ h)(X) = d(X)X$$

(iii) *The inverse map* h^{-1} *of the diffeomorphism* $h : \mathcal{H}_+ \to \mathcal{H}*_+$ *is given by*

$$h^{-1}(X^\flat) = \frac{1}{\sqrt{d^*(X^\flat)}} h^*(X^\flat)$$

Special Vinberg cones $\mathcal{V} \subset Herm_3(S)$ play an important role in N=2 Supergravity (SUGRA). The determinant hypersurface $\mathcal{V}_1 \subset \mathcal{V}$ describes the target space of the scalar multiplets in $D = 5$ SUGRA and it is called (homogeneous) very special real manifold. More generally, a very special real manifold is defined as a part $\mathcal{V}_1$ of a cubic hypersurface $d = 1$ such that the $-Hess(\log d)|_{\mathcal{V}_1}$ defines a Riemannian metric in $\mathcal{V}_1$. Then $\mathcal{V} = \mathbb{R}^+\mathcal{V}_1$ is a convex cone.

The dimensional reduction to $D = 4$ and r-map

The dimensional reduction from $D = 5$ to $D = 4$ transforms the scalar target space $\mathcal{V}_1$ of $D = 5$ theory into the scalar target space $\mathcal{S} = r(\mathcal{V}_1)$ of $D = 4$ SUGRA (which is a special Kähler manifold). The assignment $\mathcal{V}_1 \to \mathcal{S} = r(\mathcal{V}_1)$ is called the r-map (De Wit, Van Proeyen).

In fact, the image $r(\mathcal{V}_1)$ of the r-map is a special case of the Siegel domain of the first kind (Piatetski-Shapiro), which associates to a convex cone $\mathcal{V} \subset \mathbb{R}^n$ the complex tubular domain $\mathcal{S} := \mathbb{R}^n + i\mathcal{V} \subset \mathbb{C}^n$ with the Kähler metric defined by the Kähler potential

$$\mathcal{K}(z) = -\log 8d(y),\ z = x + iy \in \mathcal{S}.$$

The manifold $\mathcal{S} = r(\mathcal{V}_1)$ is (locally) completely characterized by the holomorphic function(prepotential) given in terms of the Vinberg determinant $d(y^a)$ in homogeneous coordinates by

$$F(z^I) = F(z^0, z^a) = \frac{\sum d_{abc} z^a z^b z^c}{z^0}$$

It defines an embedding of $\mathcal{S}$ as a complex Lagrangian section $dF \subset T^*\mathbb{C}^{n+1}$ of the bundle $T^*\mathbb{C}^{n+1}$.

Dimensional reduction from D=4 to D=3 and c-map

Similarly the dimensional reduction from D=4 to D=3 SUGRA transforms special Kähler manifold $\mathcal{S}$, into the special quaternionic Kähler manifold $\mathcal{Q} = c(\mathcal{S})$ which is the scalar target space of D=3 SUGRA.Thge map $c : \mathcal{S} \to \mathcal{Q} = c(\mathcal{S}) = q(\mathcal{V}_1)$ is called the c-map. De Wit and Van Proeyen defined r-map and c-map in the framework of SUGRA (1994)
see E.Lauria, A.Van Proeyen, N=2 SUGRA in dimension D= 4,5,6, 2020, but they have also purely differential geometric description.

All homogeneous very special manifolds $\mathcal{V}_1$ had been classified by B. de Wit and A. Van Proeyen 1992) and, by other method, by V. Cortes (1996).They 1-1 correspond to Clifford $Cl(q+1,1)$ modules. The associated special Kähler manifolds $\mathcal{S} = r(\mathcal{V}_1)$ are homogeneous and correspond to $Cl(q+2,2)$ modules and the associated homogeneous special quaternionic Kähler manifold $\mathcal{Q} = c(\mathcal{S}) = q(\mathcal{V}_1)$ are correspond to $Cl(q+3,3)$-modules (see Lauria, Van Proeyen, 2020).

The correspondence between quaternionic Kähler manifolds with transitive solvable isometry group and Clifford modules was establishes by D.A. (1975), where it was conjectured that they exhausted all homogeneous quaternionic Kähler manifolds with negative scalar curvature $sc < 0$ and, more generally, all Einstein homogeneous manifolds with $sc < 0$ are solvmanifolds. Both conjectures had been proven by Christoph Böhm and Ramiro Lafuente (2021). This implies that all homogeneous quaternionic Kähler manifolds with $sc < 0$ are special with two exceptions.

Black holes in $N=2, D=4$ Supergravity. Static and spherically symmetric black holes and their electro-magnetic charges

The static spherically symmetric black hole is the 4-dimensional Lorentzian manifold $(M=\mathbb{R}(t)\times\mathbb{R}^3(r,\vartheta,\varphi),g)$, where (r,ϑ,φ) are spherical ooordinates, with the Lorentz metric

$$g=-e^{2U(r)}dt^2+e^{-2U(r)}\left(dr^2+r^2\left(d\theta^2+\sin^2\theta d\varphi^2\right)\right)$$

such that g tends to the flat metric for $r\to\infty$:

$$e^{-2U(r)}\cong_{r\to0^+}\frac{C^2}{r^2},\qquad \text{for some constant } C\in\mathbb{R}. \tag{3}$$

Then there is an event horizon at $r=0$ with the area $A_H=4\pi C^2$. The geometry in the near-horizon limit $r\to0^+$ is $AdS_2\times S^2$. The value $r=0$ determines the boundary of an SO_3-invariant and time independent region $H=S_0^2$ (the colortred event horizon) of the space-like hypersurface $M_{t_o}=\mathbb{R}^3$, from which no light ray might escape.

The physical fields - electromagnetic fields and scalar fields

The physical fields are t-invariant and $SO(3)$ -invariant scalar fields z^a, $a = 1, \cdots, n$ and electromagnetic fields $\mathbb{F}^I$, $I = 0, 1, \cdots, n$.
More precisely, $z^a(r) = x^a(r) + iy^a(r)$ are depending only on the radial coordinate r components of a map $z : M \to \mathcal{S}_{\mathcal{T}}$ into a projective special Kähler manifold

$$\mathcal{S}_{\mathcal{T}} = \mathbb{R}^n + i\mathcal{V} == \mathbb{R}^n + \mathbb{R}^+\mathcal{T}$$

which is the image of a very special real manifold

$$\mathcal{T} \subset \{d(y) = d(y^1, \cdots, y^n) = 1\} \subset \mathbb{R}^n$$

under the r-map.
Roughly speaking, a very special real manifold is a part $\mathcal{T}$ of a cubic hypersurface $d(y) = 1$ s.t. $\mathcal{V} := \mathbb{R}^+\mathcal{T}$ is a convex cone equipped with the Koszul- Vinberg metric $g = -Hess \ln d(y)$.
The electromagnetic fields $\mathbb{F}^I$ are components of the curvature $\mathbb{F} = dA$ of a principal connection $A : TP\mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ of the (trivial) principal T^{n+1} -bundle $P = T^{n+1} \times M \to M$ over the space-time.

For any t_o and $r_o >> 0$, the (independent from t_o, r_o) magnetic charge p^I and the electric charge q_I of the electromagnetic fields are defined by

$$
\begin{aligned}
p^I :=\ & \frac{1}{4\pi} \int_{\mathbf{S}=\{t=t_o, r=r_o\}} \mathbb{F}^I dvol \\
=\ & \frac{1}{4\pi} \iint_{\vartheta\in[0,\pi],\varphi\in[0,2\pi]} \mathbb{F}^I_{\vartheta\varphi} d\vartheta \wedge d\varphi, \quad \text{with } \mathbb{F}^I_{\vartheta\varphi} := \mathbb{F}^I\big|_{\mathbf{S}} \left(\frac{\partial}{\partial\vartheta}, \frac{\partial}{\partial\varphi}\right), \\
q_I :=\ & \frac{1}{4\pi} \iint_{\vartheta\in[0,\pi],\varphi\in[0,2\pi]} \mathbb{G}_{I|\vartheta\varphi} d\vartheta \wedge d\varphi, \ \text{with } \mathbb{G}_{I|\vartheta\varphi} := \left(\star\mathbb{F}^I\big|_{\mathbf{S}}\right) \left(\frac{\partial}{\partial\vartheta}, \frac{\partial}{\partial\varphi}\right).
\end{aligned}
$$

The 2-form $\mathbb{G}$ is uniquely determined by the electromagnetic field $\mathbb{F}$ by means of a generalised Hodge star operator defined by the scalar fields.

These charges determines the electromagnetic fields $\mathbb{F}^I$.

BPS black hole and the expression of its entropy in term of the module of the central charge

Beside electric and magnetic charges, the black hole carries also cental charge $Z(r)$ introduced by E. Witten. It is depending only of r complex-valued function $Z : M \to \mathbb{C}$. The absolute value $|Z(r)|$ is bounded from above by the mass m of the black hole

$$m \geq |Z(r)|.$$

If $m = |Z(0)|$ the black hole is called BPS (Bogomol'ny, Prasad and Sommerfield black hole.

For BPS black hole, the Bekenstein-Hawking entropy-area formula $S = \frac{A_H}{4}$, where A_H is the area the event horizon $r = 0$, take form

$$S = \pi|Z(0)|^2 = \pi(t_o^2)$$

, where $Z(0) = t_o e^{i\vartheta_o}$.

The lift of scalar fields to the Lagrangian section defined by the prepotential F

The scalar fileds $z = z(r) : M \to \mathcal{S}$ with value in the projective scalar manifold $\mathcal{S} = \mathbb{R}^n + i\mathcal{V} \subset \mathbb{C}^n$, is lifted to a map

$$X = X(r) : M \to \mathcal{M} \subset \mathbb{C}^{n+1} , \qquad X(r) = (1, z^1(r), \ldots, z^n(r))$$

into the associated conical special Kähler manifold $\mathcal{M}$, which locally is the section $dF(X) \subset T^*\mathbb{C}^{n+1} = \mathbb{C}^{2n^2}$ of the holomorphic function (prepotential)

$$F(X) = \frac{d(X^1, X^2, \ldots X^n)}{X^0} = \frac{d_{abc}X^a X^b X^c}{X^0}.$$

(The complex coordinates $X^0, X^1, \cdots, X^n$ are homogeneous coordinates associated with affine complex coordinates $(z^1, \cdots, z^n)$.) The function $\mathcal{K}(y) = \mathcal{K}(z, \bar{z}) = -\log(8d(Im\,(z))$ is the potential of the Kähler metric $g_{\mathcal{S}_{\mathcal{T}}}$.

Inverse relations

We use subscript o to denote the value of the central charge $Z(r)$, the scalar field $z(r) \in \mathcal{S}$ and its lift $X(r) \in \mathcal{M}$ at the point $r = 0$:

$$Z_o := Z(0) \in \mathbb{C},\ z_o = z(0) \in \mathcal{S},\ X_o = (1, z^1(0), \dots, z^n(0)) \in \mathcal{M}.$$

The following fundamental "inverse relations"express the magnetic and electric charges p_I, q^I in terms of the values $Z_o = t_o e^{\vartheta_o}, X_0$.

$$\begin{gathered} p^0(t, \theta, z, \overline{z}) = \mathfrak{c} t \cos\theta, \\ p(t, \theta, z, \overline{z}) = -\mathfrak{c} t Im(e^{-i\theta} z), \\ q_0 = \mathfrak{c} t Im\big(e^{-i\vartheta} \left(d(z)\right)\big)\ , \\ q = -3\mathfrak{c} t Im\left(e^{-i\vartheta} h(z)\right) \end{gathered} \tag{4}$$

where $h : \mathcal{V} \to \mathbb{R}^{n*}$ is the quadratic map defined by $d(y)$.
The fundamental inverse relations were derived in the deep theory of the attractor mechanism.

BPS map and its inversion

The calculation of entropy is reduced to inverting the map (BPS map)

$$\mathfrak{f} : \mathbb{C}^* \times (\mathbb{R}^n + i\mathcal{V}) \subset \mathbb{C}^{n+1} \longrightarrow \mathbb{R}^{2n+2} \; ,$$
$$\mathfrak{f}(Z, z^a, \overline{Z}, \overline{z}^a) = p^K(Z, z^a, \overline{Z}, \overline{z}^a), q_L(Z, z^a, \overline{Z}, \overline{z}^a)) \; , \quad (5)$$

called the *BPS map*.
The formula for such inversion had been obtained by Shmakova.

A version of Shmakova formula for inverse BPS map

The following formula expresses the entropy $S = \pi |Z_0|^2 = \pi t_o^2$ of an BPS black hole as a function of magnetic p^I, $p^0 \neq 0$ and electric q^I charges. It is equivalent to the Shmakova formula, which used the de Witt- Van Proleyen coordinates.

$$S = \sqrt{2}\sqrt{\frac{\sin q}{p^0}\left\langle h\left(p\right) - \frac{1}{3}p^0\mathfrak{q}, h^{-1}\left(h\left(p\right) - \frac{1}{3}p^0 q\right)\right\rangle}$$

Case when $\mathcal{S}$ is a homogeneous scalar manifolds $\mathcal{S}$ associated to a special rank 3 Vinberg cone

Then the corresponding quadratic map $h|_{\mathcal{V}} : \mathcal{V} \to \mathcal{V}^* \subset \mathbb{R}^{n*}$ is globally invertible with inverse given by

$$h^{-1}(y) = \frac{1}{\sqrt{d'(y)}} h'(y)$$

with $d' = \frac{1}{k} d'_{\mathcal{V}}$ the dual invariant cubic polynomial. Using the formula for the inverse map h^{-1}, we get $S = \pi\sqrt{I_4}$,

$$I_4 = \frac{1}{(p^0)^2}[4d'\left(h(p) - \frac{1}{3}p^0 q\right) - \left((q_0 p^0 + \langle q, p\rangle)p^0 - 2d(p)\right)^2] \tag{6}$$

A remarkable fact is that, this rational function is actually a *quartic polynomial*

$$I_4 = -(q_0 p^0 + \langle q, p\rangle)^2 + 4q_0 d(p) - \frac{4}{27}p^0 d'(q) + \frac{4}{3}\langle h(p), h'(q)\rangle \tag{7}$$

Theorem

This leads to the following final result:
In ungauged $N = 2$ $D = 4$ supergravity with homogeneous scalar manifold $\mathcal{S} = \mathbb{R}^n + i\mathcal{V}$ associates to a special Vinberg cone $\mathcal{V}$, the entropy of the BPS black holes is expressed in terms of their magnetic and electric charges (p^0, p, q_0, q) by

$$S = \pi\sqrt{I_4}$$

$$= \pi\sqrt{-(q^0p^0 + \langle q, p\rangle)^2 + 4q_0 d(p) - \frac{4}{27}p^0 d'(q) + \frac{4}{3} < h(p), h'(q) >}$$

where $d(p)$, $d'(q)$ are the Vinberg determinant and its dual and $h(p)$, $h'(q)$ the associated quadratic maps.