Ref.: Stefanucci & van Leeuwen, Chapter 4

2.2 Keldysh contour

2.2.1 Time-dependent quantum averages

Goal: Compute time-dependent quantum average of operator $\hat{O}(t)$ at time t when system is initially in state $|\Psi(t_0)\rangle \equiv |\Psi_0\rangle$.

Possible solution: Propagate the state via Schrödinger equation

$$|\Psi(t)\rangle = \hat{U}(t,t_0)|\Psi_0\rangle$$

with the time evolution operator $(t>t_0)$

$$\hat{U}(t,t_0) = T\left\{e^{-i\int_{t_0}^{t} d\bar{t}\, \hat{H}(\bar{t})}\right\}.$$

$T \equiv$ time-ordering operator, later physical times to the left.

This leads to

explicit time dependence of operator

$$O(t) \equiv \langle\Psi(t)|\hat{O}(t)|\Psi(t)\rangle = \langle\Psi_0|\bar{T}\left\{e^{-i\int_{t}^{t_0} d\bar{t}\, \hat{H}(\bar{t})}\right\}\hat{O}(t) \times$$
$$\times\, T\left\{e^{-i\int_{t_0}^{t} d\bar{t}\, \hat{H}(\bar{t})}\right\}|\Psi_0\rangle.$$

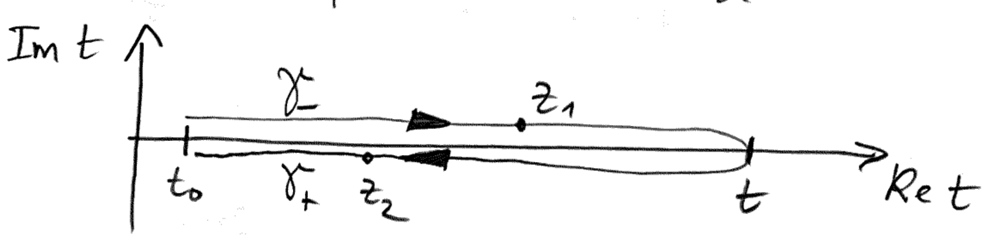
$\bar{T}$ = anti-time-ordering

$\longrightarrow$ computing an operator expectation value naturally requires forward + backward propagation in time!

Def.: We define the oriented contour γ in the complex time plane — the convenience of choosing complex time will become clear later:

$$\gamma \equiv \underbrace{(t_0, t)}_{\gamma_-} \oplus \underbrace{(t, t_0)}_{\gamma_+}$$

forward backward



Note: time arguments on both γ_- and γ_+ are real — the forward and backward branches are for bookkeeping.

Def.: Operators with contour arguments

$$\hat{A}(z') \equiv \begin{cases} \hat{A}_-(t') & \text{if } z' = t'_- \\ \hat{A}_+(t') & \text{if } z' = t'_+ \end{cases} \qquad [\hat{A}_+ \text{ and } \hat{A}_- \text{ could be different.}]$$

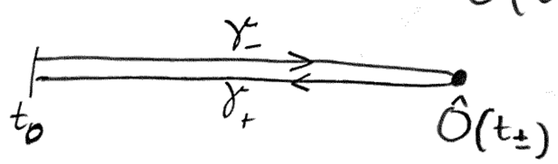
Def.: Contour time-ordering operator $\mathcal{T}$ moves operators with later contour time to the left — in particular, all operators on the γ_+-branch are to the left of all operators on the γ_--branch, and the γ_--branch-operators are physically time-ordered (T), the γ_+-branch operators are physically anti-time-ordered $(\overline{T})$ by the $\mathcal{T}$ operator.

For two operators $\hat{A}(z_1)$ and $\hat{B}(z_2)$ there are the following possibilities:

$$\mathcal{T}\{\hat{A}(z_1)\hat{B}(z_2)\} = \begin{cases} T\{\hat{A}_-(t_1)\hat{B}_-(t_2)\} & \text{if } z_1 = t_{1-},\ z_2 = t_{2-} \\ \hat{A}_+(t_1)\ \hat{B}_-(t_2) & \text{if } z_1 = t_{1+},\ z_2 = t_{2-} \\ \hat{B}_+(t_2)\ \hat{A}_-(t_1) & \text{if } z_1 = t_{1-},\ z_2 = t_{2+} \\ \overline{T}\{\hat{A}_+(t_1)\hat{B}_+(t_2)\} & \text{if } z = t_{1+},\ z_2 = t_{2+} \end{cases}$$

Now we can rewrite the operator expectation value in a more compact way:

$$O(t) = \langle \Psi_0 | \mathcal{T}\{ e^{-i\int_{\gamma_+} d\bar{z}\, \hat{H}(\bar{z})}\, \hat{O}(t_\pm)\, e^{-i\int_{\gamma_-} d\bar{z}\, \hat{H}(\bar{z})} \} | \Psi_0 \rangle$$



Inside the $\mathcal{T}$ sign we can treat all operators "as if they commute" — remember that $\mathcal{T}$ will order them contour-chronologically anyway.

$$\Rightarrow\quad O(t) = \langle \Psi_0 | \mathcal{T}\{ e^{-i\int_\gamma d\bar{z}\, \hat{H}(\bar{z})}\, \hat{O}(t_\pm)\} | \Psi_0 \rangle$$

with $\int_\gamma \equiv \int_{\gamma_-} + \int_{\gamma_+}$

All physical observables have $\hat{O}(t_\pm) \equiv \hat{O}(t)$ – the same on both γ_+ and γ_-.

Important: $\hat{O}(t)$ is not $\hat{O}_H(t) \equiv$ operator in Heisenberg picture.

The t-argument in $\hat{O}(t)$ is meant as a reminder where along γ to insert the operator.

Extend contour to infinity:

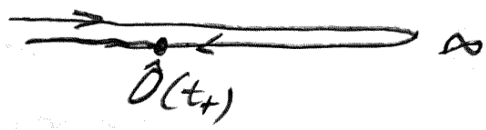
$$\mathcal{T}\left\{e^{-i\int_\gamma d\bar{z}\,\hat{H}(\bar{z})}\,\hat{O}(t_-)\right\} = \hat{U}(t_0,\infty)\hat{U}(\infty,t)\hat{O}(t)\hat{U}(t,t_0)$$

$$= \hat{U}(t_0,t)\,\hat{O}(t)\,\hat{U}(t,t_0)$$

and

$$\mathcal{T}\left\{e^{-i\int_\gamma d\bar{z}\,\hat{H}(\bar{z})}\,\hat{O}(t_+)\right\} = \hat{U}(t_0,t)\,\hat{O}(t_+)\,\hat{U}(t,\infty)\,\hat{U}(\infty,t_0)$$

$$= \hat{U}(t_0,t)\,\hat{O}(t)\,\hat{U}(t,t_0)$$



which shows that $\hat{O}(t)$ does not change if we extend the contour:

$$\boxed{O(z) = \langle\Psi_0|\,\mathcal{T}\left\{e^{-i\int_\gamma d\bar{z}\,\hat{H}(\bar{z})}\,\hat{O}(z)\right\}|\Psi_0\rangle}$$

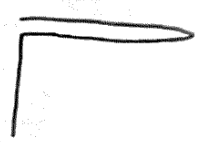
$\gamma \equiv$ Keldysh contour (Keldysh 1964)

But: Used by Schwinger already in 1961

$\Rightarrow$ also known as Schwinger–Keldysh contour.

Next step: extend the $\Longrightarrow$ contour to include imaginary times (cf. Chapter 1.8)

$\Rightarrow$ L-shaped contour



2.2.2 Time-dependent ensemble averages

Now: pure state $|\Psi_0\rangle \longrightarrow$ mixed state with probability distribution $\{w_n\}$, $w_n \in [0,1]$ and $\sum_n w_n = 1$, w_n the probability of finding the system in state $|\chi_n\rangle$ with $\langle \chi_n | \chi_n \rangle = 1$ (normalized) but $\langle \chi_m | \chi_n \rangle \neq \delta_{nm}$ (no orthogonality required).

E.g. thermal ensemble (canonical) with $w_n = \dfrac{e^{-\beta E_n}}{\sum_n e^{-\beta E_n}}$

Expectation value: $O(t_0) = \sum_n w_n \langle \chi_n | \hat{O}(t_0) | \chi_n \rangle$.

Def.: Density matrix operator

$$\hat{\rho} \equiv \sum_n w_n |\chi_n\rangle\langle \chi_n|$$

$$\hat{\rho} = \hat{\rho}^+ \quad \text{(self-adjoint)}$$

$$\langle \Psi | \hat{\rho} | \Psi \rangle = \sum_n w_n |\langle \Psi | \chi_n \rangle|^2 \geq 0 \quad \text{pos. semi-def.}$$

Let $\{|\Psi_k\rangle\}$ be a generic orthonormal basis set. Then

$$O(t_0) = \sum_{kn} w_n \langle \chi_n | \Psi_k \rangle \langle \Psi_k | \hat{O}(t_0) | \chi_n \rangle = \sum_k \langle \Psi_k | \hat{O}(t_0) \hat{\rho} | \Psi_k \rangle \equiv \mathrm{Tr}[\hat{O}(t_0)\hat{\rho}]$$

trace in Fock space = trace over complete set of many-body states.

In particular, since $\hat{O}(t_0) = \mathbb{1}$ implies $O(t_0) = 1$ and the $\{|\chi_n\rangle\}$ are normalized, one has $\mathrm{Tr}[\hat{\rho}] = 1$. (*)

We choose kets $|\Psi_k\rangle$ as eigenkets of $\hat{\rho}$: $\hat{\rho}|\Psi_k\rangle = \rho_k |\Psi_k\rangle$.

$\Rightarrow \hat{\rho} = \sum_k \rho_k |\Psi_k\rangle\langle\Psi_k|$ with $\rho_k \in [0,1]$ and $\sum_k \rho_k = 1$.

$\Rightarrow \mathrm{Tr}[\hat{\rho}^2] \leq 1$. (**)

Most general expression fulfilling (*) and (**): $\rho_k = \dfrac{e^{-x_k}}{\sum_p e^{-x_p}}$, $x_k \in \mathbb{R}$.

We can write $x_k = \beta E_k^M$ with $\beta > 0$ and define

$$\hat{H}^M = \sum_k E_k^M |\Psi_k\rangle\langle\Psi_k|.$$

$$\Rightarrow \boxed{\hat{\rho} = \sum_k \frac{e^{-\beta E_k^M}}{Z} |\Psi_k\rangle\langle\Psi_k| = \frac{e^{-\beta \hat{H}^M}}{Z}} \quad \text{with } \underbrace{Z}_{\text{partition function}} \equiv \sum_k e^{-\beta E_k^M} = \mathrm{Tr}\left[e^{-\beta \hat{H}^M}\right]$$

M = Matsubara.

For example, in a grand-canonical ensemble we choose

$$\hat{H}^M = \hat{H} - \mu \hat{N} \quad \text{and} \quad \beta = \frac{1}{k_B T},$$

$k_B \equiv$ Boltzmann constant

$T \equiv$ temperature

$\mu \equiv$ chem. pot.

$\hat{N} \equiv$ total particle number operator.

Ensemble time evolution:

Evolve each subsystem of the ensemble, then perform the weighted average. Same $\hat{H}(z)$ for all subsystems!

$$O(z) = \sum_n w_n \langle \chi_n | \hat{U}(t_0, z) \hat{O}(z) \hat{U}(z, t_0) | \chi_n \rangle = \mathrm{Tr}\left[\hat{\rho}\, \hat{U}(t_0,z) \hat{O}(z) \hat{U}(z,t_0)\right] =$$

$$= \mathrm{Tr}\left[\hat{\rho}\, \mathcal{T}\left\{e^{-i\int_\gamma d\bar{z}\, \hat{H}(\bar{z})} \hat{O}(z)\right\}\right] = \frac{\mathrm{Tr}\left[e^{-\beta \hat{H}^M} \mathcal{T}\{\dots\}\right]}{\mathrm{Tr}\left[e^{-\beta \hat{H}^M}\right]} \qquad (*)$$

We observe the following:

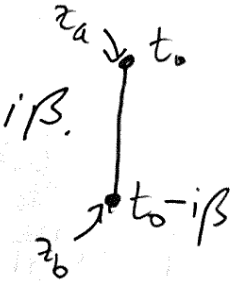
(1) $\mathcal{T}\left\{e^{-i\int_\gamma d\bar{z}\, \hat{H}(\bar{z})}\right\} = \hat{U}(t_0, \infty)\hat{U}(\infty, t_0) = \hat{\mathbb{1}}$

$\Rightarrow$ can be inserted into trace in denominator of $(*)$.

(2) The exponential can be written as

$$e^{-\beta \hat{H}^M} = e^{-i\int_{\gamma^M} d\bar{z}\, \hat{H}^M(\bar{z})}$$

where γ^M is a contour $z_a \to z_b$ with $z_b - z_a = -i\beta$.

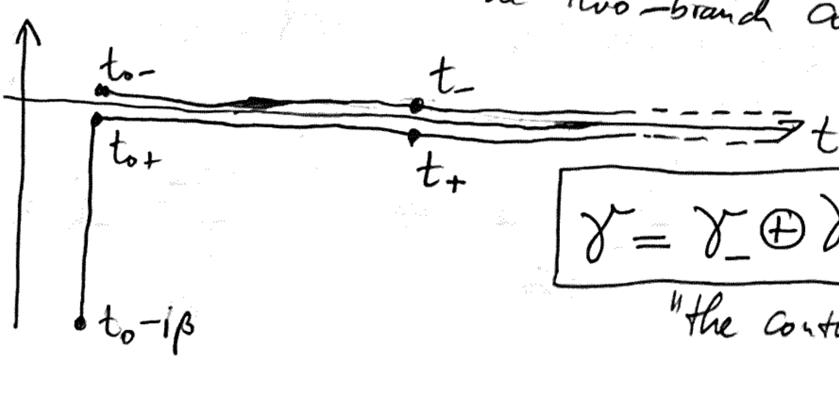


Using (1) and (2) in $(*)$ gives

$$O(z) = \frac{\mathrm{Tr}\left[e^{-i\int_{\gamma^M} d\bar{z}\, \hat{H}(\bar{z})}\, \mathcal{T}\left\{e^{-i\int_\gamma d\bar{z}\, \hat{H}(\bar{z})} \hat{O}(z)\right\}\right]}{\mathrm{Tr}\left[e^{-i\int_{\gamma^M} d\bar{z}\, \hat{H}(\bar{z})}\, \mathcal{T}\left\{e^{-i\int_\gamma d\bar{z}\, \hat{H}(\bar{z})}\right\}\right]}.$$

Note: statistical averaging amounts precisely to time propagation along the imaginary time axis γ^M.

$\Rightarrow$ we can add γ^M to the two-branch contour!



$$\boxed{\gamma = \gamma_- \oplus \gamma_+ \oplus \gamma^M}$$

"the contour"

Note: In practice γ does not extend to ∞ but has finite length depending on the maximal time at which we wish to obtain information about the system.

We finally have
$$O(z) = \frac{\mathrm{Tr}\left[\mathcal{T}\left\{e^{-i\int_\gamma d\bar{z}\,\hat{H}(\bar{z})}\,\hat{O}(z)\right\}\right]}{\mathrm{Tr}\left[\mathcal{T}\left\{e^{-i\int_\gamma d\bar{z}\,H(\bar{z})}\right\}\right]}.$$

2.3 Many-body perturbation theory — towards Feynman diagrams

2.3.1 Equations of motion on the contour

Def.: Contour evolution operator

"later on γ"

$$\hat{U}(z_2, z_1) = \begin{cases} \mathcal{T}\left\{e^{-i\int_{z_1}^{z_2} d\bar{z}\,\hat{H}(\bar{z})}\right\} & z_2 \succ z_1 \\ \bar{\mathcal{T}}\left\{e^{+i\int_{z_2}^{z_1} d\bar{z}\,\hat{H}(\bar{z})}\right\} & z_2 \prec z_1 \end{cases}$$

Properties: (1) $\hat{U}(z,z) = \mathbb{1}$

(2) $\hat{U}(z_3, z_2)\,\hat{U}(z_2, z_1) = \hat{U}(z_3, z_1)$

(3) Differential equation for $z \succ z_0$:

$$i\frac{d}{dz}\hat{U}(z,z_0) = \mathcal{T}\left\{i\frac{d}{dz}\,e^{-i\int_{z_0}^{z} d\bar{z}\,\hat{H}(\bar{z})}\right\} = \mathcal{T}\left\{\hat{H}(z)\,e^{-i\int_{z_0}^{z} d\bar{z}\,\hat{H}(\bar{z})}\right\} =$$
$$= \hat{H}(z)\,\hat{U}(z, z_0)$$

and accordingly $i\frac{d}{dz}\hat{U}(z_0, z) = -\hat{U}(z_0, z)\,\hat{H}(z)$.

Contour derivatives:

$\underline{z = t_-}$: $$\frac{d}{dz}\hat{A}(z) = \lim_{z'\to z}\frac{\hat{A}(z') - \hat{A}(z)}{z' - z} = \lim_{\varepsilon\to 0}\frac{\hat{A}_-(t+\varepsilon) - \hat{A}_-(t)}{\varepsilon} = \frac{d}{dt}\hat{A}_-(t),$$

$\underline{z = t_+}$: $\frac{d}{dz}\hat{A}(z) = \frac{d}{dt}\hat{A}_+(t)$ $\Rightarrow$ same derivatives on γ_+ and γ_- if operators are the same.

$\underline{t \in \gamma^M}$: $$\frac{d}{dz}\hat{A}(z) = \lim_{z'\to z}\frac{\hat{A}(z') - \hat{A}(z)}{z' - z} = \lim_{\varepsilon\to 0}\frac{\hat{A}(t_0 - i(\tau+\varepsilon)) - \hat{A}(t_0 - i\tau)}{-i\varepsilon} =$$
$$= i\frac{d}{d\tau}\hat{A}(t_0 - i\tau).$$

We can rewrite the ensemble average

$$O(z) = \frac{\mathrm{Tr}\left[\mathcal{T}\left\{e^{-i\int_\gamma d\bar{z}\,\hat{H}(\bar{z})}\,\hat{O}(z)\right\}\right]}{\mathrm{Tr}\left[\mathcal{T}\left\{e^{-i\int_\gamma d\bar{z}\,\hat{H}(\bar{z})}\right\}\right]}$$

using $\underbrace{z_i \equiv t_{0-}}_{\text{initial point on } \gamma}$ and $\underbrace{z_f \equiv t_0 - i\beta}_{\text{final point on } \gamma}$ as

$$O(z) = \frac{\mathrm{Tr}\left[\hat{U}(z_f,z)\,\hat{O}(z)\,\hat{U}(z,z_i)\right]}{\mathrm{Tr}\left[\hat{U}(z_f,z_i)\right]} = \frac{\mathrm{Tr}\left[\hat{U}(z_f,z_i)\overbrace{\hat{U}(z_i,z)\,\hat{O}(z)\,\hat{U}(z,z_i)}^{(*)}\right]}{\mathrm{Tr}\left[\hat{U}(z_f,z_i)\right]}$$

(*) motivates to introduce the Heisenberg picture on the contour:

$$\boxed{\hat{O}_H(z) \equiv \hat{U}(z_i,z)\,\hat{O}(z)\,\hat{U}(z_i,z)}$$

which give $\hat{O}_H(t_+) = \hat{O}_H(t_-) = \hat{O}_H(t)$ = operator in standard Heisenberg picture for real times.

Equation of motion:

$$i\frac{d}{dz}\hat{O}_H(z) = \hat{U}(z_i,z)\left[\hat{O}(z),\hat{H}(z)\right]\hat{U}(z,z_i) + i\frac{\partial}{\partial z}\hat{O}_H(z)$$

$$= \left[\hat{O}_H(z),\hat{H}(z)\right] + \underbrace{i\frac{\partial}{\partial z}\hat{O}_H(z)}_{\text{explicit time dependence}}.$$

We will consider Hamiltonians of the generic form

$$\hat{H}^M = \underbrace{\int dx \int dx'\, \psi^\dagger(x)\,\langle x|\hat{h}^M|x'\rangle\,\psi(x')}_{\equiv \hat{H}_0^M} \leftarrow \text{one-body terms}$$

$$+ \underbrace{\frac{1}{2}\int dx \int dx'\, V^M(x,x')\,\psi^\dagger(x)\,\psi^\dagger(x')\,\psi(x')\,\psi(x)}_{\hat{H}^M_{int}} \leftarrow \text{two-body interactions}$$

with field operators $\psi^\dagger, \psi$ that fulfill (anti-)commutation relations

$$\left[\psi(x),\psi^\dagger(y)\right]_\pm = \delta(x-y) \quad \text{for fermions or bosons.}$$

Using the Heisenberg EOM and the notation with a local density operator

$$\hat{n}_H(x',z) \equiv \psi_H^+(x',z)\,\psi_H(x',z)$$

one obtains the equations of motion for the field operators:

$$\boxed{\begin{aligned} i\frac{d}{dz}\psi_H(x,z) &= \int dx' \langle x|\hat{h}(z)|x'\rangle \psi_H(x',z) + \int dx'\, V(x,x',z)\,\hat{n}_H(x',z)\,\psi_H(x,z), \\ -i\frac{d}{dz}\psi_H^+(x,z) &= \int dx'\, \psi_H^+(x',z)\langle x'|\hat{h}(z)|x\rangle + \int dx'\, V(x,x',z)\,\psi_H^+(x,z)\,\hat{n}_H(x',z) \end{aligned}}$$

where $\hat{h}(z=t_\pm) = \hat{h}(t)$ $\qquad V(x,x',t_\pm) = V(x,x',t)$

$\hat{h}(z\in\gamma^M) = \hat{h}^M$ $\qquad V(x,x',z\in\gamma^M) = V^M(x,x')$.

2.3.2 Operator correlators on the contour

Our continued goal is to compute expectation values $O(t)$ on the contour. The key obstacle is to treat the time-ordered exponentials (time evolution operators) in cases where we do not a priori know all the eigenstates of the Hamiltonian. The expansion of the time-ordered exponentials yields strings of operators (operator correlators)

$$\hat{k}(z_1,\dots,z_n) = \mathcal{T}\{\hat{O}_1(z_1)\dots\hat{O}_n(z_n)\},$$

e.g., $\mathcal{T}\{n_H(x',z^+)\,\psi_H(x,z)\}$, where $z^+ > z$ infinitesimally.

$\Rightarrow$ need to find relations for operator correlators.

Abbreviation: $\hat{O}_j = \hat{O}_j(z_j)$.

Example: $\mathcal{T}\{\hat{O}_1\hat{O}_2\} = \underbrace{\Theta(z_1,z_2)}_{\gamma\text{-heaviside}}\hat{O}_1\hat{O}_2 + \Theta(z_2,z_1)\hat{O}_2\hat{O}_1$

$$\overset{\text{EOM}}{\Rightarrow}\ \frac{d}{dz_1}\mathcal{T}\{\hat{O}_1\hat{O}_2\} = \delta(z_1,z_2)\,[\hat{O}_1,\hat{O}_2]_- + \mathcal{T}\left\{\left(\frac{d}{dz_1}\hat{O}_1\right)\hat{O}_2\right\},$$

with γ - Dirac - δ function $\delta(z_1, z_2) \equiv \frac{d}{dz_1}\Theta(z_1, z_2) \equiv -\frac{d}{dz_2}\Theta(z_1, z_2)$:

$$\int_{z_i}^{z_f} d\bar{z} \;\; \delta(z, \bar{z})\, \hat{A}(\bar{z}) = \hat{A}(z).$$

Most important case: $\hat{O}_1$ and $\hat{O}_2$ field operators. For bosons the structure with commutators $[\hat{O}_1, \hat{O}_2]_-$ at equal times is convenient. For fermions we prefer an anticommutator in order to obtain simpler expressions. Hence we define fermionic time-ordering as

$$\mathcal{T}\{\hat{O}_1, \hat{O}_2\} \equiv \Theta(z_1, z_2)\, \hat{O}_1 \hat{O}_2 - \Theta(z_2, z_1)\, \hat{O}_2 \hat{O}_1,$$

which gives

$$\frac{d}{dz_1} \mathcal{T}\{\hat{O}_1, \hat{O}_2\} = \delta(z_1, z_2)\, [\hat{O}_1, \hat{O}_2]_+ + \mathcal{T}\left\{\left(\frac{d}{dz_1}\hat{O}_1\right)\hat{O}_2\right\}.$$

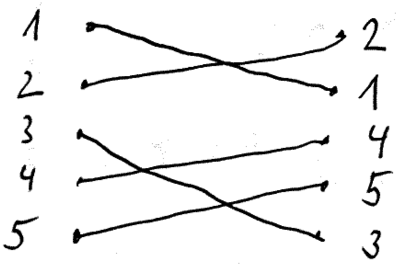
Generalized definition for strings of operators:

$$\mathcal{T}\{\hat{O}_1 \ldots \hat{O}_n\} = (\pm)^P\, \mathcal{T}\{\hat{O}_{P(1)} \ldots \hat{O}_{P(n)}\}$$

with $\pm$ bosons / fermions and P the time-ordering permutation.

Graphical way to find the sign of a permutation:
e.g., Contour-ordering with $z_2 > z_1 > z_4 > z_5 > z_3$.

Draw



and count # crossings $\equiv n_c$.

$$\Rightarrow (\pm)^P = (\pm)^{n_c}$$

$$\Rightarrow \mathcal{T}\{\hat{O}_1 \hat{O}_2 \hat{O}_3 \hat{O}_4 \hat{O}_5\} = (-1)^3\, \hat{O}_2 \hat{O}_1 \hat{O}_4 \hat{O}_5 \hat{O}_3.$$

with $P(1,2,3,4,5) = (2,1,4,5,3)$ here.

Equipped with this knowledge, we can now derive an eq. of motion for a string of operators:

(*) $$\frac{d}{dz_k}\mathcal{T}\{\hat O_1,\dots,\hat O_n\} = \partial^\Theta_{z_k}\mathcal{T}\{\hat O_1\dots\hat O_n\} + \mathcal{T}\left\{\hat O_1\dots\hat O_{k-1}\left(\frac{d}{dz_k}\hat O_k\right)\hat O_{k+1}\dots\hat O_n\right\}$$

with $$\partial^\Theta_{z_k}\mathcal{T}\{\hat O_1\dots\hat O_n\} \equiv \sum_P (\pm)^P\left(\frac{d}{dz_k}\Theta_n(z_{P(1)},\dots,z_{P(n)})\right)\hat O_{P(1)}\dots\hat O_{P(n)}.$$

More explicitly one can show that

$$\partial^\Theta_{z_k}\mathcal{T}\{\hat O_1\dots\hat O_n\} = \sum_{l=1}^{k-1}(\pm)^{k-l}\,\delta(z_k,z_l)\,\mathcal{T}\{\hat O_1,\dots,\hat O_{l-1}\hat O_{l+1}\dots\hat O_{k-1}[\hat O_k,\hat O_l]_\mp\hat O_{k+1}\dots\hat O_n\}$$

(**) $$+\sum_{l=k+1}^{n}(\pm)^{l-k-1}\,\delta(z_k,z_l)\,\mathcal{T}\{\hat O_1\dots\hat O_{k-1}[\hat O_k,\hat O_l]_\mp\hat O_{k+1}\dots\hat O_{l-1}\hat O_{l+1}\dots\hat O_n\}.$$

(**) and (*) are the n-operator generalizations of the EoM for two operators.

Example with 5 operators:

$$\begin{aligned}\frac{d}{dz_3}\mathcal{T}\{\hat O_1\hat O_2\hat O_3\hat O_4\hat O_5\} &= \delta(z_3,z_1)\,\mathcal{T}\{\hat O_2[\hat O_3,\hat O_1]_\mp\hat O_4\hat O_5\}\\
&\pm \delta(z_3,z_2)\,\mathcal{T}\{\hat O_1[\hat O_3,\hat O_2]_\mp\hat O_4\hat O_5\}\\
&+ \delta(z_3,z_4)\,\mathcal{T}\{\hat O_1\hat O_2[\hat O_3,\hat O_4]_\mp\hat O_5\}\\
&\pm \delta(z_3,z_5)\,\mathcal{T}\{\hat O_1\hat O_2[\hat O_3,\hat O_5]_\mp\hat O_4\}\\
&+ \mathcal{T}\left\{\hat O_1\hat O_2\left(\frac{d}{dz_3}\hat O_3\right)\hat O_4\hat O_5\right\}\end{aligned}$$

with signs determined by the required number of interchanges to shift $\hat O_l$ $(l=1,2,3,4,5)$ directly after $\hat O_k$ with $k=3$ here.

Specifically for field operators in the contour Heisenberg picture,

$$[\hat O_k(z),\hat O_l(z)]_\mp = \underbrace{c_{kl}(z)}_{\text{number}}\,\mathbb{1}$$

The $\mathbb{1}$ commutes with all Fock space operators and can be moved outside the contour-ordered product:

$$\partial^{\theta}_{z_k} \mathcal{T}\{\hat{O}_1 \dots \hat{O}_n\} = \sum_{l=1}^{k-1} (\pm)^{k-l} \delta(z_k, z_l) \left[\hat{O}_k, \hat{O}_l\right]_{\mp} \mathcal{T}\{\hat{O}_1 \dots \overset{\sqcap}{\hat{O}_l} \dots \overset{\sqcap}{\hat{O}_k} \dots \hat{O}_n\}$$
$$+ \sum_{l=k+1}^{n} (\pm)^{l-k-1} \delta(z_k, z_l) \left[\hat{O}_k, \hat{O}_l\right]_{\mp} \mathcal{T}\{\hat{O}_1 \dots \overset{\sqcap}{\hat{O}_k} \dots \overset{\sqcap}{\hat{O}_l} \dots \hat{O}_n\}$$

"$\sqcap$": operator missing from string.

We now work out the time derivative for a case of 4 field operators:

Define $i \equiv x_i, z_i$; $j \equiv x_j, z_j$, $i' \equiv x_i', t_i'$, $j' \equiv x_j', t_j'$ etc.

$$\delta(j,k) \equiv \delta(z_j, z_k)\, \delta(x_j - x_k).$$

Then we have

$$\frac{d}{dz_2} \mathcal{T}\{\hat{\psi}_H(1)\, \hat{\psi}_H(2)\, \hat{\psi}_H^+(3)\, \hat{\psi}_H^+(4)\} = \mathcal{T}\{\hat{\psi}_H(1) \left(\frac{d}{dz_2} \hat{\psi}_H(2)\right) \hat{\psi}_H^+(3)\, \hat{\psi}_H^+(4)\}$$
$$+ \delta(2,3)\, \mathcal{T}\{\hat{\psi}_H(1)\, \hat{\psi}_H^+(4)\} \pm \delta(2,4)\, \mathcal{T}\{\hat{\psi}_H(1)\, \hat{\psi}_H^+(3)\}.$$

We define the n-particle correlator (<u>Green's function</u>) as

$$\hat{G}_n(1, \dots, n; 1', \dots, n') \equiv \frac{1}{i^n} \mathcal{T}\{\hat{\psi}_H(1) \dots \hat{\psi}_H(n)\, \hat{\psi}_H^+(n') \dots \hat{\psi}_H^+(1')\}$$

with $\hat{G}_0 \equiv \mathbb{1}$ $(n=0)$.

We identify $\hat{O}_j = \begin{cases} \hat{\psi}_H(j) & j = 1, \dots, n \\ \hat{\psi}_H^+((2n-j+1)') & j = n+1, \dots, 2n \end{cases}$

and find

$$i \frac{d}{dz_k} \hat{G}_n(1, \dots, n; 1', \dots, n') = \frac{1}{i^n} \mathcal{T}\{\hat{\psi}_H(1) \dots \left(i \frac{d}{dz_k} \hat{\psi}_H(k)\right) \dots \hat{\psi}_H(n)\, \hat{\psi}_H^+(n') \dots \hat{\psi}_H^+(1')\}$$
$$+ \sum_{j=1}^{n} (\pm)^{k+j} \delta(k, j')\, \hat{G}_{n-1}(1, \dots \overset{\sqcap}{k} \dots, n; 1', \dots \overset{\sqcap}{j'} \dots n')$$

and

$$-i\frac{d}{dz'_k}\hat{G}_n(1,\dots n;1',\dots,n') = \frac{1}{i^n}\mathcal{T}\{\hat{\psi}_H(1)\dots\hat{\psi}_H(n)\hat{\psi}^+_H(n')\dots(-i\frac{d}{dz'_k}\hat{\psi}^+_H(k'))\dots\hat{\psi}^+_H(1')\}$$

$$+\sum_{j=1}^{n}(\pm)^{k+j}\delta(j,k')\hat{G}_{n-1}(1\dots\overset{\sqcap}{j}\dots n;1'\dots k'\dots n')$$

with $(n-j)+(n-k)$ interchanges, and $(\pm)^{n-j+(n-k)} = (\pm)^{k+j}$.

Now we assume that $\hat{h}$ is diagonal in spin space:

$$\langle x_1|\hat{h}(z_1)|x_2\rangle = h(1)\,\delta(x_1-x_2) = \delta(x_1-x_2)\,h(2).$$

The EOMs then become (for the field operators):

$$i\frac{d}{dz_k}\hat{\psi}_H(k) = h(k)\hat{\psi}_H(k) + \int d\bar{1}\, v(k;\bar{1})\,\hat{n}_H(\bar{1})\hat{\psi}_H(k)$$

$$-i\frac{d}{dz'_k}\hat{\psi}^+_H(k') = \hat{\psi}^+_H(k')h(k') + \int d\bar{1}\, v(k';\bar{1})\,\hat{\psi}^+_H(k')\hat{n}_H(\bar{1})$$

with $v(i;j) \equiv \delta(z_i,z_j)\,v(x_i,x_j,z_i)$.

Inside the $\mathcal{T}$ we can write

$$\mathcal{T}\{\dots\hat{n}_H(\bar{1})\hat{\psi}_H(k)\dots\} = \pm\mathcal{T}\{\dots\hat{\psi}_H(k)\hat{\psi}_H(\bar{1})\hat{\psi}^+_H(\bar{1}^+)\dots\}$$

where $\bar{1}^+$ has a time later on γ than $\bar{1}$.

Then we can write

$$\frac{1}{i^n}\mathcal{T}\{\hat{\psi}_H(1)\dots(i\frac{d}{dz_k}\hat{\psi}_H(k))\dots\hat{\psi}_H(n)\hat{\psi}^+_H(n')\dots\hat{\psi}^+_H(1')\}$$

$$= h(k)\hat{G}_n(1,\dots,n;n',\dots,1')$$

$$\pm\frac{1}{i^n}\int d\bar{1}\, v(k;\bar{1})\,\mathcal{T}\{\hat{\psi}_H(1)\dots\hat{\psi}_H(n)\boxed{\hat{\psi}_H(\bar{1})\hat{\psi}_H(\bar{1}^+)}\hat{\psi}^+_H(n')\dots\hat{\psi}^+_H(1')\}$$

$$= h(k)\hat{G}_n \pm i\int d\bar{1}\, v(k;\bar{1})\,\hat{G}_{n+1}(1,\dots,n,\bar{1};1',\dots,n',\bar{1}^+)$$

and analogous for the second eq. with $\bar{z}_1^-$.

Inserting into the EOMs we find

$$\left[i\frac{d}{dz_k}-h(k)\right]\hat{G}_n(1,\dots,n;1',\dots,n') = \pm i\int d\bar{1}\, v(k;\bar{1})\,\hat{G}_{n+1}(1,\dots,n,\bar{1};1',\dots,n',\bar{1}^+)$$
$$+\sum_{j=1}^{n}(\pm)^{k+j}\,\delta(k;j')\,\hat{G}_{n-1}(1,\dots\overline{k}\dots n;1'\dots\overline{j'}\dots n'),$$

$$\hat{G}_n(1,\dots,n;1',\dots,n')\left[-i\frac{\overleftarrow{d}}{dz'_k}-h(k')\right] = \pm i\int d\bar{1}\, v(k';\bar{1})\,\hat{G}_{n+1}(1,\dots,n,\bar{1};1',\dots,n',\bar{1})$$
$$+\sum_{j=1}^{n}(\pm)^{k+j}\,\delta(j;k')\,\hat{G}_{n-1}(1,\dots\overline{j}\dots n;1',\dots,\overline{k'},\dots,n').$$

- hierarchy of operator equations in Fock space
- independent of specific shape of contour
- basis for diagrammatic perturbation theory.