2.2 Keldysh contour

2.2.1 Time-dependent quantum averages

Goal: Compute time-dependent quantum average of operator \( \hat{O}(t) \) at time \( t \) when system is initially in state \( |\psi(t_0)\rangle = |\psi_0\rangle \).

Possible solution: Propagate the state via Schrödinger equation

\[
|\psi(t)\rangle = \hat{U}(t, t_0) |\psi_0\rangle
\]

with the time evolution operator \((t > t_0)\)

\[
\hat{U}(t, t_0) = T \{ e^{-i \int_{t_0}^{t} dt \hat{A}(t')} \}
\]

where:

- \( T \) = time-ordering operator, places times to the left.

This leads to explicit time dependence of operator

\[
O(t) = \langle \psi(t) | \hat{O}(t) | \psi(t) \rangle = \langle \psi_0 | \overline{T} \{ e^{i \int_{t_0}^{t} dt \hat{A}(t')} \} \hat{O}(t) \times T \{ e^{-i \int_{t_0}^{t} dt \hat{A}(t')} \} | \psi_0 \rangle,
\]

where \( T \) = anti-time-ordering.

\[
\rightarrow \text{computing an operator expectation value naturally requires forward and backward propagation in time!}
\]

Def.: We define the oriented contour \( \gamma \) in the complex time plane — the convenience of choosing complex time will become clear later:

\[
\gamma = (t_0, t) \cup (-t, t_0)
\]

Forward: \( t > t_0 \)

Backward: \( t < t_0 \)

Note: time arguments on both \( \gamma_+ \) and \( \gamma_- \) are real — the forward and backward branches are for bookkeeping.
Def. Operators with contour arguments
\[ \hat{A}(z') = \begin{cases} \hat{A}_-(z') & \text{if } z' = z_- \\ \hat{A}_+(z') & \text{if } z' = z_+ \end{cases} \] [\( \hat{A}_+ \) and \( \hat{A}_- \) could be different]

Def. Contour time-ordering operator \( \mathcal{T} \) moves operators with later contour time to the left — in particular, all operators on the \( \hat{J}_+ \)-branch are to the left of all operators on the \( \hat{J}_- \)-branch, and the \( \hat{J}_- \)-branch operators are physically time-ordered \((T)\) the \( \hat{J}_+ \)-branch operators are physically anti-time-ordered \((\overline{T})\) by the \( \mathcal{T} \) operator.

For two operators \( \hat{A}(z_1) \) and \( \hat{B}(z_2) \) there are the following possibilities:
\[
\mathcal{T} \{ \hat{A}(z_1) \hat{B}(z_2) \} = \begin{cases} \mathcal{T} \{ \hat{A}_-(z_1) \hat{B}_-(z_2) \} & \text{if } \tau_1 = \tau_{1-}, \tau_2 = \tau_{2-} \\ \hat{A}_+(z_1) \hat{B}_-(z_2) & \text{if } \tau_1 = \tau_{1+}, \tau_2 = \tau_{2-} \\ \hat{B}_+(z_2) \hat{A}_-(z_1) & \text{if } \tau_1 = \tau_{1-}, \tau_2 = \tau_{2+} \\ \mathcal{T} \{ \hat{A}_+(z_1) \hat{B}_+(z_2) \} & \text{if } \tau_1 = \tau_{1+}, \tau_2 = \tau_{2+} \end{cases}
\]

Now we can rewrite the operator expectation value in a more compact way:
\[
\mathcal{O}(t) = \langle \Psi_0 | \mathcal{T} \left\{ e^{-i \int_{t_0}^{t} d\hat{z} \hat{A}(\hat{z})} \hat{O}(t_\pm) e^{-i \int_{t_0}^{t} d\hat{z} \hat{A}(\hat{z})} \right\} | \Psi_0 \rangle
\]

Inside the \( \mathcal{T} \) sign we can treat all operators "as if they commute"—remember that \( \mathcal{T} \) will order them contour-chronologically anyway.

\[
\Rightarrow \mathcal{O}(t) = \langle \Psi_0 | \mathcal{T} \left\{ e^{-i \int_{\delta_-}^{\delta_+} d\hat{z} \hat{A}(\hat{z})} \hat{O}(t_\pm) \right\} | \Psi_0 \rangle
\]

with \( \int_{\delta} \equiv \int_{\delta_-} + \int_{\delta_+} \)
All physical observables have $\hat{O}(t_+ - t_-) = \hat{O}(t)$ — the same on both $\mathcal{H}_+$ and $\mathcal{H}_-$.

Important: $\hat{O}(t)$ is not $\hat{O}_H(t) = \text{operator in Heisenberg picture}$.

The $t$-argument in $\hat{O}(t)$ is meant as a reminder where along $\mathcal{H}$ to insert the operator.

Extend contour to infinity:

\[
\int \left\{ e^{-i \int_{\mathcal{H}_+} d\tau \hat{A}(\tau)} \hat{O}(\tau) \right\} = \hat{U}(t_0, \infty) \hat{U}(\infty, t) \hat{O}(t) \hat{U}(t, t_0) = \hat{U}(t_0, t) \hat{O}(t) \hat{U}(t, t_0)
\]

and

\[
\int \left\{ e^{-i \int_{\mathcal{H}_+} d\tau \hat{A}(\tau)} \hat{O}(\tau) \right\} = \hat{U}(t_0, t) \hat{O}(t) \hat{U}(t, \infty) \hat{U}(\infty, t_0) = \hat{U}(t_0, t) \hat{O}(t) \hat{U}(t, t_0)
\]

which shows that $\hat{O}(t)$ does not change if we extend the contour:

\[
\hat{O}(t) = \langle \psi_0 | \int \left\{ e^{-i \int_{\mathcal{H}_+} d\tau \hat{A}(\tau)} \hat{O}(\tau) \right\} | \psi_0 \rangle
\]

$\mathcal{H}$ = Keldysh contour (Keldysh, 1964)

But: Used by Schwinger already in 1961

$\Rightarrow$ also known as Schwinger-Keldysh contour.

Next step: extend the $\mathcal{H}$ contour to include imaginary times (cf. Chapter 1.8)

$\Rightarrow$ L-shaped contour
2.2.2 Time-dependent ensemble averages

Now: pure state \( |\Psi_0 \rangle \rightarrow \) mixed state with probability distribution \( \{w_n\} \), \( w_n \in [0,1] \) and \( \sum_n w_n = 1 \), \( w_n \)
the probability of finding the system in state \( |X_n \rangle \)
with \( \langle X_n | X_n \rangle = 1 \) (normalized) but \( \langle X_n | X_m \rangle \neq \delta_{mn} \)
(no orthogonality required).

E.g., Thermal ensemble with \( w_n = \frac{e^{-\beta E_n}}{\sum_n e^{-\beta E_n}} \)
(canonical)

Expectation value: \( \hat{O}(t_0) = \sum_n w_n \langle X_n | \hat{O}(t_0) | X_n \rangle \).

Def.: Density matrix operator
\[ \hat{\rho} = \sum_n w_n |X_n \rangle \langle X_n | \]
\[ \hat{\rho} = \hat{\rho}^+ \quad (\text{self-adjoint}) \]
\[ \langle \psi | \hat{\rho}^+ | \psi \rangle = \sum_n w_n |\langle \psi | X_n \rangle|^2 \geq 0 \quad \text{pos. semi-def.} \]

Let \( \{ |\Psi_k \rangle \} \) be a generic orthonormal basis set. Then
\[ \hat{O}(t_0) = \sum_{kn} w_n \langle X_n | \hat{O}(t_0) | X_k \rangle = \sum_k \langle \Psi_k | \hat{O}(t_0) \hat{\rho} | \Psi_k \rangle = \text{Tr}[\hat{O}(t_0) \hat{\rho}] \]
In particular, since \( \hat{O}(t_0) = 1 \) implies \( \hat{O}(t_0) = 1 \) and
the \( \{ |X_n \rangle \} \) are normalized, one has \( \text{Tr}[\hat{\rho}] = 1 \).
We choose \( \{ |\Psi_k \rangle \} \) as eigenkets of \( \hat{\rho} \): \( \hat{\rho} |\Psi_k \rangle = \omega_k |\Psi_k \rangle \).
\[ \Rightarrow \hat{\rho} = \sum_k \omega_k |\Psi_k \rangle \langle \Psi_k | \]
\[ \Rightarrow \text{Tr}[\hat{\rho}^2] \leq 1 \quad (\ast \ast) \]
Most general expression fulfilling \((\ast)\) and \((\ast \ast)\): \( \omega_k = \frac{e^{\frac{x_k}{\beta}}}{\sum_{\beta} e^{\frac{x_p}{\beta}}} \), \( x_k \in \mathbb{R} \).
we can write \( x_k = \beta E_k^M \) with \( \beta > 0 \) and define
\[ M = \sum_k E_k^M |\Psi_k \rangle \langle \Psi_k | \]
\[ \Rightarrow \hat{\rho} = \sum_k \frac{e^{-\beta E_k^M}}{Z} |\Psi_k \rangle \langle \Psi_k | = \frac{e^{-\beta H^M}}{Z} \]
with \( Z = \sum_k e^{-\beta E_k^M} = \text{Tr}[e^{-\beta H^M}] \)

\( M = \text{Makonbar} \).
For example, in a grand-canonical ensemble we choose
\[ \hat{H}^N = \hat{H} - \mu \hat{N} \quad \text{and} \quad \beta = \frac{1}{k_B T}, \]
where \( k_B \) is Boltzmann constant, \( T \) is temperature, \( \mu \) is chemical potential, and \( \hat{N} \) is the total particle number operator.

**Ensemble Time Evolution:**

Evolve each subsystem of the ensemble, then perform the weighted average. Same \( \mathcal{A}(\tau) \) for all subsystems!

\[
\mathcal{O}(\tau) = \sum_h w_h \langle \chi_h | \hat{U}(t, \tau) \mathcal{A}(\tau) \hat{U}(\tau, t_0) | \chi_h \rangle = \text{Tr} \left[ \hat{S} \hat{U}(t_0, \tau) \mathcal{A}(\tau) \hat{U}(\tau, t_0) \right] = \text{Tr} \left[ \hat{S} \int \left\{ e^{-i \frac{d}{dt} \hat{A}(\tau)} \hat{O}(\tau) \right\} \right] = \frac{\text{Tr} \left[ e^{-\beta \hat{H}^N} \mathcal{O}(\tau) \right]}{\text{Tr} \left[ e^{-\beta \hat{H}^N} \right]} \quad (\ast)
\]

We observe the following:

1. \( \int \left\{ e^{-i \frac{d}{dt} \hat{A}(\tau)} \right\} = \hat{U}(t_0, \infty) \hat{U}(\infty, t_0) = \hat{I} \)
   
   \( \Rightarrow \) can be inserted into trace in denominator of (\ast).

2. The exponential can be written as
   
   \[ e^{-\beta \hat{H}^N} = e^{-i \int \sigma_m dt \hat{A}^m(\tau)} \]

   where \( \gamma^m \) is a contour \( z_a \rightarrow z_b \) with \( z_b - z_a = -i \beta \).

Using (1) and (2) in (\ast) gives

\[
\mathcal{O}(\tau) = \frac{\text{Tr} \left[ e^{-i \int \sigma_m dt \hat{A}^m(\tau)} \mathcal{O}(\tau) \right]}{\text{Tr} \left[ e^{-i \int \sigma_m dt \hat{A}^m(\tau)} \right]}.
\]

Note: Statistical averaging amounts precisely to time propagation along the imaginary time axis \( \gamma^m \).

\( \Rightarrow \) we can add \( \gamma^m \) to the two-branch contour!

\[ \gamma = \gamma_- \oplus \gamma_+ \oplus \gamma^m \]

"the contour"
Note: In practice \( Y \) does not extend to \( \infty \) but has finite length, depending on the maximal time at which we wish to obtain information about the system.

We finally have
\[
\mathcal{O}(t) = \frac{\text{Tr} \left[ \oint \{ e^{-i \int_{t_0}^{t} \, d\tau \hat{A}(\tau) \} \hat{\mathcal{O}}(\tau) \} \right]}{\text{Tr} \left[ \oint \{ e^{-i \int_{t_0}^{t} \, d\tau \hat{H}(\tau) \} \} \right]}.
\]

2.3 Many-body perturbation theory — towards Feynman diagrams

2.3.1 Equations of motion on the contour

**Def.** Contour evolution operator
\[
\hat{U}(t_2, t_1) = \begin{cases} \int \{ e^{-i \int_{t_1}^{t_2} \, d\tau \hat{A}(\tau) \} \} & \text{if } t_2 > t_1, \\ \int \{ e^{i \int_{t_1}^{t_2} \, d\tau \hat{A}(\tau) \} \} & \text{if } t_2 < t_1. \end{cases}
\]

**Properties:**
1. \( \hat{U}(t_2, t_2) = \mathbb{1} \)
2. \( \hat{U}(t_3, t_2) \hat{U}(t_2, t_1) = \hat{U}(t_3, t_1) \)
3. **Differential equation** for \( t > t_0 \):
\[
i \frac{d}{dt} \hat{U}(t, t_0) = \int \left\{ i \frac{d}{dt} e^{-i \int_{t_0}^{t} \, d\tau \hat{A}(\tau)} \right\} = \int \left\{ \hat{A}(t) e^{-i \int_{t_0}^{t} \, d\tau \hat{A}(\tau)} \right\} = \hat{A}(t) \hat{U}(t, t_0)
\]
and accordingly
\[
i \frac{d}{dt} \hat{U}(t_0, t) = -\hat{U}(t_0, t) \hat{A}(t).
\]

**Contour derivatives:**
\[
\begin{align*}
\dot{Z} = t_-: & \quad \frac{d}{dt} \hat{A}(t) \lim_{t_+ \rightarrow t} = \lim_{t_+ \rightarrow t} = \frac{\hat{A}(t) - \hat{A}(t_+)}{t_+ - t} = \frac{\hat{A}(t) - \hat{A}(t_+)}{t_+ - t} = \frac{d}{dt} \hat{A}(t), \\
\dot{Z} = t_+: & \quad \frac{d}{dt} \hat{A}(t) = \frac{d}{dt} \hat{A}(t) \rightarrow \text{some derivatives on } \mathcal{X} \text{ and } \mathcal{Y} \\
t \in \mathcal{Y}: & \quad \frac{d}{dt} \hat{A}(t) = \lim_{t_+ \rightarrow t} = \frac{\hat{A}(t) - \hat{A}(t_+)}{t_+ - t} = \lim_{t_+ \rightarrow t} = \frac{\hat{A}(t) - \hat{A}(t_+)}{t_+ - t} = \frac{d}{dt} \hat{A}(t_0 - i \tau).
\end{align*}
\]
We can rewrite the ensemble average
\[ O(t) = \frac{\text{Tr} \left[ e^{-i \int_0^t dz A(z) B(z)} \right]}{\text{Tr} \left[ e^{-i \int_0^t dz A(z)} \right]} \]

using \( z_i = t_0 \) as initial point and \( z_f = t_0 - i \beta \) as final point on \( \gamma \)

\[ O(t) = \frac{\text{Tr} \left[ \hat{U}(z_f, z) \hat{B}(z) \hat{U}(z, z_i) \right]}{\text{Tr} \left[ \hat{U}(z_f, z_i) \right]} = \frac{\text{Tr} \left[ \hat{U}(z_f, z) \hat{B}(z) \hat{U}(z, z_i) \hat{U}(z, z_i) \right]}{\text{Tr} \left[ \hat{U}(z_f, z_i) \right]} \]

(x) motivates to introduce the Heisenberg picture on the contour:

\[ \hat{\mathcal{O}}_H(z) = \hat{U}(z_f, z) \hat{\mathcal{O}}(z) \hat{U}(z, z_i) \]

which gives \( \hat{\mathcal{O}}_H(t_f) = \hat{\mathcal{O}}_H(t_i) = \hat{\mathcal{O}}_H(t) \)

= operator in standard Heisenberg picture for real times.

Equation of motion:
\[ i \frac{d}{dt} \hat{\mathcal{O}}_H(z) = \hat{U}(z_f, z) \left[ \hat{\mathcal{O}}(z), \hat{\mathcal{H}}(z) \right] \hat{U}(z, z_i) + i \frac{\partial}{\partial z} \hat{\mathcal{O}}_H(z) \]

\[ = \left[ \hat{\mathcal{O}}_H(z), \hat{\mathcal{H}}(z) \right] + i \frac{\partial}{\partial z} \hat{\mathcal{O}}_H(z) \]

explicit time dependence.

We will consider Hamiltonians of the generic form
\[ \hat{\mathcal{H}}^M = \int \text{d}x \int \text{d}x' \left\langle x' | \hat{\mathcal{H}}^M | x' \right\rangle \psi(x) \psi^+(x') \equiv \hat{\mathcal{H}}_0^M \]

\[ + \frac{1}{2} \int \text{d}x \int \text{d}x' \left\langle x' | \hat{\mathcal{V}}^M(x, x') \psi(x) \psi^+(x') \psi(x) \psi^+(x') \right\rangle \]

\[ \equiv \hat{\mathcal{H}}_\text{int}^M \]

With field operators \( \psi, \psi^+ \) that fulfill (anti-)commutation relations

\[ [\psi(x), \psi^+(y)] = \delta(x-y) \]

for fermions or bosons.
Using the Heisenberg EOM and the notation with a local density operator
\[ \hat{\psi}_H(x', t) = \hat{\gamma}^+(x', t) \hat{\gamma}_H(x', t) \]

one obtains the equations of motion for the field operators:

\[
\begin{align*}
 i \frac{d}{dt} \psi_H(x, t) &= \int dx' \langle x | \hat{h}(t) | x' \rangle \psi_H(x', t) + \int dx' V(x, x', t) \hat{\gamma}_H(x', t) \psi_H(x', t) \\
- i \frac{d}{dx} \psi^+_H(x, t) &= \int dx' \psi^+_H(x', t) \langle x' | \hat{h}(t) | x \rangle + \int dx' V(x, x', t) \psi^+_H(x', t) \hat{\gamma}_H(x', t) \\
\end{align*}
\]

Where
\[ \hat{h}(t = t^\pm) = \hat{h}(t) \quad V(x, x', t^\pm) = V(x, x') \quad V(x, x', t^\pm) V^M(x, x') = V^M(x, x') \]

2.3.2 Operator Correlators on the Contour

Our continued goal is to compute expectation values \( \langle O(t) \rangle \) on the contour. The key obstacle is to treat the time-ordered exponentials (time evolution operators) in cases where we do not a priori know all the eigenstates of the Hamiltonian. The expansion of the time-ordered exponentials yields strings of operators (operator correlators)
\[ \hat{\psi}(z_1, \ldots, z_n) = \mathcal{T} \{ \hat{\gamma}(z_1) \ldots \hat{\gamma}(z_n) \}, \]

e.g., \( \mathcal{T} \{ \psi_H(x, t^+), \psi_H(x, t^-) \} \), where \( t^+ < t^- \) infinitesimally.

We need to find relations for operator correlators.

Abbreviation: \( \delta_j = \delta_j(t^\pm) \).

Example: \( \mathcal{T} \{ \delta_j, \delta_k \} = \Theta(t^\pm, t^\pm) \delta_j \delta_k + \Theta(t^\pm, t^\pm) \delta_k \delta_j \)

EOM
\[
\begin{align*}
\frac{d}{dt} \mathcal{T} \{ \delta_j, \delta_k \} &= \delta_\Gamma(t^\pm, t^\pm) \{ \delta_j, \delta_k \} + \mathcal{T} \{ \frac{d}{dt} \delta_j, \delta_k \} \\
\end{align*}
\]
with \( \gamma = \text{Dirac spinor} \) \( \delta (z_1, z_2) \equiv \frac{d}{dz} \Theta (z_1, z_2) = -\frac{d}{dz} \Theta (z_1, z_2) \)

\[
\int_{z_1}^{z_2} \delta (\bar{z}, z) \hat{A} (\bar{z}) = \hat{A} (z)
\]

Most important case: \( \delta \), and \( \delta \), field operators. For bosons the structure with commutators \([\hat{\delta}, \hat{\delta}]\) at equal time is convenient. For fermions, we prefer an anticommutator in order to obtain simpler expressions. Hence we define fermionic time-ordered as

\[
T \{ \delta_1, \delta_2 \} = \Theta (z_1, z_2) \delta_1 \delta_2 - \Theta (z_2, z_1) \delta_2 \delta_1
\]

which gives

\[
\frac{d}{dz_2} T \{ \delta_1, \delta_2 \} = \delta (z_1, z_2) [\delta_1, \delta_2] + T \left\{ (\frac{d}{dz_1}) \delta_2 \right\}
\]

Generalized definition for strings of operators:

\[
T \{ \delta_1, ..., \delta_n \} = (\pm) P T \{ \delta_{p(1)}, ..., \delta_{p(n)} \}
\]

with \( \pm \) bosons and \( P \) the time-ordering permutation.

Graphical way to find the sign of a permutation:

e.g., contour ordering with \( z_2 > z_1 > z_4 > z_5 > z_3 \).

Draw \( 1 \)

\( 2 \)

\( 3 \)

\( 4 \)

\( 5 \)

\( \Rightarrow \) \( T \{ \delta_1, \delta_2, \delta_3, \delta_4, \delta_5 \} = (-1)^2 \delta_2 \delta_3 \delta_4 \delta_5 \delta_1 \delta_2 \).

\( \text{with } P(1,2,3,4,5) = (2,1,4,5,3) \) here.

Equipped with this knowledge, we can now derive eq. of motion for a string of operators.
\[ \frac{d}{dt} \mathcal{J} \{ \delta_1, \ldots, \delta_n \} = 2 \Theta \mathcal{J} \{ \delta_1, \ldots, \delta_n \} + \mathcal{J} \{ \delta_1 \ldots \delta_{k-1} \left( \frac{d}{dt_k} \delta_k \right) \delta_{k+1} \ldots \delta_n \} \]

with \[ 2 \Theta \mathcal{J} \{ \delta_1, \ldots, \delta_n \} = \sum_p (\pm) \left( \frac{d}{dt_k} \left( 2 \Theta_{p(n)} \ldots 2 \Theta_{m} \right) \delta_{p(n)} \ldots \delta_{p(m)} \right) \]

More explicitly, one can show that

\[ 2 \Theta \mathcal{J} \{ \delta_1, \ldots, \delta_n \} = \sum_{\ell=1}^{k-1} (\pm) \delta(t_k, t_\ell) \mathcal{J} \{ \delta_1 \ldots \delta_{k-1} [\delta_k, \delta_\ell] \delta_{k+1} \ldots \delta_n \} \]

\[ + \sum_{\ell=1}^{n} (\pm) \delta(t_k, t_\ell) \mathcal{J} \{ \delta_1 \ldots \delta_{k-1} [\delta_k, \delta_\ell] \delta_{k+1} \ldots \delta_n \} \]

\((***)\) and \((*)\) are the n-operator generalizations of the Ehf for two operators.

Example with 5 operators:

\[ \frac{d}{dt_3} \mathcal{J} \{ \delta_1, \delta_2, \delta_3, \delta_4, \delta_5 \} = \delta(t_3, t_4) \mathcal{J} \{ \delta_3 \left[ \delta_2, \delta_1 \right]_+ \delta_4 \delta_5 \} \]

\[ \pm \delta(t_3, t_4) \mathcal{J} \{ \delta_3 \left[ \delta_2, \delta_1 \right]_+ \delta_4 \delta_5 \} \]

\[ + \delta(t_3, t_4) \mathcal{J} \{ \delta_3 \delta_4 \left[ \delta_2, \delta_5 \right] \delta_1 \delta_5 \} \]

\[ \pm \delta(t_3, t_4) \mathcal{J} \{ \delta_3 \delta_4 \left[ \delta_2, \delta_5 \right] \delta_1 \delta_5 \} \]

\[ + \mathcal{J} \{ \delta_3, \delta_4 \left( \frac{d}{dt_3} \delta_3 \right) \delta_1 \delta_5 \} \]

with signs determined by the required number of interchanges to shift \( \delta_k \) \((l=1, 2, 3, 4, 5)\) directly after \( \delta_k \) with \( k=3 \) here.

Specifically, for field operators in the contour Heisenberg picture,

\[ \left[ \delta_k(t), \delta_l(t) \right]_+ = \frac{\mathcal{C}_{\text{He}}(t)}{\text{number}} \]

The \( \mathcal{A} \) commute with all Fock space operators and can be
We now work out the time derivative for a case of field operators:

Define $i' = x_i, t_i, j' = x_j, t_j, i'' = x_i', t_i', j'' = x_j', t_j', \text{ etc.}$

Then we have

$$\frac{d}{dt} \mathcal{T} \{ \Phi^+(1) \Phi^+(2) \Phi^+(3) \Phi^+(4) \} = \mathcal{T} \{ \Phi^+(1) \left( \frac{d}{dt} \Phi^+(2) \right) \Phi^+(3) \Phi^+(4) \}$$

$$+ \delta^{(2, 2)} \mathcal{T} \{ \Phi^+(1) \Phi^+(4) \} \pm \delta^{(2, 4)} \mathcal{T} \{ \Phi^+(1) \Phi^+(3) \}.$$ 

We define the $n$-particle correlator (Green's function) as

$$\hat{G}_n (1, ..., n; 1', ..., n') = \frac{1}{i^n} \mathcal{T} \{ \Phi^+_H (1) \cdots \Phi^+_H (n) \Phi^+_H (n) \cdots \Phi^+_H (1) \}$$

with $\hat{G}_0 = 1$ ($n=0$).

We identify $\delta_j = \begin{cases} \Phi^+_H (j) & \text{for } j = 1, \ldots, n \\ \Phi^+_H (2n-j+1) & \text{for } j = n+1, \ldots, 2n \end{cases}$

and

$$i \frac{d}{dx_k} \hat{G}_n (1, ..., n; 1', ..., n') = \frac{1}{i^n} \mathcal{T} \{ \Phi^+_H (1) \cdots \left( i \frac{d}{dx_k} \Phi^+_H (k) \right) \cdots \Phi^+_H (n) \Phi^+_H (n) \cdots \Phi^+_H (1) \}$$

$$+ \sum_{j=1}^n (\pm)^{k+j} \delta(k, j) \hat{G}_{n-1} (1, ..., k, ..., n; 1', ..., j', ..., n')$$

and
\[-i \frac{d}{dz_k} \hat{G}_n (1, \ldots, n; 1', \ldots, n') = \frac{1}{i^n} \sum_{(\pm)} \hat{\Psi}_H (k) \hat{\Phi}_H (n) \hat{\Phi}_H^+ (n) \cdots (i \frac{d}{dz_k} \hat{\Phi}_H^+ (1') \cdots \hat{\Phi}_H^+ (1)) \]

\[+ \sum_{j=1}^{n} (\pm) \hat{\Psi}_H (j) \delta (j, k') \hat{G}_{n-1} (1, \ldots, n; j, 1', \ldots, n') \]

with \((n-j)+ (n-k)\) interchanges, and \((\pm)^{n-j+(n-k)} = (\pm)^{k+j}\).

Now we assume that \(\hat{h}\) is diagonal in spin space:
\[
\langle x_i | \hat{h} (x) | x_\ell \rangle = h (1) \delta (x_i - x_\ell) = \delta (x_i - x_\ell) h (2).
\]

The EOMs then become (for the field operators):
\[
i \frac{d}{dz_k} \hat{\Phi}_H (k) = h (k) \hat{\Phi}_H (k) + \int d\tau \nabla (k, \tau) \hat{\Phi}_H (\tau) \hat{\Phi}_H (k)
\]

\[-i \frac{d}{dz_k} \hat{\Phi}_H^+ (k') = \hat{\Phi}_H^+ (k') h (k') + \int d\tau \nabla (k', \tau) \hat{\Phi}_H^+ (\tau) \hat{\Phi}_H (k')
\]

with \(\nabla (i, j) = \delta (i, j) \nabla (x_i - x_j, \xi_j).\)

Inside the \(\int\) we can write
\[\int \ldots \hat{\Phi}_H (\tau) \hat{\Phi}_H (k) \ldots \rangle = \pm \int \ldots \hat{\Psi}_H (k') \hat{\Phi}_H (\tau) \hat{\Phi}_H^+ (k) \ldots \rangle
\]
where \(\hat{1}\) has a time later on \(\tau\) than \(\hat{1}\).

Then we can write
\[
\frac{1}{i^n} \int_\mathbb{R} \{ \hat{\Phi}_H (1) \cdots (i \frac{d}{dz_k} \hat{\Phi}_H (k)) \cdots \hat{\Phi}_H (n) \hat{\Phi}_H^+ (n) \cdots \hat{\Phi}_H^+ (1) \}
\]

\[= h (k) \hat{G}_n (1, \ldots, n; n', 1') \]
\[\pm \frac{1}{i^n} \int d\tau \nabla (k, \tau) \int \{ \hat{\Phi}_H (1) \cdots \hat{\Psi}_H (n) \hat{\Phi}_H (\tau) \hat{\Phi}_H (k') \hat{\Phi}_H^+ (k) \cdots \hat{\Phi}_H^+ (1) \}
\]

\[-h (k) \hat{G}_n \pm i \int d\tau \nabla (k, \tau) \hat{G}_{n+1} (1, \ldots, n; 1', 1', \ldots, n', \tau')
\]
and analogous for the second eq. with \(\hat{\eta}\).

Inserting into the EOMs we find
\[
\left[ i \frac{d}{d \omega} - h(\omega) \right] \widehat{G}_n (1, \ldots, n; \bar{1}, \ldots, \bar{n}) = \pm i \int d\bar{\omega} V(\omega, \bar{\omega}) \widehat{G}_{n+1} (1, \ldots, n, \bar{1}, \ldots, \bar{n}, \bar{\omega}) + \sum_{j=1}^{n} \left( \pm \delta_{ij} \right) \widehat{G}_{n-1} (1, \ldots, \bar{1}, \ldots, \bar{n}, \bar{1}, \ldots, \bar{n})
\]

- hierarchy of operator equations in Fock space
- independent of specific shape of contour
- basis for diagrammatic perturbation theory