

Topological Band Theory

- References:
- Bernevig & Hughes, Topological Insulators and Superconductors
 - Lecture Notes by C. Kane
 - edX online course: Topology in condensed matter : tying quantum knots (Delft X)

Plan for the lecture:

- ① 0D examples, topology, and symmetry
- ② 1D example: Bulk-edge Correspondence in Kitaev chain
- ③ Charge pumping
- ④ Quantum Hall Effect
- ⑤ Chern Insulators
- ⑥ From quantum spin Hall effect to topological insulators
- ⑦ Kubo formula and TKNN invariant

① 0D examples, topology and symmetry

Topology : Discrete things that cannot change
continuously

Interesting if discreteness has measurable physical consequences. Topology can be used to classify physical systems with an excitation energy gap.

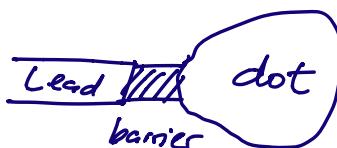
Zero-dimensional quantum systems

Consider a quantum system with N states

\Rightarrow Hamiltonian represented by $N \times N$ matrix H

$$H|n\rangle = E_n|n\rangle$$

e.g., a small quantum dot



Fermi level E_F in metallic lead

\Rightarrow states with $E_n < E_F$ are filled in dot
(set $E_F = 0$)

Topology and gapped systems

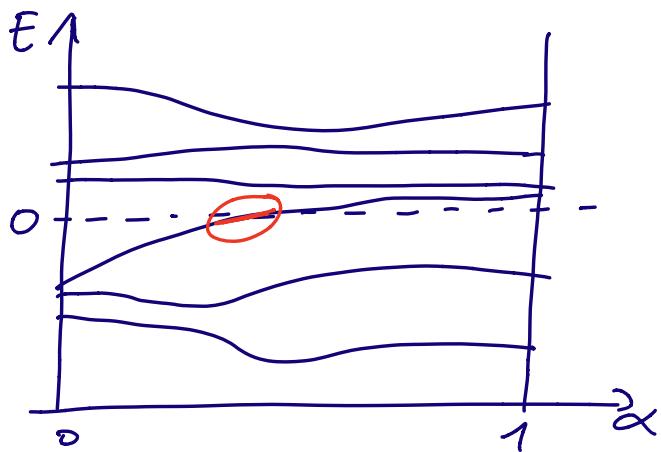
Def.: two systems are topologically equivalent if they can be continuously deformed into each other without closing the energy gap.

Choose a random real symmetric H and deform it into H' : $H(\alpha) = \alpha H' + (1-\alpha)H$

$\alpha=0$: initial H , $\alpha=1$: final H'

/2

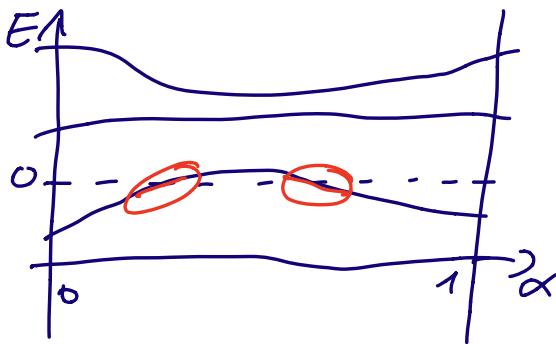
For example:



zero-energy crossing : breaks "gap condition".

Are H and H' topologically equivalent according to the above definition?

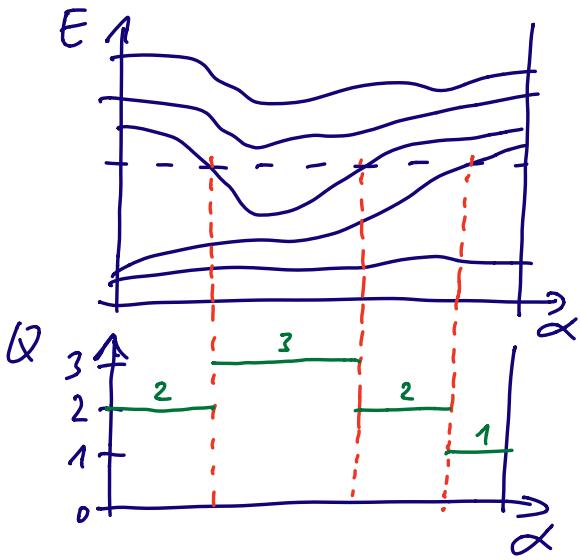
other example:



two crossings - but maybe we can change the path between H and H' to push this level below zero.
=> need an easier way to figure out topological equivalence.

Concept: topological invariant

Idea: Count # levels below $E=0$ = # filled states
= topological invariant Q



\Rightarrow no need to keep track of all filled states! It is sufficient to keep track of zero-energy crossings!

Role of Conservation Laws

Consider a Hamiltonian with a symmetry constraint:

\exists a unitary U , e.g., $U = \sigma_z \otimes \mathbb{1}$, such that

$$U^\dagger H U = H.$$

\Rightarrow H can be block-diagonalized due to a conservation law.

$$H_{4 \times 4} = \begin{bmatrix} H_1 & & \\ & \ddots & \\ & & H_2 \end{bmatrix}_{4 \times 4}$$

\Rightarrow look at Q for each subblock separately

$$Q[H] = Q[H_1] + Q[H_2]$$

\Rightarrow unitary symmetries are useful to reduce dimensionality, but otherwise boring.

Other symmetries are more interesting, e.g., time-reversal symmetry.

Time-reversal symmetry (trs)

Are real matrices special? Yes: they are manifestations of trs.

trs is represented by an anti-unitary operator that can be written as $\tilde{T} = U K$

\uparrow \uparrow
unitary complex
 conjugation

For real matrices, $H = H^*$, and they commute with \tilde{T} since complex conjugation does nothing.

For a random complex Hamiltonian, nothing changes when $\tilde{T} = K$ ($U = \mathbb{1}$) — Q changes as energy levels cross zero.

Important case where trs makes a difference: spin 1/2 systems. For these: $\tilde{T} = i \sigma_y K$, $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$
 $\Rightarrow \tilde{T}^2 = -1$.

Then trs means $H = \sigma_y H^* \sigma_y$

\Rightarrow every energy eigenvalue is doubly degenerate (Kramers degeneracy)

$\Rightarrow Q$ can only take even numbers

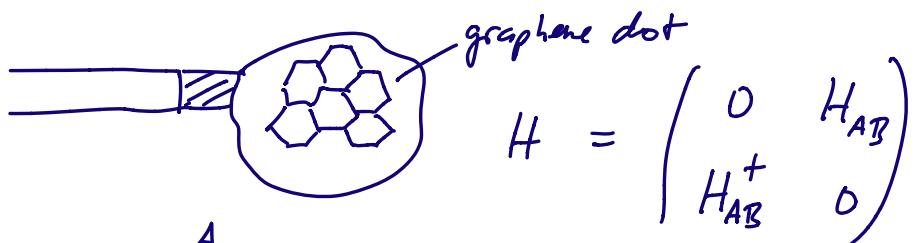
$$Q = 0, 2, 4, \dots$$

Why?
→ Kramers theorem

Lecture: Time-reversal symmetry and Kramers theorem

\Rightarrow example for how discrete symmetries influence topology / 5

Sublattice symmetry



only nonzero matrix elements between sublattices A and B, which are degenerate.

\Rightarrow introduce diagonal matrix σ_z which is +1 for sublattice A and -1 for B:

$$-H = \sigma_z H \sigma_z$$

\Rightarrow if $(\psi_A, \psi_B)^T$ is an eigenvector with energy E , then $(\psi_A, -\psi_B)^T$ is an eigenvector with energy $-E$. \Rightarrow symmetric spectrum due to sublattice symmetry

\Rightarrow the topological invariant Q never changes for systems with sublattice symmetry

\Rightarrow extra symmetry may render topological classification trivial!

Test: which symmetry does not restrict the possible values of Q for quantum dots?

- (A) spinless trs (B) sublattice symmetry (C) spinful trs /6

Particle-hole symmetry

Example : Superconductors (SC)

SC has pairing terms

$$H = \sum_{nm} H_{nm} c_n^+ c_m + \frac{1}{2} (\Delta_{nm} c_n^+ c_m^+ + \Delta_{nm}^* c_m c_n)$$

c_n^+, c_n : fermionic creation/annihilation op's

$$c_n^+ c_m + c_m c_n^+ = \delta_{nm}$$

Δ_{nm} : antisymmetric matrix

H_{nn} : dot Hamiltonian without SC pairing

H does not preserve particle number, but preserves its parity, i.e., whether # electrons is even or odd.

Using $C = (c_1, c_2, \dots, c_n, c_1^+, c_2^+, \dots, c_n^+)^T$ we

write

$$H = \frac{1}{2} C^+ H_{BdG} C$$

with the Bogoliubov - de - Gennes Hamiltonian

$$H_{BdG} = \begin{pmatrix} H & \Delta \\ -\Delta^* & -H^* \end{pmatrix} = \begin{pmatrix} \text{electrons} & \text{pairing} \\ \text{pairing} & \text{holes} \end{pmatrix}$$

Since holes and electrons are related, H_{BdG} automatically has an extra symmetry exchanging electrons and holes:

antiunitary $P = T_x K$, where $T_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
acts on the particle and hole blocks:

$$\mathcal{P} H_{\text{BdG}} \mathcal{P}^{-1} = -H_{\text{BdG}}$$

Particle-hole symmetry is represented by an antiunitary operator that anticommutes with the Hamiltonian.

Because of the minus sign in front of H_{BdG} for phs, the spectrum of H_{BdG} must be symmetric around energy zero (Fermi level) : for every eigenvector $\Psi = (u, v)^T$ of H_{BdG} with energy E , there is a ph-symmetric eigenvector $\mathcal{P}\Psi = (v^*, u^*)^T$ with energy $-E$.

\Rightarrow as for sublattice symmetry, phs forces \mathcal{Q} to remain the same ; but, for H_{BdG} zero-energy crossings can occur for phs — what happens there ?

Fermion parity switches

For H_{BdG} we had to double the number of states.

The pairs at $\pm E$ do not correspond to different quantum states, but to a single quantum state.

Bogoliubov quasiparticle = coherent superposition of electrons & holes ; energy E , operator $a^+ = u c^+ + v c$.

Populating the partner state at $-E$ is the same as emptying the $+E$ state. /8

When a pair of Bogoliubov states crosses zero, the excitation energy E changes sign and it becomes favorable to add a Bogoliubov quasiparticle to/ remove it from the dot. Level crossings = changes in fermionic parity in dot ground state from even to odd or vice versa. = fermion parity switches.

H_{BdG} preserves fermion parity if there are no Bogoliubov quasiparticles crossing zero energy: ground state fermion parity is the topological invariant of the system.

The topological invariant depends on the nature (symmetry) of the system under consideration!

Non-superconductors : $Q = \#$ of negative energy eigenvalues

Superconductors: level crossings can occur but $\#$ of negative eigenvalues does not change, but fermion parity changes. $Q =$ parity is a suitable topological invariant.

Can we compute the parity directly from H_{BdG} ?

The Pfaffian invariant

Basis transformation: $\tilde{H}_{BdG} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} H_{BdG} \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}$

$$\Rightarrow \tilde{H}_{BdG} = \frac{1}{2} \begin{pmatrix} H - H^* + \Delta - \Delta^* & -i(H - iH^* + i\Delta + i\Delta^*) \\ i(H + iH^* + i\Delta + i\Delta^*) & H - H^* - \Delta + \Delta^* \end{pmatrix} \underbrace{\text{antisymmetric}}_q$$

Δ is anti-symmetric. Since H is Hermitian, $H - H^*$ is also anti-symmetric and $H + H^*$ is symmetric.

$\Rightarrow \tilde{H}_{BdG}$ is anti-symmetric.

Pfaffian is defined for anti-symmetric matrices.

Basic idea: eigenvalues of anti-symmetric matrices come in pairs. For \tilde{H}_{BdG} , these are $\pm E_n$.

determinant : $\prod_n (-E_n^2)$.

The Pfaffian allows to take the $\sqrt{\dots}$ of the determinant : $\pm i \prod_n E_n$, in such a way that the sign of the product is uniquely defined. At a fermion parity switch, a single E_n changes sign, such that the Pfaffian changes sign while the determinant remains the same!

$$Q_{BdG} \equiv \text{sign} \left[\text{Pf} \left(i \tilde{H}_{BdG} \right) \right] \quad \begin{matrix} \leftarrow \\ \text{i for convenience to} \\ \text{make the Pfaffian} \\ \text{real} \end{matrix}$$

$$\text{Pf}(A) = \sqrt{\det(\tilde{A})'}$$

Test: What happens to the topological invariant when we take $\Delta = 0$ in H_{BdG} ?

- (A) Pfaffian still captures all topological properties
- (B) H loses ph-symmetry and becomes topologically trivial
- (C) H has a new conservation law \Rightarrow exists two blocks each with their own invariant
- (D) $\Delta = 0$ not allowed / No

Summary of ①:

- simplest topological invariant of zero-dim. systems:
number of negative-energy states = matrix
signature of Hamiltonian (zeroth Chern number)
- Conservation law (unitary) \Rightarrow H block-diagonal
 \Rightarrow study topology of individual blocks
- number of filled states becomes even for spinful time-reversal symmetry
- sublattice symmetry: number of filled states becomes constant
- particle-hole (charge conjugation) symmetry: makes signature constant like sublattice symmetry but generates a new kind of invariant: the sign of the Pfaffian which can only take two values: $\pm 1 \equiv$ parity of the electron number in the ground state

Symmetries and conservation laws define the type and existence of topological invariants.

② 1D example : Bulk-edge correspondence in the Kitaev chain

Fermionic operators c, c^\dagger : $cc^\dagger + c^\dagger c = 1$
 $c^2 = 0$
 $c^{\dagger 2} = 0$

Two states : $|0\rangle, |1\rangle$ with $c^\dagger |0\rangle = |1\rangle \quad c|0\rangle = 0$
 $c^\dagger |1\rangle = |0\rangle \quad c^\dagger |1\rangle = 0$

$$c^\dagger \underbrace{\downarrow c}_{|0\rangle} \begin{matrix} |1\rangle \\ |0\rangle \end{matrix}$$

Rewrite fermions using Majorana operators γ_1, γ_2 :

$$c^\dagger = \frac{1}{2}(\gamma_1 + i\gamma_2)$$

$$c = \frac{1}{2}(\gamma_1 - i\gamma_2)$$

Inverse transformation:

$$\begin{aligned} \gamma_1 &= c + c^\dagger & \gamma_1^+ &= c^\dagger + c = \gamma_1 \\ \gamma_2 &= i(c - c^\dagger) & \gamma_2^+ &= -i(c^\dagger - c) = \gamma_2 \end{aligned} \quad (\star)$$

\Rightarrow there is no "occupation number operator" for Majoranas !

For normal (Dirac) fermions, $c \neq c^\dagger$, and $\hat{n} = c^\dagger c$ counts the occupation of a fermionic state :

$$\begin{aligned} \langle 0 | \hat{n} | 0 \rangle &= 0 & \hat{\gamma}_1^2 &= \underbrace{c^\dagger c}_{=0} + \underbrace{c c^\dagger}_{=1} = 1 \\ \langle 1 | \hat{n} | 1 \rangle &= 1 \end{aligned}$$

For Majoranas : $\gamma_1^+ \gamma_1 \stackrel{*}{=} \gamma_1^2 = 1 \Rightarrow \underline{\text{operator identity}}$

It means that $\langle 4 | \gamma_1^+ \gamma_1 | 4 \rangle = 1$ independent of $|4\rangle$ / 12

And we have $\gamma_1\gamma_2 + \gamma_2\gamma_1 = 0$ — Majoranas for different modes anticommute. That is why we call them "fermions".

Since $c^+ = \frac{1}{2}(\gamma_1 + i\gamma_2)$ we can view two

$\begin{matrix} \uparrow & \uparrow & \uparrow \\ \text{complex} & \text{real} & \text{real} \\ & \text{(hermitian)} & \end{matrix}$

Majoranas as the real and imaginary parts of one fermion.

Majoranas must come in pairs.

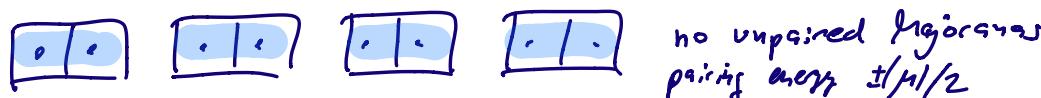
Can two Majoranas be separated?

Domino model

γ_1	γ_2	γ_3	γ_4	γ_5	γ_6	γ_7	γ_8
• •	• •	• •	• •	• •	• •	• •	• •
$n=1$	$n=2$	$n=3$	$n=4$				

$$H_\mu = -\mu \hat{N} = -\mu \sum_{n=1}^N c_n^\dagger c_n \stackrel{\cong}{=} \text{fermion filling}$$

$$\stackrel{\cong}{=} \frac{i}{2}\mu \sum_{n=1}^N \gamma_{2n-1} \gamma_{2n} \stackrel{\cong}{=} \text{Majorana pairing}$$



But we would like a pairing like so: \Rightarrow excitations gapped



unpaired Majoranas

$$\Rightarrow H_t = it \sum_{n=1}^{N-1} \gamma_{2n} \gamma_{2n+1} \stackrel{\cong}{=} \text{fermion hopping}$$

$\Rightarrow \gamma_1$ and γ_{2N} do not appear in H_t

$\Rightarrow H_t$ has two zero-energy states
localized at the edges

\Rightarrow all other states have energy $\pm |t|$, independently
of chain length

\Rightarrow 1D system with gapped bulk and zero-
energy edge states

Kitaev chain model

$$\text{use } \gamma_{2n+1} = (c_n^+ + c_n)$$

$$\gamma_{2n} = -i(c_n^+ - c_n)$$

$$c_n^+ = \frac{1}{2}(\gamma_{2n+1} + i\gamma_{2n})$$

$$c_n^- = \frac{1}{2}(\gamma_{2n+1} - i\gamma_{2n})$$

$$c_{n+1}^+ = \frac{1}{2}(\gamma_{2n+3} + i\gamma_{2n+2})$$

$$c_{n+1}^- = \frac{1}{2}(\gamma_{2n+3} - i\gamma_{2n+2})$$

tight-binding model for 1D superconducting (SC) wire:

$$H = -\mu \sum_n c_n^+ c_n^- - t \sum_n (c_{n+1}^+ c_n^- + h.c.) + \Delta \sum_n (c_n^- c_{n+1}^+ + h.c.)$$

3 real parameters: chemical potential μ
hopping integral t
SC pairing energy Δ

Pairing terms in Majorana's:

$$\begin{aligned} \Delta c_n c_{n+1} + h.c. &= \frac{\Delta}{4} (\cancel{\gamma_{2n+1}\gamma_{2n+1}} - \cancel{\gamma_{2n}\gamma_{2n+2}} - i\gamma_{2n}\gamma_{2n+1} - i\gamma_{2n+1}\gamma_{2n+2}) \\ &\quad + \frac{\Delta}{4} (\cancel{\gamma_{2n+1}\gamma_{2n+1}} - \cancel{\gamma_{2n+2}\gamma_{2n}} + i\gamma_{2n+1}\gamma_{2n} + i\gamma_{2n+2}\gamma_{2n+1}) \\ &= \frac{i\Delta}{2} (\gamma_{2n+1}\gamma_{2n} + \gamma_{2n+2}\gamma_{2n+1}) \end{aligned}$$

Hopping:

$$\begin{aligned} -t c_{n+1}^+ c_n + h.c. &= -\frac{t}{4} (\cancel{\gamma_{2n+1}\gamma_{2n-1}} + \cancel{\gamma_{2n+2}\gamma_{2n}} + i\gamma_{2n+2}\gamma_{2n-1} - i\gamma_{2n+1}\gamma_{2n}) \\ &\quad - \frac{t}{4} (\cancel{\gamma_{2n-1}\gamma_{2n+1}} + \cancel{\gamma_{2n}\gamma_{2n+2}} - i\gamma_{2n-1}\gamma_{2n+2} + i\gamma_{2n}\gamma_{2n+1}) \\ &= \frac{it}{2} (\gamma_{2n+1}\gamma_{2n} - \gamma_{2n+2}\gamma_{2n-1}) \end{aligned}$$

\Rightarrow if $\Delta = t$, the $\gamma_{2n+2} \gamma_{2n-1}$ terms vanish.

$\Rightarrow \Delta = t \neq 0, \mu = 0$ creates exactly the unpinned Majorana edge mode model

trivial case (fully paired): $\mu \neq 0, \Delta = t = 0$

\Rightarrow there must be a phase transition in between!

Remember: It is useful to write a superconducting Hamiltonian in Bogoliubov-de-Gennes form:

$$H = \frac{1}{2} C^+ H_{\text{Rdeg}} C$$

$$C \equiv (c_1, \dots, c_N, c_1^+, \dots, c_N^+)^T$$

$2N \times 2N$ matrix H_{BdG} can be written using Pauli matrices in particle-hole space, and $|n\rangle = (0, 1, 0, 0)^T$ a column vector corresponding to the n -th site of the chain. For example: $C^+ T_z |n\rangle \langle n| C = 2C_n + c_i - 1$.

$$H_{\text{BdG}} = - \sum_n \mu_{T_z} |n\rangle \langle n| - \sum_n \left[(t_{T_z + i\Delta T_y})_n |n+1\rangle \langle n+1| + h.c. \right]$$

H_{DdG} acts on states in a basis $|n\rangle|\tau\rangle$, with $\tau = \pm 1$ corresponding to electron and hole states.

$$H_{BDG} \text{ has phys: } \mathcal{P} H_{BDG} \mathcal{P}^{-1} = -H_{BDG}, \quad \mathcal{P} = T_x K$$

Complex conj. 15

check:

$$\begin{aligned} \mathcal{P} H_{\text{BDG}} \mathcal{P}^{-1} &= - \sum_n \mu \underbrace{T_x T_z T_x}_{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}} |n\rangle \langle n| \\ &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = T_y \\ &= \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = -T_z \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \underbrace{\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{T_y} \\ &\quad - \sum_n \left[(t \underbrace{T_x T_z T_x}_{-T_z} - i \Delta \underbrace{T_x T_y^* T_x}_{T_y}) |n\rangle \langle n+1| + \text{h.c.} \right] \\ &= -H_{\text{BDG}} \checkmark \end{aligned}$$

Topological protection of Majorana edge modes

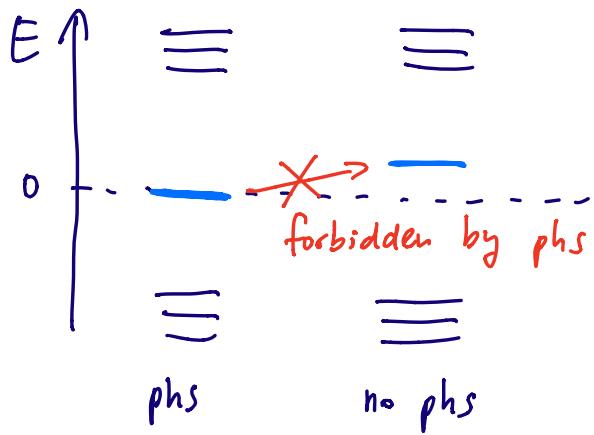
Is finetuning ($\mu=0$) necessary to have unpaired Majoranas? What happens when $\mu \neq 0$?

→ demo $N=25$ chain

Majoranas persist until $\mu \approx 2t$, where the bulk gap closes.

⇒ Majoranas only merge when the higher-energy states in the bulk, originally at $2t$, come close to zero energy. No FINETUNING REQUIRED!

Majorana edge modes are protected by the bulk energy gap, and by particle-hole symmetry!



Only way to move Majorana zero modes (MZM) away from zero is to couple them to each other! This is impossible as long as they are spatially separated and the bulk gap persists.

\Rightarrow one needs to close the gap to destroy MZM
($\mu = 2t$: gap closes)

Hint from this: Information about the edge modes is already contained in the bulk states!

There is a deep reason: bulk-edge correspondence

Test: what happens if we remove a single Majorana site in the topological phase?

- (A) we get a chain with a single unpaired MZM
- (B) impossible: electrons (pairs of Majoranas) are the only physical degree of freedom in the outside world
- (C) H becomes topologically trivial
- (D) Removing a single Majorana site is forbidden by particle-hole symmetry

Topological phases from the bulk

Momentum space (band structure):

close the chain with periodic boundary conditions (PBC)

$$|k\rangle = (N)^{-1/2} \sum_{n=1}^N e^{-ikn} |n\rangle$$

$$\text{PBC: } \langle k|n=0\rangle = \langle k|n=N\rangle$$

$\Rightarrow k$ good quantum number, takes values $\frac{2\pi}{N}\varphi$,

$$\varphi = 0, \dots, N-1$$

$N \rightarrow \infty : k \in [-\pi, \pi] = 1\text{st Brillouin zone}$

$$H_{\text{BDG}} = \sum_k H(k) |k\rangle \langle k|$$

$$H(k) \equiv \langle k | H_{\text{BDG}} | k \rangle = (-2t \cos k - \mu) T_z + 2\Delta \sin k T_y$$

$$N \rightarrow \infty : \sum_k \rightarrow \int \frac{dk}{2\pi}$$

particle-hole symmetry in momentum space:

Careful with anti-unitary operators under basis transformations because of the action of K on the e^{-ikn} : $Ke^{-ikn} = e^{+ikn}$

$$\text{Here: } P |k\rangle |T\rangle = \left(\sum_n e^{-ikn} \right)^* |n\rangle T_x |T\rangle^* = |k\rangle T_x |T\rangle^*$$

$\Rightarrow P$ changes k to $-k$!

$$P H_{\text{BDG}} P^{-1} = \sum_k T_x H^*(k) T_x |k\rangle \langle -k| = \sum_k \underset{\substack{\uparrow \\ \text{momenta come in pairs } (-k, k)}}{T_x H^*(-k)} T_x |k\rangle \langle k|$$

+ symmetric points $\frac{k=0}{k=\pi} / 18$

Particle-hole symmetry $\mathcal{P} H_{\text{BdG}} \mathcal{P}^{-1} \stackrel{!}{=} -H_{\text{BdG}}$ \iff

$$H(k) \stackrel{!}{=} -T_x H^*(k) T_x$$

This is true because

$$\begin{aligned} T_x H^*(-k) T_x &= (+2t \cos k + \mu) T_z - 2\Delta \sin k T_y \\ &= -H(k) \quad \checkmark \end{aligned}$$

particle-hole symmetry in k space

\Rightarrow given a solution with E and k , there is also a solution with $-E$ and $-k$

Band structure:

$$\text{Diagonalize } H(k) = \begin{pmatrix} -2t \cos k - \mu & -2i\Delta \sin k \\ +2i\Delta \sin k & +2t \cos k + \mu \end{pmatrix}$$

$$\Rightarrow E(k) = \pm \sqrt{(2t \cos k + \mu)^2 + 4\Delta^2 \sin^2 k}$$

\rightarrow DEMO

\Rightarrow gapped at all k for $\mu = 0$ [why? before we had Majorana zero modes at $\mu = 0$]

\Rightarrow bulk gap closes at $\mu = \pm 2t$

+ : band touching at $k = \pi$

- : band touching at $k = 0$

Is there a phase transition? First: Write down a simple effective model near the transition

Effective Dirac model:

Focus on $\mu = -2t$ at $k=0$.

Linear Taylor expansion:

$$H(k) \simeq m T_2 + 2 \Delta k T_3$$

Dirac-type Hamiltonian

$$m \equiv -\mu - 2t \quad \text{mass term}$$

$$\Rightarrow E(k) = \pm \sqrt{m^2 + 4\Delta^2 k^2}$$

Mass term is crucial: It determines the gap and

- $m < 0$ for $\mu > -2t$: topological phase
- $m > 0$ for $\mu < -2t$: trivial phase

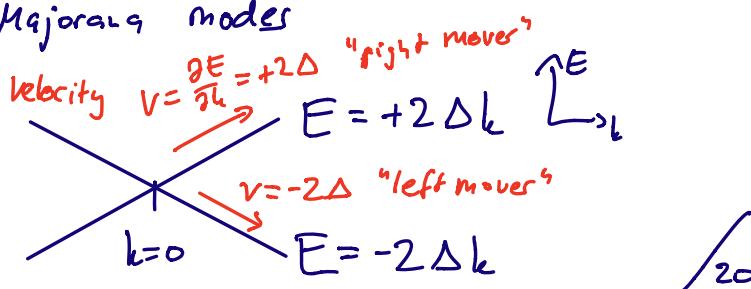
Mass term changes sign at $\mu = -2t$!

$m=0$: $H(k)$ has two eigenstates $E = \pm 2\Delta k$,

which are eigenstates of T_y

\Rightarrow equal-weight superpositions of electron + hole

\Rightarrow these are Majorana modes



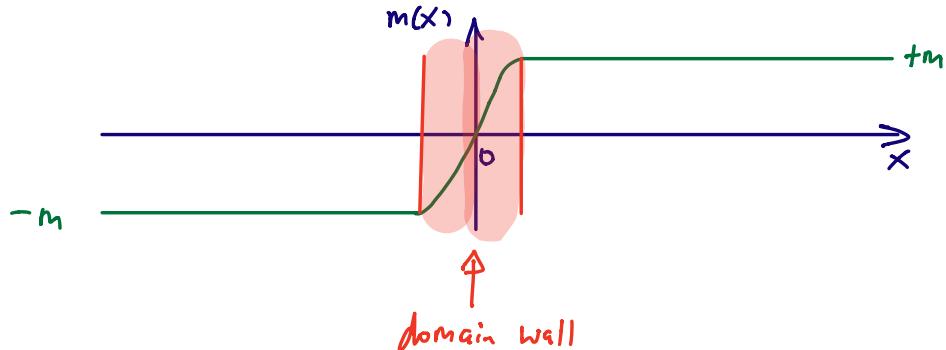
For $m=0$, there are two branches of free-moving Majorana fermions!

Majorana modes at domain walls

What if m changes sign in real space?

$$H(x) = -V T_y i \partial_x + m(x) T_z$$

Choose $m(x) \rightarrow \pm m$ for $x \rightarrow \pm \infty$, $m(x=0) = 0$



Look for $M^2 M$: $H\Psi = 0$

$$\Leftrightarrow \partial_x \Psi(x) = \frac{1}{V} m(x) T_x \Psi(x)$$

$$\Rightarrow \Psi(x) = \exp \left(T_x \int_0^x \frac{m(x')}{V} dx' \right) \Psi(0)$$

\Rightarrow two linearly independent solutions via eigenstates of T_x :

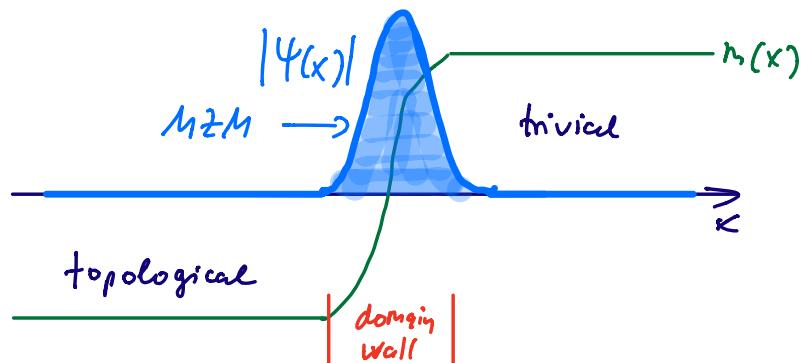
$$\Psi(x) = \exp \left(\pm \int_0^x \frac{m(x')}{V} dx' \right) \begin{pmatrix} 1 \\ \pm 1 \end{pmatrix}$$

\Rightarrow only one of them is normalizable (+ branch is divergent)

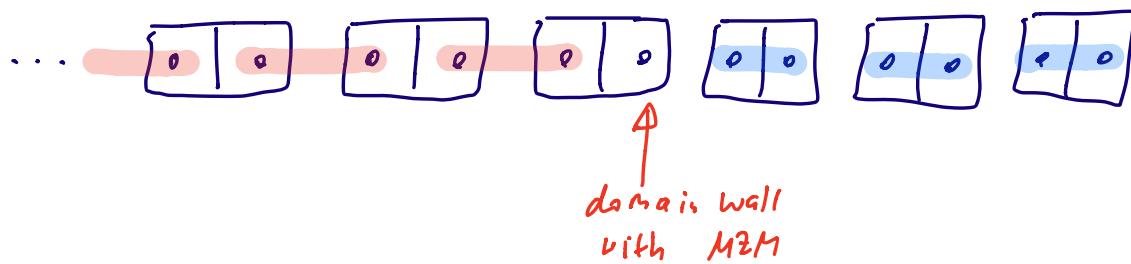
\Rightarrow bound state localized near the domain wall
thanks to sign change of $n(x)$ — otherwise
no normalizable M2M would exist

Physics: $x < 0$: topological phase
 $x > 0$: trivial phase

At the interface there must be a zero mode!



Domino-tile representation:



Bulk topological invariant:

Effective model: Sign of mass term helped understand topology. What can we use as topological invariant for a more general $H(k)$?

Heuristic derivation:

- BdG Hamiltonian : for quantum dots we used the sign of the Pfaffian as topological invariant in the presence of particle-hole symmetry; the Pfaffian changes sign at every gap closing.

Link m to a Pfaffian?

- H_{BdG} : large matrix with phs.
 \Rightarrow put H_{BdG} in antisymmetric form and compute its Pfaffian?
- Pfaffian sign only changes when an eigenvalue passes through zero. phs $\Rightarrow E(k)$ comes with $-E(-k)$. Only two exceptions:
 $k=0$ and $k=\pi$ are mapped onto themselves for $k \rightarrow -k$.

$$\text{phs} : T_x H^*(0) T_x = -H(0)$$

$$T_x H^*(\pi) T_x = -H(\pi)$$

$\Rightarrow H(0)$ and $H(\pi)$ can be anti-symmetrized individually.

- gap closings happen exactly at $k=0$ and $k=\pi$, too.

→ focus on 0 and π !

Anti-symmetrization (cf. Chapter 1):

$$\tilde{H}(0) = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \begin{pmatrix} -2t-\mu & 0 \\ 0 & 2t+\mu \end{pmatrix} \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} = -i \begin{pmatrix} 0 & -2t-\mu \\ 2t+\mu & 0 \end{pmatrix}$$

$$\tilde{H}(\pi) = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \begin{pmatrix} 2t-\mu & 0 \\ 0 & -2t+\mu \end{pmatrix} \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} = -i \begin{pmatrix} 0 & 2t-\mu \\ -2t+\mu & 0 \end{pmatrix}$$

$$\Rightarrow \text{Pf}[iH(0)] = -2t-\mu \rightarrow \text{sign charge at } \mu = -2t$$

$$\text{Pf}[iH(\pi)] = 2t-\mu \rightarrow " \text{---} \mu = +2t$$

Topological invariant: $Q = \text{sign} (\text{Pf}[iH(0)] \text{Pf}[iH(\pi)])$

$Q = -1$: bulk is in topological phase

$Q = +1$: " trivial "

The topological invariant Q cannot change under continuous deformations of the Hamiltonian unless the gap closes.

Connection between bulk invariant and edge modes

Physical meaning of Q ?

Sign of Pfaffian of BdG Hamiltonian: ground state fermion parity

Product of $\text{Pf}[iH(0)] \text{Pf}[iH(\pi)]$: we are comparing the fermion parities of the $k=0$ and $k=\pi$ states:

$Q = -1$ only if they are different.

\Rightarrow if we continuously deform $H(0)$ into $H(\pi)$ without breaking phs, we must encounter a zero-energy level crossing (fermion parity switch).

Implementation: Change from PBC to anti-periodic BC:

$$\langle k | n=0 \rangle = -\langle k | n=N \rangle \quad \left. \begin{array}{l} \\ \Rightarrow \text{allowed momenta } k = \frac{2\pi}{N} p + \frac{\pi}{N} \end{array} \right\} \begin{array}{l} \text{APBC} \\ p=0, \dots, N-1 \end{array}$$

\Rightarrow what is the parity difference between the PBC chain and the APBC chain?

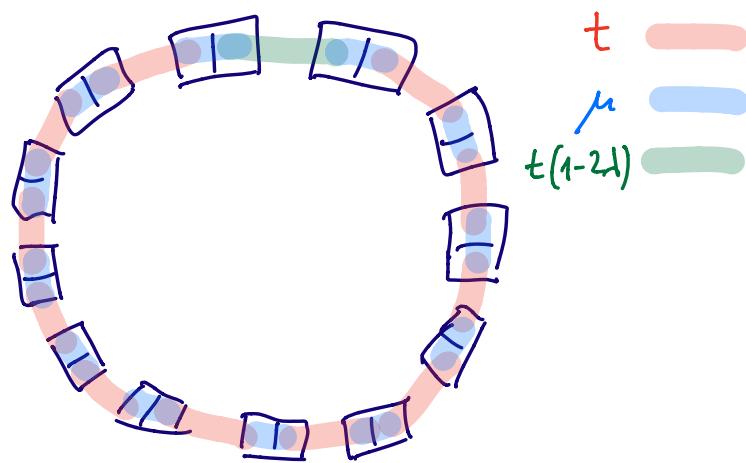
$k=0$ is always present in PBC chain

$$k = \frac{2\pi p}{N}, \quad p=0, \dots, N-1$$

$k=\pi$ is present for $\begin{cases} N \text{ even in PBC} \\ N \text{ odd in APBC} \end{cases}$

\Rightarrow in either case, the difference in ground-state fermion parities between the PBC and APBC chains is equal to Q !

Check: Consider Kitaev ring with $J=t$. Change of hopping from t to $-t$ on the last bond between $n=N-1$ and $n=0 \Leftrightarrow$ going from PBC to APBC chain. This can be done continuously w/o breaking phs by setting the last hopping equal to $t(1-2\lambda)$ with $\lambda \in (0, 1)$:



\Rightarrow DEMO energy spectrum $E(l)$ as μ varies through the gap closing

$\mu < 2t$: $E(l)$ crosses 0 at $l = \frac{1}{2}$

\Rightarrow different parity at $l=0$ and $l=1$

$l = \frac{1}{2}$: cut in chain ($t=0$ on last bond)

\Rightarrow two MZM in topological phase!

$\mu > 2t$: no level crossing through 0

\Rightarrow same ground state fermion parity at $l=0$ and $l=1$

$l = \frac{1}{2}$: no MZM \Rightarrow consistent with trivial open chain

Essence of bulk-boundary correspondence: $Q = -1$ nontrivial bulk invariant for closed chain implies existence of unpaired Majoranas in the open chain.

Connection of Q to measurable quantity: groundstate fermion parity / 26

Test: What happens when we take a 100-site topological Kitaev chain and change μ to a very large negative (trivial) value for the last 50 sites?
(select all that match)

- A gap at last 50 sites closes, then reopens
- B Majoranas get destroyed
- C One of the Majoranas moves from the end to the middle of the chain

③ Charge pumping

Thouless pumps and winding invariant

Hamiltonians with parameters:

Consider a Hamiltonian that is periodic in a parameter:

$$H(t+T) = H(t) \quad \text{for period } T$$

Examples: • band structure $H(k)$ has period 2π (in 1D)

- time-dependent Hamiltonian under periodic driving

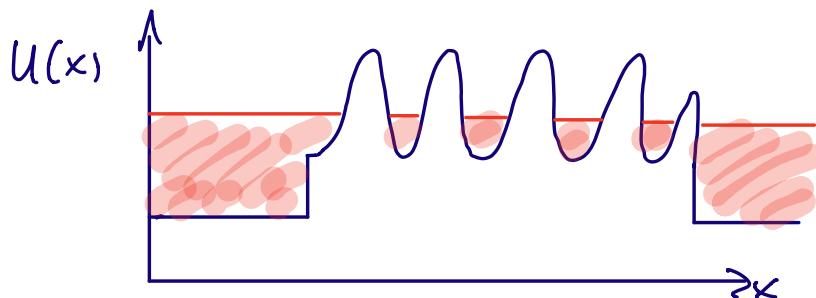
We want to study $H(t)$ with slow changes in t , such that the system remains in the ground state (gapped system).
Slow : adiabatic !

Note: $T \rightarrow \infty$ is possible, $H(-\infty) = H(+\infty)$.

Important: parameter space must be compact to define topological invariants.

Quantum pumps:

Consider a 1D region coupled to two electrodes with a sine-shaped confining potential.



$$H(t) = \frac{\hbar^2}{2m} + A \left[1 - \cos\left(\frac{x}{\lambda} + 2\pi t/T\right) \right]$$

for x in the central region.

Choose $A \gg 1/m\lambda^2 \rightarrow$ strong potential

$\mu \ll A \rightarrow$ states bound in minima have small overlap

Near minima: periodic potential $\rightarrow E_n = (n + \frac{1}{2})\hbar\omega_c$
 $\omega_c = \sqrt{A/m\lambda^2}$

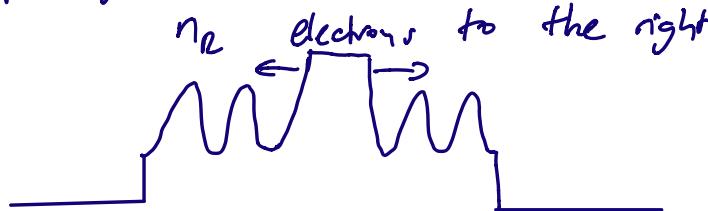
Each minimum: integer number of electrons depending on chemical potential μ

One time period: an integer number N of electrons is pumped from the left to the right electrode

Quantization of pumped charge

Is the integrality of pumped charge a topological effect, or is it just due to finetuning (because the wells are deep)?

Thought experiment: "dry out" pump by emptying the middle region pushing n_L electrons to the left



- emptying middle region: sides disconnected
 \rightarrow number of electrons must be integer on either side

2. adiabaticity : system always in an eigenstate
3. drying out \rightarrow pump integer number of charges
4. adiabatic manipulation only possible if H is always gapped

\rightarrow the number of electrons N pumped per cycle is integer as long as the bulk of the pump remains gapped.

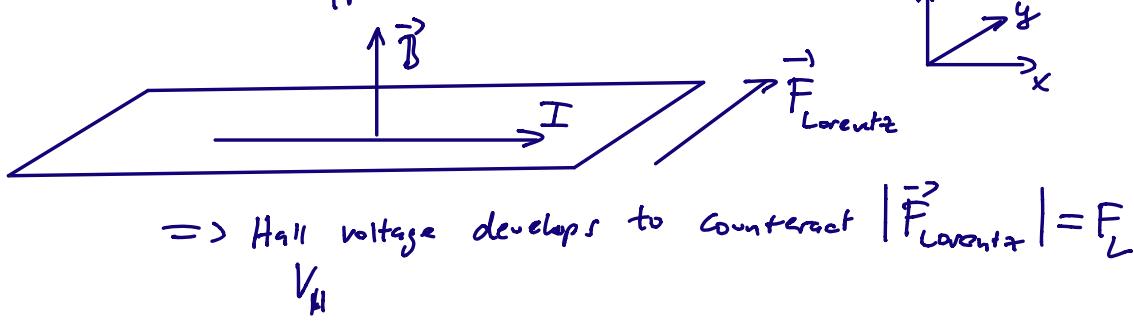
$\Rightarrow N$: topological invariant

In fact, N is a Chern number (TKNN invariant), which will be introduced later.

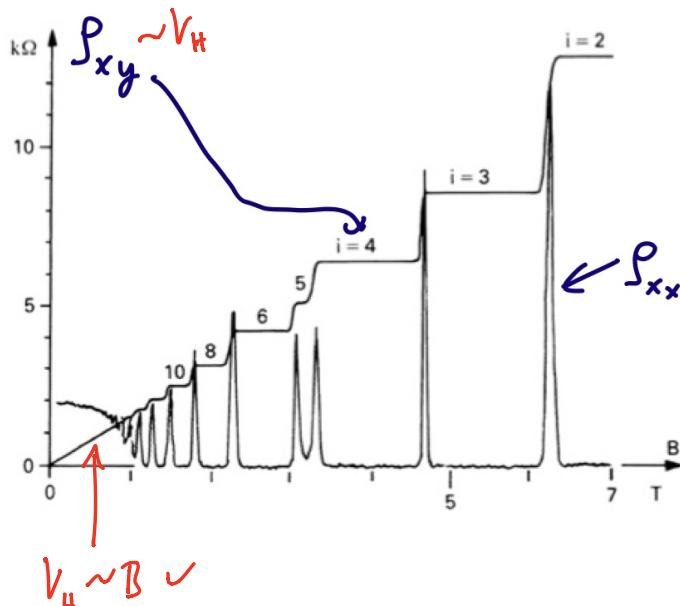
④ Quantum Hall Effect (QHE)

Laughlin argument for quantization:

Classical Hall Effect:



$$\text{Since } F_L \sim B \Rightarrow V_H \sim B$$



but: at larger B there are plateaux! to one part in a billion!!!

ρ_{xy} (the Hall resistivity) DOES NOT CHANGE and

is quantized in inverse integers of $\frac{h}{e^2}$

$$\rho_{xy} = \frac{1}{r} \frac{h}{e^2}, \quad \frac{h}{e^2} = 25,812.807 \Omega$$

$\equiv R_K$, von Klitzing constant

Found by Klaus von Klitzing 1982.

→ integer quantum Hall effect (IQHE).

Thouless, Kohmoto, Nightingale, den Nijs (TKNN):

IQHE is a topological phenomenon related to a topological invariant (Chern number).

At the same time: longitudinal resistivity ρ_{xx} vanishes.

Reason: current and electric field are \perp to each other.

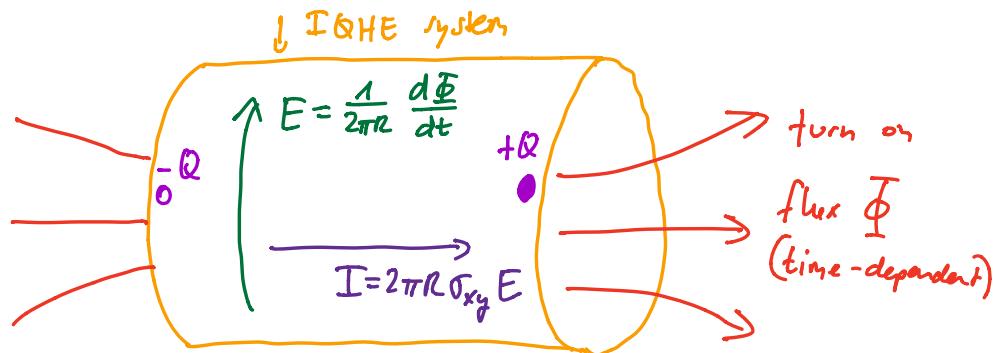
Curiously, this implies that the longitudinal conductivity σ_{xx} also vanishes.

Hall conductivity is quantized: $\sigma_{xy} = \nu \frac{e^2}{h}$.

At higher temperature: ρ_{xx} and σ_{xx} show activated behavior $\sim e^{-T_0/T}$. → system is gapped in the bulk

Twist: Gap must close at the sample edges.

Consider a cylinder of IQHE material (radius R)



flux → azimuthal electric field → current

→ polarization of charge \propto between the edges.

Charge cannot relax because $\sigma_{xx} = 0$!

→ charge accumulates — it needs states to do this!

Since the perturbation (magnetic field) is slow (low-frequency) and small (just one flux quantum), it is low-energy and scales like $1/\text{system size}$.

→ these excited states must be close to the chemical potential.

→ the gap must close at the edge!

At the interface between two regions with different Hall conductivities, there must be a gap closure.

- 3 pieces:
- left edge ("1D conductor")
 - interior ("2D insulator")
 - right edge ("1D conductor")

But: Edges are not like 1D wires because charge in them is not conserved! If an edge is peeled off, the smaller cylinder will again have gapless edge states.

Quantum mechanics: Spectrum of cylinder must be periodic in the flux with periodicity $\frac{\hbar}{e} = \Phi_0$ (flux quantum)
→ Aharonov-Bohm effect.

start with ground state of cylinder at $\Phi = 0$.

Change flux to $\Phi = \Phi_0$. \rightarrow system still in eigenstate.
Not ground state because charge has flowed.

For noninteracting electrons: eigenstates = products of single-particle states; occupied + unoccupied.

Ground state: fill states below chemical potential.

After one flux quantum: again eigenstate, some filled states are empty, some empty ones are now filled.

Only available states close to μ are at edges:

An integer number of electrons gets pumped between the edges!

\hookrightarrow Hall conductivity is quantized in units of e^2/h

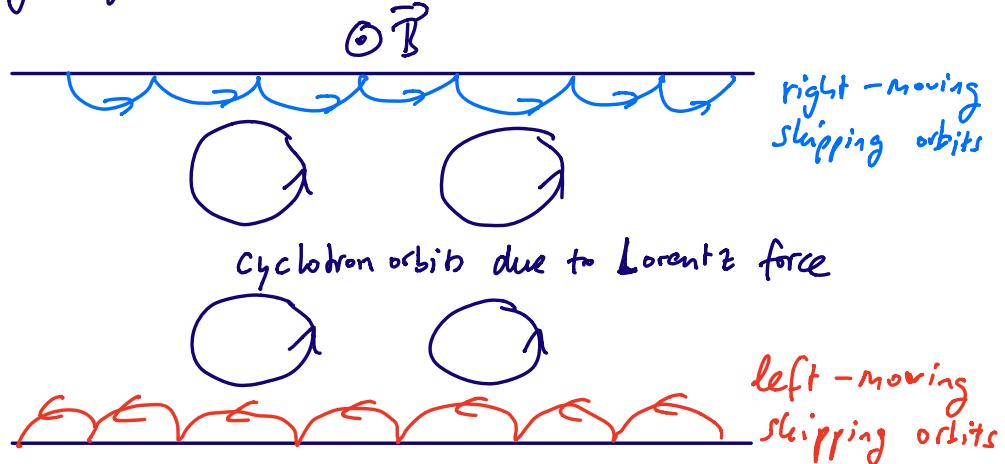
This is again a Thouless charge pump.

Underlying physics: electrons in Landau levels, cyclotron orbits. Role of disorder: In clean sample, Landau levels get filled in continuously; with disorder, there are localized states in between Landau levels; when these are filled, the plateaus appear.

Important ingredient to QHE: time-reversal symmetry breaking! Here: magnetic field. But there are other cases, too. / 34

Chiral edge states:

We have seen that the charge-pumping argument implies conducting edge states. Intuitive quasi-classical picture:



- chiral edge states: right and left movers on each edge
- chirality due to transmission determined by direction of \vec{B} -field
- another instance of bulk-boundary correspondence

⑤ Chern Insulators

Strategy to construct a lattice model for the QHE:

Idea: Kitaev chain model helped us understand Majoranas. QHE: One way to get QHE is to place electrons in a magnetic field. Now we would like to find a simple tight-binding model for the QHE which also allows us to get rid of the magnetic field.

Such models are called Chern insulators (CI).

QHE without magnetic field : Quantum Anomalous Hall Effect (QAHE)

First CI model: Haldane 1988 - dressed graphene lattice

Now: more natural approach - dominoes again :-)

Two key aspects:

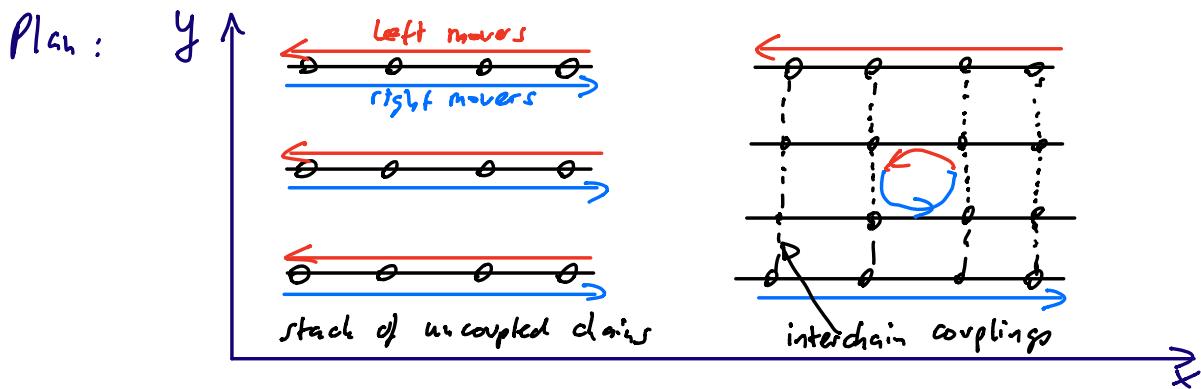
- QHE edge cannot exist in isolation from bulk, relies on bulk-boundary correspondence ("chiral anomaly").

Kitaev chain: M2M

QHE : chiral edge states

- find a lower-dimensional building block
Kitaev chain: fermionic sites = pairs of Majoranas

- First:
- find 1D system that can host pair of chiral edge states
 - left + right movers, on top of each other
 - put 1D lego pieces on top of each other and couple them like we did for the Kitaev chain \rightarrow 2D system with spatially separated chiral edge states = QAHE!



1D wire with pair of edge states:

Reminder, Dirac model $H = \Delta \sin k_y T_x$ of the Kitaev chain at the critical point (topological phase transition).

Kitaev model: $H(k) = -(2t \cos k + \mu) T_x + \Delta \sin k T_y$

Critical point $\mu = -2t$: $H(k) = -2t(\cos k - 1) T_x + \Delta \sin k T_y$

Near $k=0$: pair of states with wave functions related to eigenvalues ± 1 of T_y , and with opposite velocities.

"Problem": Kitaev model is superconducting with phs,
and T matrices refer to particles and
holes. \rightarrow should not enter for the QHE

Solution: re-interpret T matrices as acting on the
space of left and right movers.

Generic strategy: phase transition points of lower-dim. models
are good starting points to construct higher-
dim. topological models.

Test: quantum Hall effect and Kitaev chain can have
chiral states. What is the fundamental difference
apart from dimensionality?

- (A) QHE edge states go in opposite directions while Kitaev
states go in the same.
- (B) QHE edge states go in same direction while Kitaev
states go in opposite directions.
- (C) QHE edge states always cross zero at $k=0$
while Kitaev states don't
- (D) Kitaev chiral states only exist at specific parameter
values while QHE edge states are more robust

Model Hamiltonian:

- stack x -oriented chains together along y .
- Chain index n_y
- replace $k \rightarrow k_x$
- single chain: $\left[-(2t \cos k_x + \mu) T_z + \Delta \sin k_x T_y \right] \otimes |n_y\rangle \langle n_y|$
- couple $T_y = -1$ branch of one chain to $T_y = +1$ branch of a neighboring chain \rightarrow QHE
- Write this as

$$Y \underbrace{|n_y\rangle \langle n_y+1|}_{\substack{\text{Coupling} \\ \text{strength}}} \otimes \underbrace{(T_z + iT_x)}_{\substack{\text{Couple} \\ \text{neighboring} \\ \text{chains}}} \begin{array}{l} \text{turns left} \\ \text{moves into} \\ \text{right moves} \\ \text{when particle} \\ \text{hops} \checkmark \end{array}$$

$$T_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$|\Psi_R\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} -i \\ 1 \end{pmatrix} \quad T_y |\Psi_R\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} -i \\ \pm 1 \end{pmatrix} = \pm |\Psi_{R/L}\rangle$$

$$\begin{aligned} (T_z + iT_x) |\Psi_{R/L}\rangle &= \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} -i \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \mp i + i \\ \pm 1 - 1 \end{pmatrix} \\ &= \begin{cases} 0 & \text{for } R \\ -2 |\Psi_R\rangle & \text{for } L \end{cases} \end{aligned}$$

- complete H : sum over n_y

/79

$$\begin{aligned}
 H &= \sum_{n_y} \sum_{k_x} \left\{ \left[-(2t \cos k_x + \mu) T_z + \Delta \sin k_x T_y \right] \otimes |n_y\rangle \langle n_y| \right. \\
 &\quad \left. = \sum_{k_x} -\gamma |n_y\rangle \langle n_y+1| \otimes (T_z + iT_x) + \text{h.c.} \right\} \\
 \rightarrow \text{should suffice to produce a quantum Hall state!} \\
 &\quad (\text{need to keep in mind to use } \mu = -2t \text{ for criticality of chains, but keep } \mu \text{ free for now})
 \end{aligned}$$

Gap and edge states:

Finite # chains $n_y = 1, \dots, N$.

Tune $\mu = -2t$: check that at $k_x \approx 0$ H has one right-moving edge state for $n_y = N$ with eigenvalue $\approx \Delta k_x$, and one left-moving edge state for $n_y = 1$ with eigenvalue $\approx -\Delta k_x$.

How: Write H as matrix ($N \times N$) in $|n_y\rangle$ basis

for $2t \cos k_x + \mu \approx 0$, $\sin k_x \approx k_x$:

$$H(k_x \approx 0) = \begin{bmatrix} \Delta k_x T_y & -\gamma(T_z + iT_x) & & & \\ -\gamma(T_z - iT_x) & \Delta k_x T_y & -\gamma(T_z + iT_x) & & \\ & -\gamma(T_z - iT_x) & \Delta k_x T_y & -\gamma(T_z + iT_x) & \\ & & -\gamma(T_z - iT_x) & \Delta k_x T_y & -\gamma(T_z + iT_x) \\ & & & -\gamma(T_z - iT_x) & \Delta k_x T_y \end{bmatrix}$$

e.g. $N=4$

Exercise: find eigenstates and eigenvalues for $N=2$ and identify the edge states

Next: Are these edge states the only low-energy eigenstates — is the bulk gapped?

Exercise: Infinite stack = bulk only

→ go to momentum space along y also!

→ find $H(k_x, k_y)$, plot the dispersion $E(k_x, k_y)$

Solution: 2D Block Hamiltonian for $t=1.0$, $\Delta=0.3$, $\gamma=-0.5$, $\mu \in [-2, 0]$

$$H(k_x, k_y) = [-(2t \cos k_x + \mu) T_z + \Delta \sin k_x T_y] \\ - 2\gamma [\cos k_y T_z + \sin k_y T_x]$$

$$E(k_x, k_y) = \pm \sqrt{\Delta^2 \sin^2 k_x + (2\gamma \cos k_y + \mu + 2t \cos k_x)^2 + 4\gamma^2 \sin^2 k_y}$$

→ gapped except at $\mu = -2t - 2\gamma$

Finite ribbon: Plot all eigenvalues $E(k_x)$ and check if there are edge states (Dirac-like crossing at $k_x = 0$)!

Details such as bulk spectrum and edge dispersion are different from the QHE with magnetic field; but bulk-edge correspondence tells us that our edge states are as robust as those of the QHE!

Test: How does our lattice model differ from the original quantum Hall effect?

(A) Since there is no magnetic field, the lattice QHE model preserves time-reversal symmetry

- Ⓐ QHE in magnetic field has Landau levels that do not disperse in k (flat bands) while the bulk states disperse in the lattice
- Ⓑ QHE in lattice has no chiral edge states, which arise from skipping orbits in magnetic field
- Ⓒ in a magnetic field the filling fraction is fixed to integer per flux quantum, while in the lattice the filling fraction per unit cell is arbitrary

Dirac equation at the phase transition:

Effective bulk Hamiltonian near transition, $\mathcal{H} = -\vec{k}_x^2 + 2t + 2\gamma$.

Define mass $m \equiv -(\mu + 2t + 2\gamma)$

$$H_{\text{Dirac}}(\vec{k}) = \Delta k_x T_y - 2\gamma k_y T_x + m T_z$$

transition : $m=0$ massless Dirac model !

$m>0$: topological phase

$m<0$: trivial phase

Remember Kitelev chain : $m(x)$ with sign change of m
 \rightarrow domain wall
 \rightarrow nondegenerate zero-mode

Here: fix $k_y = 0$ \rightarrow identical Hamiltonian as in Ⓚ
zero mode: eigenstate of T_x with eigenvalue +1

But here: zero mode not stationary $E(k_y) \approx -2\gamma k_y$ at $k_x = 0$ /42

Summary :

- relation between Kitaev chain and lattice quantum Hall model (Chern insulator, quantum anomalous Hall effect)
 - drive lower-dim. topological state to critical point \rightarrow massless state
 \rightarrow couple massless states to get higher-dim. topological states
 - QHE can be realized in translationally invariant lattice models without magnetic field
 - chiral edge states cannot exist by themselves in 1D : 1D model always has pairs of left + right movers \rightarrow only way to realize only left movers is on the boundary of a higher-dimensional state
- general principle: forbidden states in a lower dimension and nontrivial topological states in a higher dimension are related
- prototype lattice model for all sorts of topological states

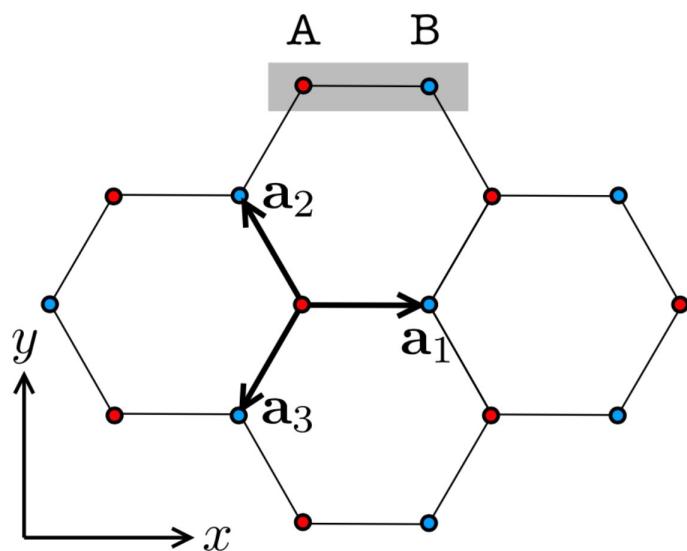
Haldane model, Berry curvature, and Chern number

Recipe to obtain QHE: massless Dirac model + mass term that can change sign

Real 2D Dirac material: **Graphene!**

Haldane 1988 (before graphene was synthesized): give graphene a gap (mass term).

Graphene: 2D honeycomb lattice — triangular lattice with basis A & B



Wave function in unit cell: vector $(\psi_A, \psi_B)^T$ of amplitudes on A, B sites.

Simple hopping model with hopping strength t_1 along bonds between A and B sites:

(A and B sublattices)

$$H_0(k) = \begin{pmatrix} 0 & h(k) \\ h^+(k) & 0 \end{pmatrix}, \quad k = (k_x, k_y)$$

$$h(k) = t_1 \sum_{i=1,2,3} \exp(i \vec{k} \cdot \vec{q}_i), \quad \vec{q}_i \text{ see figure}$$

Using 2×2 Pauli matrices σ which act on sublattice space:

$$H_0(k) = t_1 \sum_i \left[\sigma_x \cos(\vec{k} \cdot \vec{q}_i) - \sigma_y \sin(\vec{k} \cdot \vec{q}_i) \right]$$

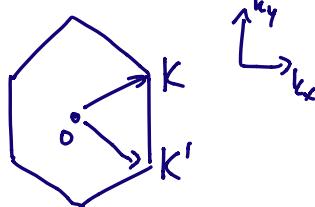
\uparrow \uparrow
Re and Im of $\exp(i \vec{k} \cdot \vec{q}_i)$

Energy dispersion:

$$E(k) = \pm |h(k)| \rightarrow \text{graphene band structure}$$

$$\text{with Dirac cones near } K = \left(\frac{2\pi}{3}, \frac{2\pi}{3\sqrt{3}} \right)$$

$$K' = \left(\frac{2\pi}{3}, -\frac{2\pi}{3\sqrt{3}} \right)$$



$$\sigma_z \sigma_x \sigma_z = -\sigma_x$$

$$\sigma_z \sigma_y \sigma_z = -\sigma_y$$

$$\sigma_z \sigma_z \sigma_z = +\sigma_z$$

block structure

$$\begin{pmatrix} 0 & h \\ h^+ & 0 \end{pmatrix}$$

Symmetries of graphene:

- sublattice symmetry $\sigma_z H_0(k) \sigma_z = -H_0(k)$

only approximate and consequence of only nearest-neighbor hopping; not important for Dirac cones

- inversion symmetry: $A \leftrightarrow \bar{D}$ important for Dirac cones

- 3-fold rotational symmetry around center of unit cell, important for Dirac cones but not in the following

- time-reversal symmetry : spinless \rightarrow only complex conjugation in k space

$$H_0(k) = H_0^*(-k)$$

\rightarrow exchanges the two Dirac cones

- product of sublattice (approximate in real graphene) and time-reversal symmetry
 \Rightarrow particle-hole symmetry

$$\sigma_z H_0^*(-k) \sigma_z = -H_0(k)$$

Make graphene topological :

\rightarrow need bulk gapped

\rightarrow break inversion and/or time-reversal symmetry

- 1) broken inversion symmetry : onsite potential $\pm M$ on A/B

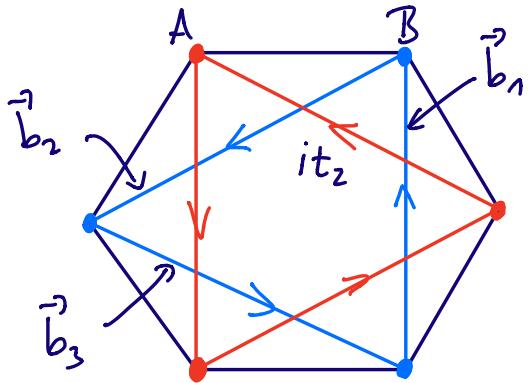
$$H_0(k) + M\sigma_z$$

$$\rightarrow E(k) = \pm \sqrt{|h(k)|^2 + M^2} \text{ gapped}$$

but : boring ; for $M \gg t$, the electronic states are localized on either A or B sublattices ;
no trace of edge states

why? $M\sigma_z$ preserves time-reversal symmetry

- 2) add imaginary second-neighbor hoppings :



arrows: direction in which hopping is $+it_2$
 (opposite direction: $-it_2$)

- it_2 hoppings are purely imaginary and have chirality (handedness)

- they couple $A - A$ and $B - \bar{B}$

→ it_2 terms break time-reversal and sublattice symmetry:

$$H(k) = H_0(k) + M\sigma_z + 2t_2 \sum_i \sigma_z \sin(\vec{k} \cdot \vec{b}_i)$$

$\uparrow \equiv H_{t_2}$

changes sign under time reversal (breaks it)

$$H_{t_2}^*(-k) = 2t_2 \sum_i \sigma_z \sin(-\vec{k} \cdot \vec{b}_i)$$

$$= \textcolor{red}{-} H_{t_2}(k)$$

Note: there is also a $\sin(\vec{k} \cdot \vec{a}_i)$ term in $H_0(k)$ but it goes with σ_y , which also changes sign under time reversal: $\sigma_y^* = -\sigma_y$
 $\Rightarrow H_0(k)$ does not break trs!

→ band structure demo

$M \neq 0, t_2 = 0$: boring gapped phase

(there might be edge states depending on lattice termination, but these are not chiral, dispersionless, and boring; also they do not connect valence and conduction bands)

t_2 passes through $\pm M/\sqrt{3}$: gap closes and changes sign at one of the two Dirac points —
at K' for $t_2 = +M/\sqrt{3}$
at K for $t_2 = -M/\sqrt{3}$

→ chiral edge states appear!

→ we have created a Chern insulator

Test: What happens if we take a topological Haldane model and turn on a weak magnetic field?

- (A) → Landau levels, which change the number of edge states
- (B) if mag. field is weak, nothing changes if it does not close the gap
- (C) bulk gap closes and edge states disappear
- (D) gap does not close but edge states may change propagation direction depending on sign of field

Pumping and Berry phase:

QHE with Laughlin argument: adiabatically pierce flux $\vec{\Phi}$ through QHE - cylinder so that Hamiltonian returns to itself.

$$H(\vec{k}) \rightarrow H(\vec{k} + e\vec{A}) \text{ with } \vec{A} = \gamma \frac{\vec{\Phi}}{L}$$

flux pumping \leftrightarrow change in momentum

flux change by integer number of flux quanta:

momentum \vec{k} changes by reciprocal lattice vector
 \rightarrow Bloch Hamiltonian returns to itself

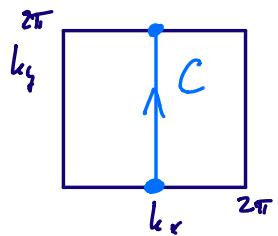
Simplicity: Use a 2D square lattice zone

$(k_x, k_y) \in [0, 2\pi]^2$; all arguments apply equally to the hexagonal graphene zone

Adiabatic time evolution of an eigenstate $|\Psi(\vec{k})\rangle$ of $H(\vec{k})$ with energy $E(\vec{k})$ as \vec{k} is changed slowly.

$|\Psi(\vec{k})\rangle$ should remain nondegenerate (true for Haldane model)
 \rightarrow can follow adiabatically

Choose path $k(t)$ with $k(0) = k(T)$ — periodic



What is the final quantum state at T ?

$$|\psi(k_x, k_y + 2\pi)\rangle \stackrel{?}{=} |\psi(k_x, k_y)\rangle e^{-i \int_0^T dt E(k(t)) dt}$$

No! Berry: Additional phase $\gamma(C) = \oint_C \vec{A}(\vec{k}) \cdot d\vec{k}$

$$\vec{A}(\vec{k}) = i \langle \psi(\vec{k}) | \vec{\nabla}_{\vec{k}} \psi(\vec{k}) \rangle$$

"Berry connection" *"Berry phase"*

In our example γ will depend on k_x , which depends on the path C but not on $\vec{k}(t)$, the actual "time evolution". *Berry phase = geometric phase*

Later: γ is related to the topological character of the Hamiltonian and its ground state wavefunction.

Flux pumping and Chern number:

$\gamma(k_x)$: info about charge pumped during adiabatic cycle.

We know: pumped charge is invariant as long as energy gap is preserved.

→ we can change $E(k_x, k_y)$ if we do not close the gap

choice: flat E along k_x for fixed k_y

→ localized state in single unit cell in x direction since all wave functions have same energy independent of k_x

$$|\psi(n, t=0)\rangle = \int dk_x e^{ik_x n} |\psi(k_x, k_y=0)\rangle$$

$$\Rightarrow |\psi(n, t=T)\rangle = \int_0^{2\pi} dk_x e^{ik_x n} e^{[i\gamma(k_x) - i\theta(k_x)]} |\psi(k_x, k_y=0)\rangle$$

$$\Theta(k_x) = \int_0^T dt E(k_x, k_y(t)) \quad -\text{dynamical phase}$$

strange: while $\Theta(k_x)$ truly periodic in k_x because $E(k_x) = E(k_x + 2\pi)$, the only restriction on $\gamma(k_x)$ is to be periodic modulo 2π :

$$\gamma(k_x + 2\pi) = \gamma(k_x) + 2\pi w, \quad w \in \mathbb{Z} \text{ integer}$$

Deform dispersion along k_y to make $\gamma(k_x) - \theta(k_x)$ as large as possible:

$$\gamma(k_x) - \theta(k_x) = wh_x$$

$$\rightarrow |\psi(n, t=T)\rangle = \int dk_x e^{ik_x(n+w)} |\psi(k_x, k_y=0)\rangle$$

\Rightarrow wave function shifted over by w unit cells.

\Rightarrow pump w units of charge if Berry phase satisfies

$$\gamma(k_x + 2\pi) - \gamma(k_x) = 2\pi w$$

w : Chern number = topological invariant

characterizing the band structure of 2D

Quantum Hall systems.

In fact: w = bulk topological invariant
of all insulators with broken time-reversal
symmetry. $w=0$: trivial insulator, no chiral edge states
 $w=n$: n chiral edge states, Chern insulator

Chern number from Berry curvature

Berry connection $\vec{A}(\vec{k}) \leftrightarrow$ vector potential $\vec{A}(\vec{r})$?

Similarities: • $\vec{A}(\vec{k})$ is gauge-dependent

$$|\psi(\vec{k})\rangle \rightarrow e^{i\vec{A}(\vec{k})} |\psi(\vec{k})\rangle$$

$$\Rightarrow \vec{A}(\vec{k}) \rightarrow \vec{A}(\vec{k}) + \vec{\nabla}_k$$

- Berry phase is gauge-independent for closed paths

Idea: In electromagnetism, $\vec{\nabla} \times \vec{A}(\vec{r})$ is the magnetic field, which is measurable.

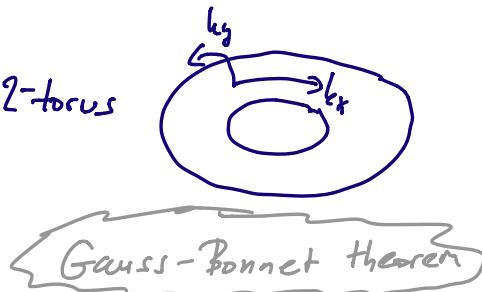
Here: $\vec{\nabla}_k \times \vec{A}(\vec{k}) = \vec{\Omega}(\vec{k})$ Berry curvature
 $= i \left[\left\langle \frac{\partial \Psi}{\partial k_x} \left| \frac{\partial \Psi}{\partial k_y} \right. \right\rangle - \left\langle \frac{\partial \Psi}{\partial k_y} \left| \frac{\partial \Psi}{\partial k_x} \right. \right\rangle \right]$ (z-component
only for 2D)

gauge-independent

Stokes theorem: Brillouin zone = 2-torus
(Bt)

$$2\pi \omega = \gamma(2\pi) - \gamma(0)$$

$$= \iint_{Bt} \vec{\Omega}(\vec{k}) \cdot d\vec{s}$$



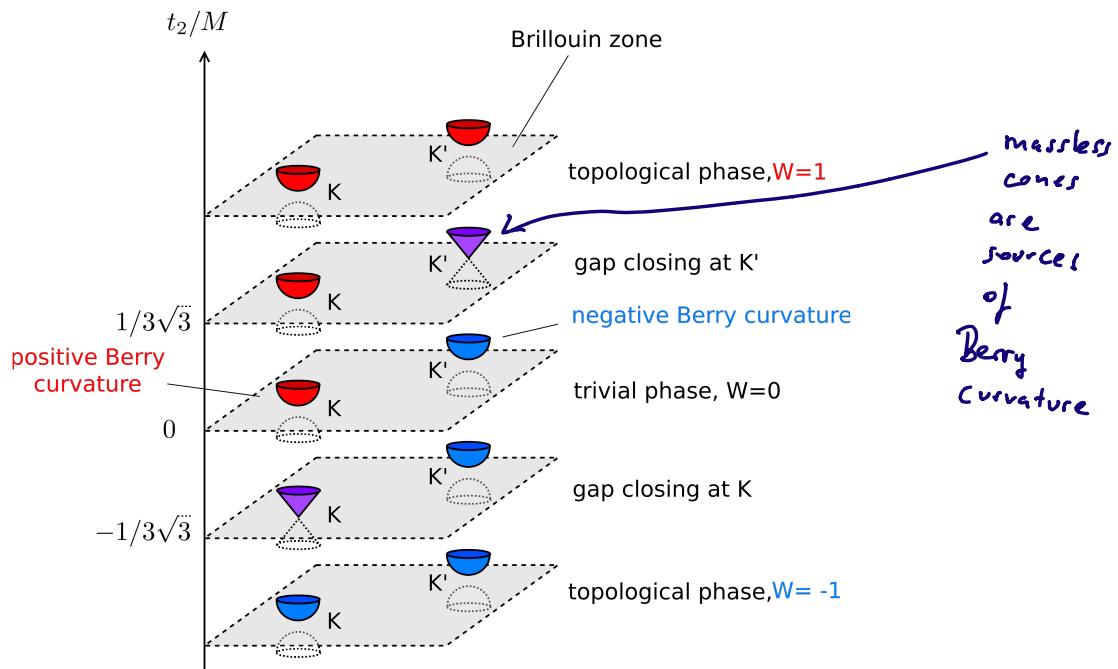
If Chern number defined in terms of $|\Psi(\vec{k})\rangle$ only,
closed-form expression in terms of derivatives of $|\Psi(\vec{k})\rangle$
via Berry curvature

$\omega \neq 0$: "flux" coming out of a closed surface

→ "magnetic monopole" (Dirac monopole)

→ What are the sources of Berry curvature?

Gap closings are sources of Berry curvature



⑥ From Quantum Spin Hall effect to topological insulators

Adding symmetry to a topological insulator

Approaches to discover novel topological states:

- (i) lower-D to higher-D to create edge states
(cf. Kitaev, Chern insulator)
- (ii) topology (K-theory): calculate topological classification from dimensionality and symmetries
 - powerful but hard
- (iii) take one topological model and enforce additional symmetry
 - this will be used now

Example: • quantum dot with H_0

- $\mathbb{Q} = \# \text{filled levels}$
- enforce particle-hole symmetry

$$H_{\text{BdG}} = \begin{pmatrix} H_0 & \\ & -H_0^* \end{pmatrix}$$

→ topologically trivial w.r.t. $\mathbb{Q} = \text{const}$ here
but there are level crossings

- are crossings protected when SC Δ is added to couple the blocks?
- answer (see ①): yes! Pfaffian invariant

Now: add time-reversal symmetry to Chern insulator

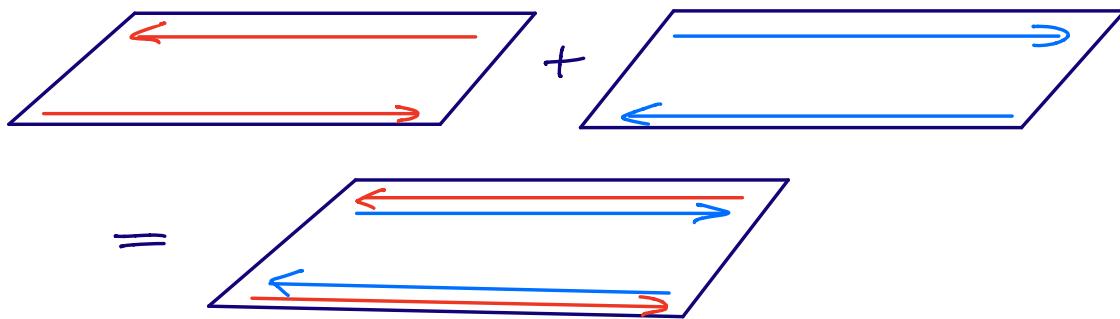
CI: chiral edge states that are flipped under time-reversal \mathcal{T}

add trs: $H = \begin{pmatrix} H_0 & 0 \\ 0 & \mathcal{T}H_0\mathcal{T}^{-1} \end{pmatrix}$ (time-reversal simply flips the two blocks)

H_0 : CI with N edge states

H : N pairs of counterpropagating edge states that transform into each other under time-reversal

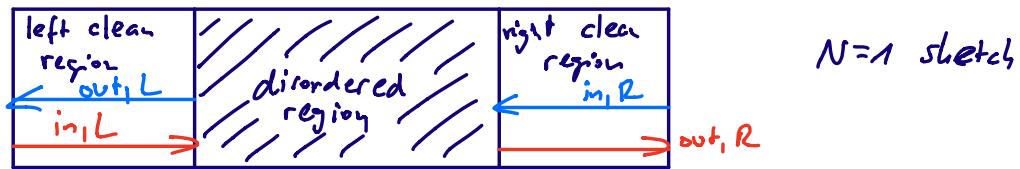
$N=1$ sketch:



next: are these edge states robust when the blocks are coupled?

SS

A perfectly transmitted channel and Kramers degeneracy:



Scattering states:

$$\begin{aligned} \text{incoming} \quad & |n_1, L\rangle \quad n=1, \dots, N \\ & |n_1, R\rangle \end{aligned}$$

outgoing $\mathcal{T}|n_1, L\rangle$ time-reversed partners of incoming states
 $\mathcal{T}|n_1, R\rangle$

Scattering states:

$$\begin{aligned} |\Psi_{1L}\rangle &= \sum_{n=1}^N (\alpha_{n,L} |n_1, L\rangle + \beta_{n,L} \mathcal{T}|n_1, L\rangle) \\ |\Psi_{1R}\rangle &= \sum_{n=1}^N (\alpha_{n,R} |n_1, R\rangle + \beta_{n,R} \mathcal{T}|n_1, R\rangle) \end{aligned}$$

Define vectors : $\alpha_L \equiv (\alpha_{1,L}, \dots, \alpha_{N,L})$ etc.

Relation between incoming and outgoing modes: scattering matrix S of disordered region

$$\begin{pmatrix} \beta_L \\ \beta_R \end{pmatrix} = S \begin{pmatrix} \alpha_L \\ \alpha_R \end{pmatrix}, \quad S \text{ is } 2N \times 2N \text{ matrix}$$

$S = S^*$ unitary
(no losses)

Split S into $N \times N$ reflection and transmission blocks:

$$S = \begin{pmatrix} r & t \\ t' & r' \end{pmatrix}$$

Can there be no transmission at all?

This would mean $t = t' = 0$ due to backscattering via disorder, gapped or edge states, ...

If yes: all modes perfectly reflected back:

$$r^+ r = r'^+ r' = \mathbb{1} \quad (\text{unitary blocks})$$

→ need to understand constraints imposed by time-reversal symmetry.

Scattering matrices with time-reversal symmetry:

time reversal: antiunitary operator \mathcal{T} $\begin{pmatrix} \langle J\psi | J\phi \rangle \\ = \langle \psi | \phi^* \end{pmatrix}$
which commutes with H

two flavors: $\mathcal{T}^2 = \pm \mathbb{1}$

+ case: systems with no or integer spin

$\mathcal{T} = K$ complex conjugation

- case: systems with half-integer spin

simplest case $S=\frac{1}{2}$: $\mathcal{T} = i\sigma_y K$

Apply to scattering states:

$$\mathcal{T} |\psi_{,L}\rangle = \sum_n \alpha_{n,L}^* \mathcal{T} |n_{,L}\rangle + \beta_{n,L}^* \mathcal{T}^2 |n_{,L}\rangle$$

$$\mathcal{T} |\psi_{,R}\rangle = \sum_n \alpha_{n,R}^* \mathcal{T} |n_{,R}\rangle + \beta_{n,R}^* \mathcal{T}^2 |n_{,R}\rangle$$

\mathcal{T} does not change the energy of states

→ same scattering matrix as for $|\psi_{,L}\rangle, |\psi_{,R}\rangle$

but roles of α, β are now reversed

$$S \mathcal{T}^2 \begin{pmatrix} \beta_L^* \\ \beta_R^* \end{pmatrix} = \begin{pmatrix} \alpha_L^* \\ \alpha_R^* \end{pmatrix}$$

Multiply both sides by $J^2 S^T$ and take C.C.:

$$\begin{pmatrix} \beta_L \\ \beta_R \end{pmatrix} = J^2 S^T \begin{pmatrix} \alpha_L \\ \alpha_R \end{pmatrix}$$

$$\Rightarrow S \stackrel{!}{=} J^2 S^T$$

if $J^2 = +1$: $S = S^T$ symmetric

if $J^2 = -1$: $S = -S^T$ antisymmetric

Now try to set $t=t'=0$:

For $S=S^T$ we cannot tell

But for $S=-S^T$ and $t=t'=0$, we have

$$r^+ r = 1 \quad \text{unitary}$$

$$r = -r^T \quad \text{antisymmetric}$$

If N odd: impossible — any odd-dimensional antisymmetric matrix must have a single zero eigenvalue, while unitary matrices only have eigenvalues with unit norm!

→ $t=0$ is impossible!

→ zero eigenvalue of r means that there is always a single mode that is transmitted with unit probability!!!

If $T^2 = -1$ and number of edge states going in one direction is odd, they cannot be gapped out.

If there is an even number of edge states, they can be gapped out.

Since these are the only two options, the topological invariant of a Chern insulator is reduced to a ± 1 invariant ("Z₂ invariant") in the presence of time-reversal symmetry. The topologically protected counterpropagating edge states are called helical edge states.

Helical edge states are Kramers pairs

Why is $T^2 = -1$ special?

Kramers Theorem:

For every energy eigenstate of a time-reversal symmetric system with half-integer spin, there is at least one more eigenstate with the same energy.

Proof: trs $\Rightarrow [H, T] = 0$. If $|n\rangle$ eigenstate with $H|n\rangle = E_n|n\rangle$, then $T|n\rangle$ eigenstate and $H T|n\rangle = T H|n\rangle = T E_n|n\rangle = E_n T|n\rangle$.

Crucial ingredient: $T|n\rangle$ and $|n\rangle$ are different / 59

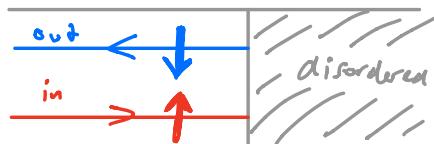
states for half-integer spin! \mathcal{T} reverses all angular momenta, and the magnetic quantum number (e.g., S_z for $S = \frac{1}{2}$) is never zero! Hence, $S_z = \frac{1}{2} \rightarrow S_z = -\frac{1}{2}$ under \mathcal{T} . Q.E.D.

$|\psi\rangle$ and $\mathcal{T}|\psi\rangle$ are orthogonal : $\langle \psi | \mathcal{T} | \psi \rangle = 0$

Kramers pair

\Rightarrow it is impossible to introduce any backscattering between them without breaking TRS!

Example: $N=1, S = \frac{1}{2} \Rightarrow S_z = \pm \frac{1}{2}$ ↑ and ↓



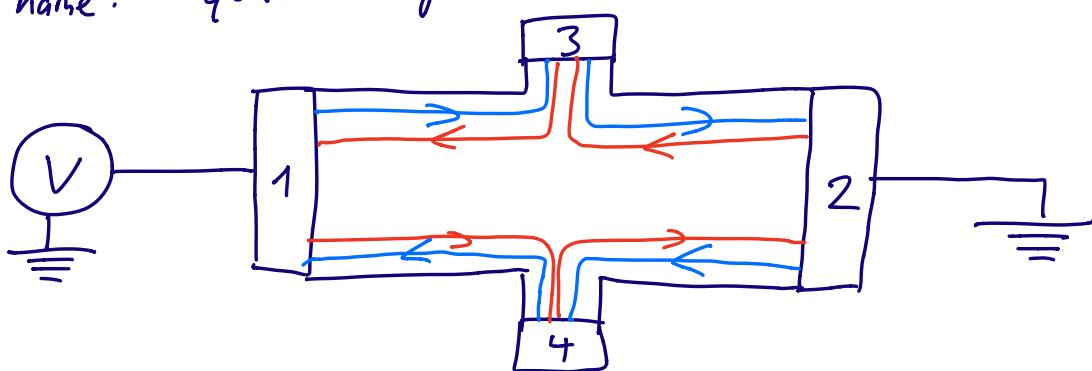
- \Rightarrow reflection requires spin-flip scattering
- \Rightarrow forbidden by TRS
- \Rightarrow perfect transmission

The quantum spin Hall effect

2D topological insulator with time-reversal symmetry

= " \mathbb{Z}_2 topological insulator" — only indicates
that topological invariant can only take two values ± 1

better name: "quantum spin Hall insulator"



- electrons injected from left (1)
- same # of channels connecting 1-3 and 1-4
 \Rightarrow no net Hall current
 $\Rightarrow J_{\text{Hall}} = 0$ as expected for trs system
- but: \rightarrow and \leftarrow have opposite spin
 \Rightarrow there may be a net spin current
- if S_2 preserved:
 - \rightarrow have \uparrow -spin
 - \rightarrow have \downarrow -spin $\Rightarrow \uparrow$ go from 1 to 4
 \downarrow go from 1 to 3
 \Rightarrow quantized spin current between 3 and 4

\Rightarrow quantum spin Hall effect

Not a general property of \mathbb{Z}_2 topological insulator —
only when spin conservation law is present.

Model for quantum spin Hall (QSH) insulator

Bernevig - Hughes - Zhang (BHZ) model:

2 copies of Chern insulator on square lattice

↑
spin

$$H_{\text{BHZ}}(\vec{k}) = \begin{pmatrix} h(\vec{k}) & 0 \\ 0 & h^*(-\vec{k}) \end{pmatrix}$$

$$h(\vec{k}) = \varepsilon(\vec{k}) \mathbb{1} + \vec{d}(\vec{k}) \cdot \vec{\sigma}$$

$\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$, Pauli matrices act on electron/
hole degrees of freedom of the Chern insulator.

$$\varepsilon(\vec{k}) = C - D (k_x^2 + k_y^2)$$

$$\vec{d}(\vec{k}) = (Ak_x, -Ak_y, M(\vec{k}))$$

$$M(\vec{k}) = M - B(k_x^2 + k_y^2)$$

$A k_x \sigma_x - A k_y \sigma_y$: linear coupling between holes
and electrons

$M(\vec{k})$: momentum-dependent mass

M changes sign: gap closing at $\vec{k} = 0$.
trivial \rightarrow topological insulator.

Dirac model applies to semiconductor sandwich materials involving strong spin-orbit coupling.

(7) Kubo formula and TKNN invariant

Before we have derived the Chern number W as a topological invariant related to the Berry curvature of filled energy bands of periodic systems. We have also argued that W counts the number of chiral edge modes in a Chern insulator.

Here: Explicit calculation of the Hall conductivity

$$\sigma_{xy} = W \frac{e^2}{h}$$

quantized in units of the conductance quantum $\frac{e^2}{h}$ from the Kubo formula (linear response).

Linear response in a nutshell:

$$H = H_0 + \Delta H$$

↑ ↑
system perturbation

Energy eigenstates of H_0 : $H_0 |n\rangle = E_n |n\rangle$ (formal! In general: unknown)

Electric field: $\vec{E} = -\partial_t \vec{A}$

$$\rightarrow \Delta H = -\vec{j} \cdot \vec{A}$$

Choose: $\vec{A}(t) = \frac{\vec{E}}{i\omega} e^{-i\omega t}$ (take DC limit $\omega \rightarrow 0$)
ad the end

Note: \vec{A}^2 terms only matter for longitudinal transport, neglected here.

K1

Goal: Write $\langle \vec{J}(\omega) \rangle = \overleftarrow{\Gamma}(\omega) \vec{E}(\omega)$

\downarrow only depends on H_0 $\overrightarrow{\Gamma}$ linear in field

$\overleftrightarrow{\Gamma}(\omega)$: tensor of optical conductivity

More specific: $\Gamma_{xy}(\omega)$ Hall conductivity

Interaction picture: $O(t) = V^{-1} \partial V, V = e^{-iH_0 t/\hbar}$

$$|\Psi(t)\rangle_I = U(t, t_0) |\Psi(t_0)\rangle_I$$

$$U(t, t_0) = \overleftarrow{\Gamma} \exp\left(-\frac{i}{\hbar} \int_{t_0}^t \Delta H(t') dt'\right)$$

time-ordering: $i\hbar \dot{U} = \Delta H U$

Write

$U(t) = U(t, t_0 \rightarrow -\infty)$; prepare system in $|0\rangle$ at $t_0 \rightarrow -\infty$:
(usually: ground state)

$$\begin{aligned} \langle \vec{J}(t) \rangle &= \langle 0 | \vec{J}(t) | 0 \rangle \\ &= \langle 0 | U^{-1}(t) \vec{J}(t) U(t) | 0 \rangle \\ &\stackrel{U}{\approx} \langle 0 | \left(\vec{J}(t) + \frac{i}{\hbar} \int_{-\infty}^t dt' [\Delta H(t'), \vec{J}(t)] \right) | 0 \rangle \end{aligned}$$

Expansion of $U(t)$ keeping only leading terms.

Assume: $\langle 0 | \vec{J}(t) | 0 \rangle = 0$ no current in ground state

$$\langle J_i(t) \rangle = \frac{1}{i\omega} \int_{-\infty}^t dt' \langle 0 | [J_i(t'), J_i(t)] | 0 \rangle E_j e^{-i\omega t'} = \dots$$

time-transl. inv.: depends only on $t'' = t - t'$

$$\dots = \frac{1}{i\omega} \left(\int_0^\infty dt'' e^{i\omega t''} \langle 0 | [J_i(0), J_i(t'')] | 0 \rangle \right) E_j e^{-i\omega t}$$

Actually: $\omega \rightarrow \omega + i\alpha^+$ for convergence

/K2

Hall conductivity:

$$\sigma_{xy}(\omega) = \frac{1}{\hbar\omega} \int_0^\infty dt e^{i\omega t} \langle 0 | [J_y(t), J_x(0)] \rangle$$

Kubo formula

Inserting $H_0 |n\rangle = E_n |n\rangle$ eigenstates:

$$\sigma_{xy}(\omega) = -\frac{i}{\omega} \sum_{n \neq 0} \left[\frac{\langle 0 | J_y | n \rangle \langle n | J_x | 0 \rangle}{\hbar\omega + E_n - E_0} - \frac{\langle 0 | J_x | n \rangle \langle n | J_y | 0 \rangle}{\hbar\omega + E_0 - E_n} \right]$$

DC $\omega \rightarrow 0$ limit:

$$\frac{1}{\hbar\omega + E_n - E_0} \approx \frac{1}{E_n - E_0} - \frac{\hbar\omega}{(E_n - E_0)^2} + \mathcal{O}(\omega^2)$$

\uparrow
only important for longitudinal conductivity
vanishes for Hall

$$\sigma_{xy} = i\hbar \sum_{n \neq 0} \frac{\langle 0 | J_y | n \rangle \langle 0 | J_x | 0 \rangle - \langle 0 | J_x | n \rangle \langle n | J_y | 0 \rangle}{(E_n - E_0)^2}$$

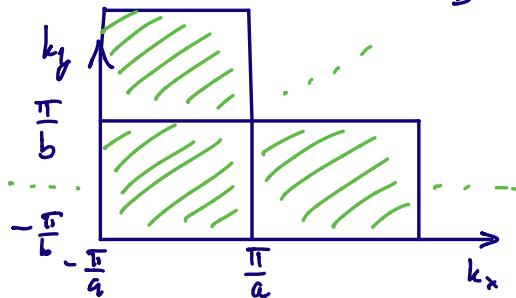
Kubo formula for Hall Conductivity

Role of topology

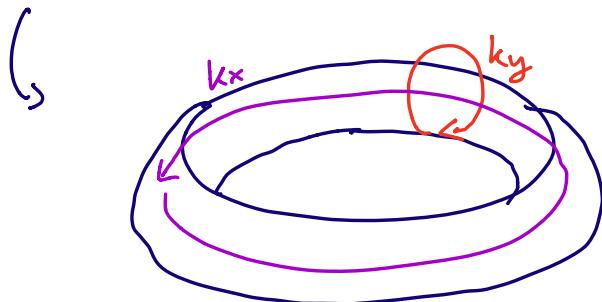
Consider particles on lattice. Brillouin zone:

$$- \frac{\pi}{a} < k_x \leq \frac{\pi}{a}, \quad - \frac{\pi}{b} < k_y \leq \frac{\pi}{b}$$

Since reciprocal
lattice periodic:
torus π^2



L3

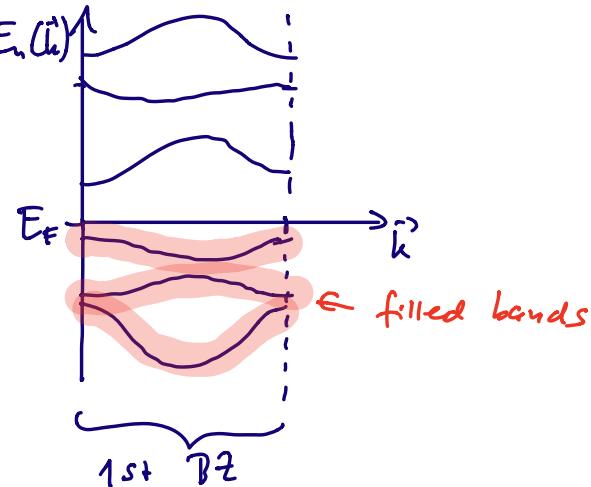


Wave functions (Bloch) : $\Psi_{\vec{k}}(\vec{x}) = e^{i\vec{k}\cdot\vec{x}} u_{\vec{k}}(\vec{x})$

\downarrow
periodic in \vec{k}

- assume that single-particle spectrum decomposes into separate bands $E_n(\vec{k})$
- noninteracting electrons : fill single-particle states employing only Pauli principle
- assume excitation gap between bands and Fermi energy E_F in one of these gaps

→ band insulator



\Rightarrow assign integer-valued topological invariant $w \in \mathbb{Z}$ to each band

14

$U(n)$ Berry connection defined over \mathbb{T}^2 :

$$A_j(\vec{k}) = -i \langle u_{\vec{k}} | \frac{\partial}{\partial k^j} | u_{\vec{k}} \rangle$$

→ field strength (Berry curvature):

$$\tilde{F}_{xy} = \frac{\partial A_x}{\partial k^y} - \frac{\partial A_y}{\partial k^x} = -i \left\langle \frac{\partial u}{\partial k^x} \middle| \frac{\partial u}{\partial k^y} \right\rangle + i \left\langle \frac{\partial u}{\partial k^y} \middle| \frac{\partial u}{\partial k^x} \right\rangle$$

→ 1st Chern number:

$$(*) \quad \omega = \frac{1}{2\pi} \int_{\mathbb{T}^2} d^2k \quad \tilde{F}_{xy} \quad \text{TKNN invariant}$$

Thouless, Kohmoto, Nightingale, den Nijs, PRL 49, 405 (1982)

Kubo formula for Hall conductivity using tensor products
of single-particle wavefunctions:

$$\sigma_{xy} = i\hbar \sum_{E_\alpha < E_F < E_\beta} \int \frac{d^2k}{\pi^2} \frac{\langle u_k^\alpha | j_y | u_k^\beta \rangle \langle u_k^\beta | j_x | u_k^\alpha \rangle - (x \leftrightarrow y)}{(E_\beta(k) - E_\alpha(k))^2}$$

α : filled bands

(actually: separate momentum

β : empty bands

integrals ... but will not matter here)

Current: $H | \psi_k \rangle = E_k | \psi_k \rangle$

$$\Rightarrow (e^{-ikx} H e^{ikx}) | \psi_k \rangle = E_k | \psi_k \rangle$$

$$\Rightarrow \hat{H}(k) | \psi_k \rangle = E_k | \psi_k \rangle, \quad \hat{H}(k) = e^{-ikx} H e^{ikx}$$

Current $\vec{j} = \frac{e}{\hbar} \frac{\partial \hat{H}}{\partial \vec{k}}$ ← group velocity of wavepackets

Check: Continuum $H = (\vec{p} + e\vec{A})^2/2m$

/k5

$$\rightarrow \tilde{H} = (\rho + \frac{e}{\hbar} k + eA)^2 / 2m \rightarrow \vec{j} = e \vec{x} \quad \checkmark$$

$$\sigma_{xy} = \frac{ie^2}{\hbar} \sum_{E_\alpha < E_F < E_\beta} \int_{\frac{d^2k}{(2\pi)^2}} \frac{\langle u_k^\alpha | \partial_y \tilde{H} | u_k^\beta \rangle \langle u_k^\beta | \partial_x \tilde{H} | u_k^\alpha \rangle - (x \leftrightarrow y)}{(E_\beta(k) - E_\alpha(k))^2}$$

∂_x, ∂_y means $\frac{\partial}{\partial k_x}, \frac{\partial}{\partial k_y}$

$$\begin{aligned} \langle u_k^\alpha | \partial_i \tilde{H} | u_k^\beta \rangle &= \langle u_k^\alpha | \partial_i (\tilde{H} | u_k^\beta \rangle) - \langle u_k^\alpha | \tilde{H} | \partial_i u_k^\beta \rangle \\ (\partial_i E_\beta) \underbrace{\langle u_k^\alpha | u_k^\beta \rangle}_{=0} &\cong (E_\beta(k) - E_\alpha(k)) \langle u_k^\alpha | \partial_i u_k^\beta \rangle \\ &= -(E_\beta(k) - E_\alpha(k)) \langle \partial_i u_k^\alpha | u_k^\beta \rangle \end{aligned}$$

$$\rightarrow \sigma_{xy} = \frac{ie^2}{\hbar} \sum_{\alpha \neq \beta} \int_{\frac{d^2k}{(2\pi)^2}} \langle \partial_y u_k^\alpha | u_k^\beta \rangle \langle u_k^\beta | \partial_x u_k^\alpha \rangle - (x \leftrightarrow y)$$

Sum over empty bands : $\sum_{\beta} |u_k^\beta \rangle \langle u_k^\beta| = \mathbb{1} - \sum_{\alpha} |u_k^\alpha \rangle \langle u_k^\alpha|$

\downarrow
vanishes in σ_{xy}
by symmetry

$$\rightarrow \boxed{\sigma_{xy} = \frac{ie^2}{\hbar} \sum_{\alpha \text{ filled}} \int_{\frac{d^2k}{(2\pi)^2}} \frac{d^2k}{(2\pi)^2} \langle \partial_y u_k^\alpha | \partial_x u_k^\alpha \rangle - \langle \partial_x u_k^\alpha | \partial_y u_k^\alpha \rangle}$$

$$(*) \Rightarrow \sigma_{xy} = \frac{e^2}{h} \sum_{\alpha} W_{\alpha}$$

Kubo = Chern \rightarrow Japanese physicist = Chinese mathematician
TKNN formula: Hall conductivity = sum of \oint

Chern number of filled bands = number of chiral edge modes of Chern insulator/QH insulator $/ h$

Chern insulator revisited

$$\tilde{H}(\vec{k}) = \vec{h}(\vec{k}) \cdot \vec{\sigma} + \varepsilon(\vec{k}) \mathbb{1}$$

$\vec{k} \in \mathbb{T}^2$, $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ Pauli matrices

energies : $E(\vec{k}) = \varepsilon(\vec{k}) \pm |\vec{h}(\vec{k})|$

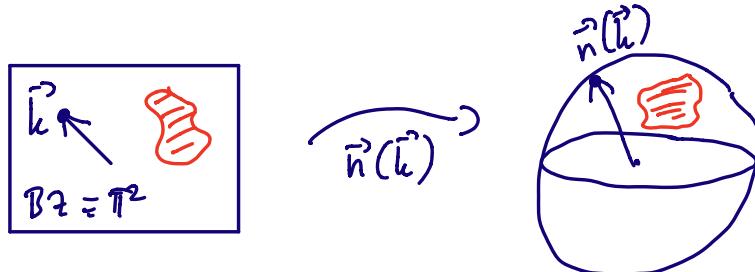
For example : $\tilde{H}(\vec{k}) = \sin k_x \sigma_x + \sin k_y \sigma_y$

Xiao-Liang Qi, Yong-Shi Wu,
Shou-Cheng Zhang, cond-mat/0505308

Pseudo spin vector : $\vec{n}(\vec{k}) = \frac{\vec{h}(\vec{k})}{|\vec{h}(\vec{k})|}$

$\vec{n} \in \mathbb{S}^2 \equiv 2\text{-sphere} \equiv \text{Bloch sphere}$:

Mapping from $\mathbb{T}^2 \rightarrow \mathbb{S}^2$



Final exercise:

- find band closings for $\tilde{H}(\vec{k})$ (values of m)

- compute the Chern number in the different phases
- what does this have to do with the winding of \vec{n} around the Bloch sphere? / 47