(*) Kinetic equations for electron-phonon systems

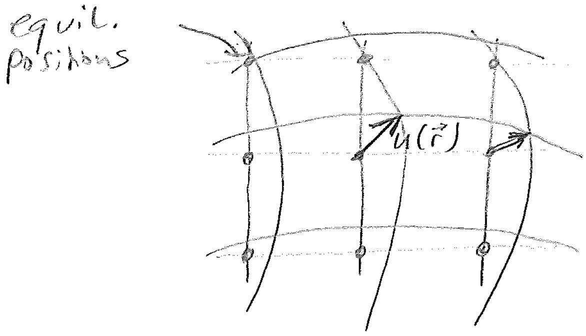
1. Electron-phonon interaction (in a nutshell)

1.1 Phonons

Phonons = harmonic oscillations of atoms in a crystal lattice

acoustic phonons : "quantum theory of sound"

Consider the distortion of an elastic medium:



$u(\vec{r})$: displacement vector

Elastic energy can be shown to be given by

$$\boxed{E_{el}[u] = \frac{1}{2} \int d^3r \sum_{\substack{ijkl \\ = x,y,z}} \epsilon_{ij}(\vec{r})\, C_{ijkl}\, \epsilon_{kl}(\vec{r})}$$

$\underset{=}{C}$: 4-th rank elastic tensor

$\epsilon_{ij} = \frac{1}{2}\left(\partial_i u_j + \partial_j u_i\right)$: strain tensor (2nd rank)

$\underset{=}{C}$ depends on symmetry of material.

Isotropic medium: only two independent coefficients

$$\boxed{\begin{gathered} C_{ijkl} = \lambda\, \delta_{ij}\, \delta_{kl} + \mu\left(\delta_{ik}\, \delta_{jl} + \delta_{il}\, \delta_{jk}\right) \\ \lambda, \mu : \text{Lame coefficients.} \end{gathered}}$$

Elastic waves:

For simplicity we consider first only compression waves ($\mu = 0$).

- Kinetic energy: $E_{kin}[\dot{u}] = \int d^3r \; \underbrace{\frac{\rho_0}{2}}_{\text{density}} \left(\frac{\partial \vec{u}}{\partial t}\right)^2$
- Lagrange density $\mathcal{L}[u, \dot{u}] = E_{kin}[\dot{u}] - \underbrace{E_{el}[u]}_{\frac{\lambda}{2}\int d^3r (\vec{\nabla} u)^2}$
- EOM: $\partial_t \frac{\delta \mathcal{L}}{\partial(\partial_t u_j)} + \sum_{i=x,y,z} \partial_i \frac{\partial \mathcal{L}}{\partial(\partial_i u_j)} - \frac{\partial \mathcal{L}}{\partial u_j} = 0$

$$\Rightarrow \boxed{\frac{1}{c^2} \partial_t^2 \vec{u} - \vec{\nabla}(\vec{\nabla} \cdot \vec{u}) = 0 \; , \; c^2 = \frac{\lambda}{\rho_0}}$$

- solution via expansion in normal modes:

$$\vec{u}(\vec{r}, t) = \frac{1}{\sqrt{V}} \sum_k \vec{\epsilon}_k \left(q_k(t) e^{i\vec{k}\vec{r}} + q_k^*(t) e^{-i\vec{k}\vec{r}} \right)$$

some unit polarization vector

leads to $\boxed{\frac{d^2}{dt^2} q_k + \underbrace{c^2 k^2}_{\equiv \omega_k^2} q_k = 0}$ $\Rightarrow$ harmonic oscillator

- Hamiltonian representation:

$$Q_k \equiv \sqrt{\rho_0} (q_k + q_k^*)$$

$$P_k \equiv \frac{dQ_k}{dt} = -i\omega_k \sqrt{\rho_0} (q_k - q_k^*)$$

$$H = E_{kin} + E_{el} = \sum_k 2 \rho_0 \omega_k^2 q_k(t) q_k^*(t) =$$

$$\boxed{H = \frac{1}{2} \sum_k P_k^2 + \omega_k^2 Q_k^2}$$ Hamiltonian function of independent oscillators

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⇒ canonical quantization

$$\boxed{\hat{H} = \frac{1}{2}\sum_k \left(\hat{P}_k^2 + \omega_k^2 \hat{Q}_k^2\right) \qquad [\hat{Q}_k, \hat{P}_{k'}] = i\hbar\delta_{kk'}}$$

⇒ solution by raising/lowering operators:

$$b_k = \frac{1}{\sqrt{2\hbar\omega_k}}\left(\omega_k \hat{Q}_k + i\hat{P}_k\right)$$

$$b_k^+ = \frac{1}{\sqrt{2\hbar\omega_k}}\left(\omega_k \hat{Q}_k - i\hat{P}_k\right)$$

$$\Rightarrow \quad [b_k, b_{k'}^+] = \delta_{kk'} \qquad [b_k, b_{k'}] = 0 \qquad [b_k^+, b_{k'}^+] = 0$$

<u>bosons</u>

$$\boxed{\hat{H} = \sum_k \hbar\omega_k \left(b_k^+ b_k + \frac{1}{2}\right)}$$

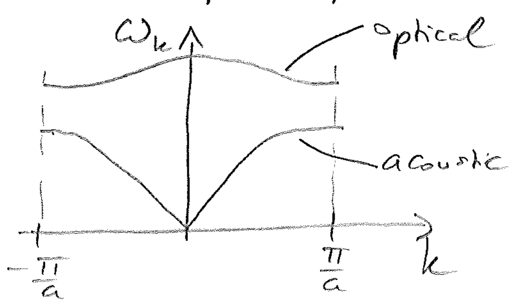
Hamiltonian for independent bosons with dispersion $\omega_k = c|\vec{k}|$ → acoustic phonons

inserting definitions back into $\hat{u}(\vec{r})$:

$$\hat{u}(\vec{r}) = \frac{1}{\sqrt{V}}\sum_k \hat{\varepsilon}_{\tilde{k}} \sqrt{\frac{\hbar}{2\varrho_0\omega_k}}\left(\hat{b}_k e^{i\vec{k}\vec{r}} + b_k^+ e^{-i\vec{k}\vec{r}}\right).$$

<u>Note:</u> For more than one atom per unit cell there are also phonons with $\omega_k \xrightarrow[|k|\to 0]{} \text{const} > 0$ → optical phonons

For example: diatomic chain



1.2 Electron-phonon interaction (minimal description)

For simplicity we consider the continuum ("jellium") model. Interaction between elastic medium and electrons:

$$H_{el-ph} = \int d^3r \int d^3r' \sum_\sigma \underbrace{\psi_\sigma(\vec{r})^\dagger \psi_\sigma(\vec{r})}_{\text{electronic density for spin } \sigma} \overbrace{V(\vec{r}-\vec{r}')}^{\text{screened Coulomb interaction}} \underbrace{\vec{\nabla}\cdot\vec{u}(\vec{r}')\, \overbrace{n_0}^{\frac{\#\text{ions}}{\text{volume}}}}_{\text{ion density modulation}}$$

Using the displacement operator ($\omega_q = \omega_{-q}$) $\qquad \rho_0 = \frac{NM}{V}$

$$\vec{u}(\vec{r}) = \frac{1}{\sqrt{V}} \sum_{q\alpha} \hat{\epsilon}_{q\alpha} \sqrt{\frac{\hbar}{2\rho_0 \omega_{q\alpha}}}\, e^{i\vec{q}\vec{r}} \left(b_{q\alpha} + b^\dagger_{-q\alpha}\right)$$

and the Fourier decomposition $\psi_\sigma(\vec{r}) = \frac{1}{\sqrt{V}} \sum_k c_{k\sigma} e^{i\vec{k}\vec{r}}$

we obtain the electron-phonon interaction

$$\boxed{H_{el-ph} = \sum_{kq\alpha\sigma} M_{q\alpha}\, c^\dagger_{k+q,\sigma} c_{k\sigma} \left(b_{q\alpha} + b^\dagger_{-q\alpha}\right)}$$

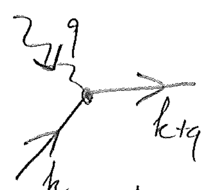
$$M_{q\alpha} = i \left(\vec{q}\cdot\hat{\epsilon}_{q\alpha}\right) \sqrt{\frac{\hbar N}{2M\omega_{q\alpha}}}\, V_q$$

$$V_q = \frac{1}{V} \int d^3r\, e^{-i\vec{q}\vec{r}}\, V(\vec{r})$$

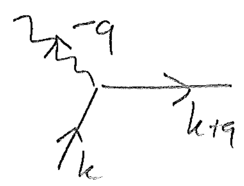
(Special form of the Fröhlich Hamiltonian)

Interpretation of terms in H_{el-ph}:

$$c^\dagger_{k+q} c_k b_q$$



phonon absorption



phonon emission

Consequences of electron-phonon interaction:

- renormalization of phonon dispersion (Kohn anomalies)
- renormalization of electronic dispersion ("kinks" in dispersion, effective mass, polaronic states)
- effective interaction between electrons ⟹ superconductivity
- electron-phonon scattering ⟹ electric resistivity
- laser control of electronic properties: resonant laser excitation of phonons can lead to ultrafast control of electronic properties in materials — light-induced superconductivity?! ⟹ ongoing research at MPSD!

But now: electron-phonon diagrammatics

1.3 Perturbation theory for electron-phonon interaction

El-ph: $\hat{H} = \hat{H}_{el} + \hat{H}_{ph} + \hat{H}_{el-ph} \equiv \hat{H}_0 + \hat{H}_{el-ph}$

suppress spin, only one band

$$\hat{H}_{el-ph} = \sum_{kq} M_q \, c^+_{k+q} c_k (b_q + b^+_{-q}) \equiv \hat{H}_{int}$$

Reminder:

Expansion in powers of H_{int}

$$G_n = \frac{1}{i^n} \frac{\sum_{k=0}^{\infty} \frac{(-i)^k}{k!} \int_\gamma dz_1 \ldots dz_k \langle \mathcal{T} \langle \hat{H}_{int}(z_1) \ldots \hat{H}_{int}(z_k) \hat{\psi}(1) \ldots \hat{\psi}^+(1') \rangle_0}{\mathcal{Z}}$$

$n=1$:

$$G(a,b) = \frac{1}{i} \frac{\sum_{k=0}^{\infty} \frac{(-i)^k}{k!} \int_\gamma dz_1 \ldots dz_k \langle \mathcal{T} \langle \hat{H}_{int}(z_1) \ldots \hat{H}_{int}(z_k) \hat{\psi}(a) \hat{\psi}^+(b) \rangle_0}{\mathcal{Z}}$$

Here: We want specifically $G_k(z_a, z_b) \equiv G(a,b)$ with

$\hat{\psi}(a) \to \hat{c}_k(z_a)$

$\hat{\psi}^+(b) \to \hat{c}^+_k(z_b)$

$$G_k(z_a,z_b) = \frac{1}{i\mathcal{Z}} \sum_{\ell=0}^{\infty} \frac{1}{\ell!} (-i)^\ell \int_\gamma dz_1 \ldots dz_\ell \sum_{k_1 q_1} \ldots \sum_{k_\ell q_\ell} M^{(z_1)}_{q_1} \ldots M^{(z_\ell)}_{q_\ell} \times$$

$$\times \langle \mathcal{T} \Big\{ c^+_{k_1+q_1}(z_1^+) c_{k_1}(z_1) [b_{q_1}(z_1) + b^+_{-q_1}(z_1)] \ldots c^+_{k_\ell+q_\ell}(z_\ell^+) c_{k_\ell}(z_\ell) \cdot [b_{q_\ell}(z_\ell) + b^+_{-q_\ell}(z_\ell)] \times c_k(z_a) c^+_k(z_b) \Big\} \rangle_0$$

reordering:

$$\langle \mathcal{T} \Big\{ c_k(z_a) c_{k_1}(z_1) \ldots c_{k_\ell}(z_\ell) c^+_{k_\ell+q_\ell}(z_\ell^+) \ldots c^+_{k_1+q_1}(z_1^+) c^+_k(z_b) \times [b_{q_1}(z_1) + b^+_{q_1}(z_1)] \times \ldots [b_{q_\ell}(z_\ell) + b^+_{-q_\ell}(z_\ell)] \Big\} \rangle_0$$

Wick theorem for fermions ✓

Wick theorem for bosons ✓

Wick theorem for $b + b^+$: multiply out

$l=1$: no terms since $\langle b \rangle_0 = \langle b^\dagger \rangle_0 = 0$

$l=2$: $\langle b_{q_1}(z_1)\, b^+_{-q_2}(z_2) \rangle_0 + \langle b^+_{-q_1}(z_1)\, b_{q_2}(z_2) \rangle_0 = \ldots$ (*)

$l=3$: no terms again $\rightarrow$ only even orders contribute

$l=4$: $\left(\langle b_{q_1}(z_1)\, b^+_{-q_2}(z_2) \rangle_0 + \langle b^+_{-q_1}(z_1)\, b_{q_2}(z_2) \rangle_0\right) \times$

$\left(\langle b_{q_3}(z_3)\, b^+_{-q_4}(z_4) \rangle_0 + \langle b^+_{-q_3}(z_3)\, b_{q_4}(z_4) \rangle_0\right) + \ldots$

Shorthand: (Momentum conservation: $q_1 = -q_2$, $q_3 = -q_4$)

$$\left(\langle b_1\, b_2^+ \rangle_0 + \langle b_1^+\, b_2 \rangle_0\right)\left(\langle b_3\, b_4^+ \rangle_0 + \langle b_3^+\, b_4 \rangle_0\right)$$
$$+\left(\langle b_1\, b_4^+ \rangle_0 + \langle b_1^+\, b_4 \rangle_0\right)\left(\langle b_3\, b_2^+ \rangle_0 + \langle b_3^+\, b_2 \rangle_0\right)$$
$$+\left(\langle b_1\, b_3^+ \rangle_0 + \langle b_1^+\, b_3 \rangle_0\right)\left(\langle b_2\, b_4^+ \rangle_0 + \langle b_2^+\, b_4 \rangle_0\right) = \ldots (**)$$

Define: $D_0(1;1') \equiv \frac{1}{i} \langle \mathcal{T} X_1\, X^+_{1'} \rangle_0$

$$\equiv \frac{1}{i} \langle \mathcal{T} (b_1 + b_1^+)(b_{1'} + b_{1'}^+) \rangle_0$$

$$= \frac{1}{i}\left(\langle \mathcal{T}\, b_1 b_{1'}^+ \rangle_0 + \langle \mathcal{T}\, b_1^+ b_{1'} \rangle_0\right)$$

$l=2$: (*) $= i\, D_0(1;2)$ $\qquad b_1 \equiv b_{\vec{q}_1}(z_1)$, $b_1^+ \equiv b_{-\vec{q}_1}(z_1)$

$l=4$: (**) $= i^2 \left[D_0(1;2)\, D_0(3;4) + D_0(1;3)\, D_0(2;4) + D_0(1;4)\, D_0(2;3)\right]$

First order : $\quad l = 2$

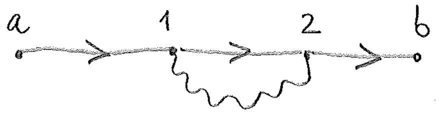
$$\frac{1}{iZ} \; \frac{1}{2}(-i)^2 \int dz_1 \int dz_2 \sum_{k_1 q_1} \sum_{k_2 q_2} M_{q_1} M_{q_2} \quad \times$$

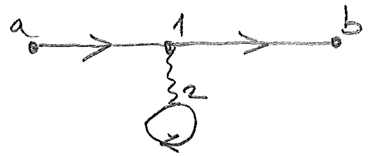
$$\times \; i^5 Z_0 \begin{vmatrix} G_0(a;b) & G_0(a;1) & G_0(a;2) \\ G_0(1;b) & G_0(1;1) & G_0(1;2) \\ G_0(2;b) & G_0(2;1) & G_0(2;2) \end{vmatrix}_- \; i D_0(1;2)$$

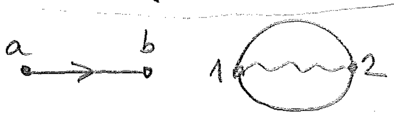
$D_0(1;2)$ enforces $\quad q_1 = -q_2$

6 terms :

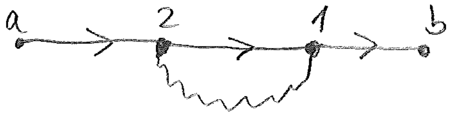
+ $\qquad\qquad$ −



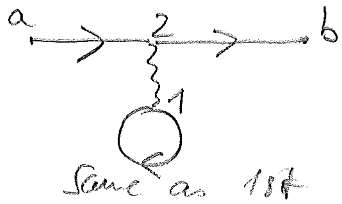




will drop out against denominator (disconnected)

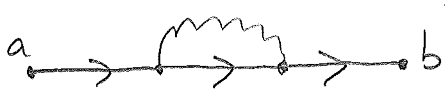


same as 1st with interchanged labels 1,2



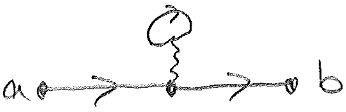
Same as 1st with $1 \leftrightarrow 2$

=> 2 distinct diagrams (multiplicity 2):



"Fock"

[actually : reverse all arrows ...]



"Hartree"

With  reinterpreted as $M_1 D_0(1;2) M_2$

"effective interaction between electrons mediated by phonons"

=> use same diagram rules as before

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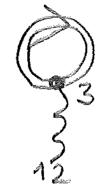
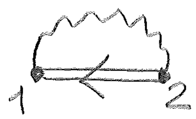
We will use the Hartree-Fock / 1st Born approximation for the derivation of kinetic equations for electron-phonon scattering.

$$\Sigma(1;2) = -i\,\delta(1;2)\int d3\; v(1;3)\, G(3;3^+)$$



$$+\, i\, v(1;2)\, G(1;2^+)$$



$$v(1;2) = M_1\, D_0(1;2)\, M_2$$

(with summation over q indices written out in detail below)

Note: This is the partially self-consistent Born approximation. For nonequilibrium dynamics, this means that we assume that the phonons are not affected by the electronic dynamics and remain at equilibrium. "heat bath approximation".

1.4 The phonon propagator

$$D_0(\vec{q}, z, z') = \frac{1}{i} \left\langle \mathcal{T}\, \hat{X}_{\vec{q}}(z)\, \hat{X}_{-\vec{q}}(z') \right\rangle_0$$

with $\hat{H}_0 = \sum_{\vec{q}} \omega_{\vec{q}}\, b^{\dagger}_{\vec{q}} b_{\vec{q}}$

$$\hat{X}_{\vec{q}} \equiv b_{\vec{q}} + b^{\dagger}_{-\vec{q}}$$

What is the structure of the phonon propagator?

Using $\langle b^{\dagger}_q b_q \rangle_0 = N_B(\omega_q) = \dfrac{1}{e^{\beta\omega_q} - 1}$ we obtain

Contour propagator:

$$D_0(\vec{q}, z, z') = -i\left[N_B(\omega_q) + 1 - \Theta(z, z')\right] e^{i\omega_q(t-t')}$$
$$-i\left[N_B(\omega_q) + \Theta(z, z')\right] e^{-i\omega_q(t-t')}$$

Greater, lesser, retarded:

$$D_0^{>}(\vec{q}, t, t') = -i\left(N_B(\omega_q)\, e^{i\omega_q(t-t')} + \left(N_B(\omega_q)+1\right) e^{-i\omega_q(t-t')}\right)$$

$$D_0^{<}(\vec{q}, t, t') = -i\left(\left(N_B(\omega_q)+1\right) e^{i\omega_q(t-t')} + N_B(\omega_q)\, e^{-i\omega_q(t-t')}\right)$$

$$D_0^{R}(\vec{q}, t, t') = \Theta(t, t')\left(D^{>} - D^{<}\right)$$
$$= +i\,\Theta(t, t')\left[e^{i\omega_q(t-t')} - e^{-i\omega_q(t-t')}\right]$$

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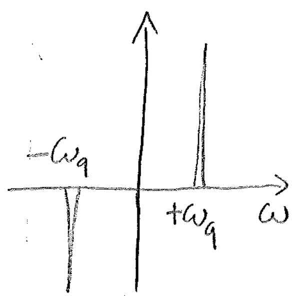
Fourier with "$+i\eta$" convergence factor:

$$D_0^{R}(\vec{q}, \omega) = \left.\frac{1}{\omega - \omega_q + i\eta} - \frac{1}{\omega + \omega_q + i\eta}\right|_{\eta \to 0^+}$$

Spectrum:

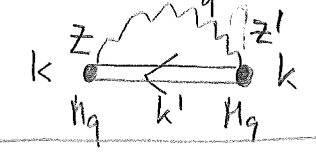
$\frac{1}{\pi} \operatorname{Im} D_0^R$



1.5 The electron-phonon self-energy in 1st Born approximation

In practice we will need here only the Fock diagram, since the instantaneous Hartree term does not contribute to retardation effects, in particular energy dissipation.

We will use

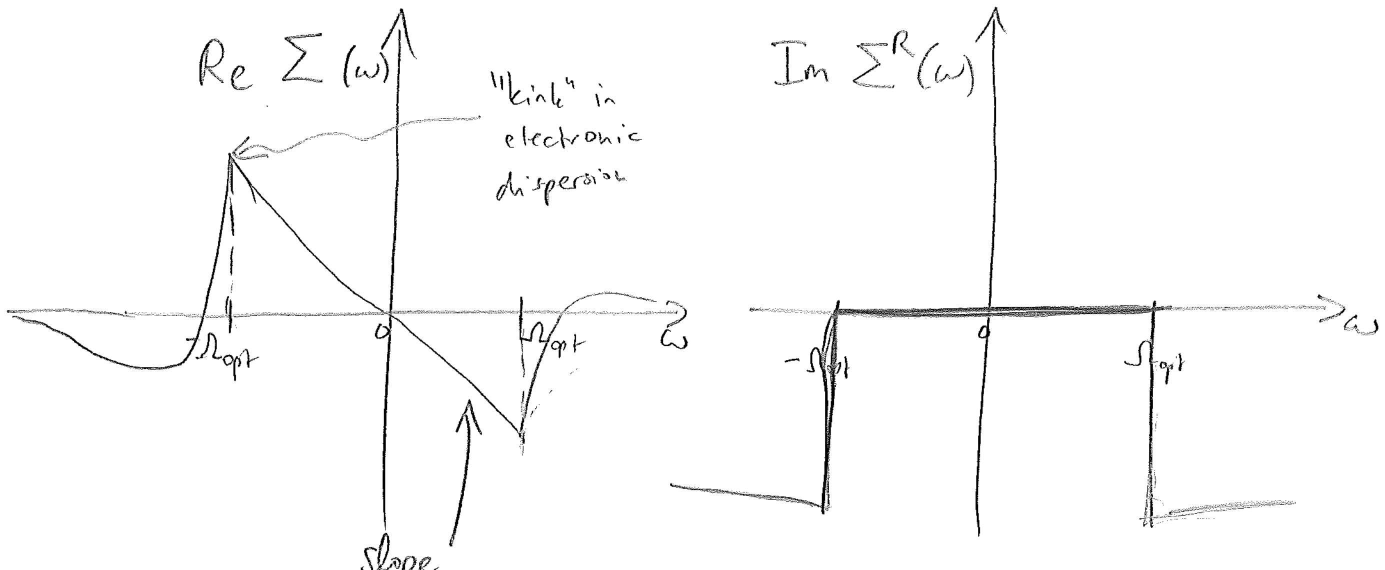


$$\longrightarrow \boxed{\Sigma^{\text{Fock}}_{el\text{-}ph}(z,z') = i \sum_{k'q} |M_q|^2 \, \delta(k-k'-q)\, D_0(q,z,z')\, G(k',z,z')},$$

which is also known as "Migdal approximation" (partially self-consistent).

This self-energy also forms the basis of electron-phonon superconductivity (Gor'kov, Migdal, Eliashberg).

For an optical phonon with little dispersion, $\omega_q \approx \Omega_{opt}\ \forall q$, the retarded self-energy has a particular structure:



$$Z = \frac{1}{1-\frac{\partial \mathrm{Re}\Sigma}{\partial \omega}} \qquad \left.\frac{\partial\, \mathrm{Re}\Sigma}{\partial \omega}\right|_{\omega=0} = 1 - \frac{1}{Z}$$

QP weight

1.6 Boltzmann equation à la Prange & Kadanoff 1964:

$$[E-\varepsilon_k-\mathrm{Re}\,\Sigma, g^>]-[\Sigma^>, \mathrm{Re}\,g]$$

(Eq. 15) $$= -[E-\varepsilon_k-\mathrm{Re}\,\Sigma, g^<]+[\Sigma^<, \mathrm{Re}\,g]$$

$$= -\Sigma^<(\vec{k},E)\,g^>(\vec{k},E)+\Sigma^>(\vec{k},E)\,g^<(\vec{k},E)$$

$$[A,B] \equiv \frac{\partial A}{\partial E}\frac{\partial B}{\partial t}-\frac{\partial A}{\partial t}\frac{\partial B}{\partial E}+\nabla_r A\cdot\nabla_k B-\nabla_k B\cdot\nabla_r A$$

$$\mathrm{Re}\,g \equiv \mathrm{Re}\,g(\vec{k},E;\vec{r},t) = \mathcal{P}\int\frac{dE'}{2\pi}\frac{A(\vec{k},E',\vec{r},t)}{E-E'}$$

$$g(\vec{k},z) = \int\frac{dE'}{2\pi}\frac{g^>(\vec{k},E')+g^<(\vec{k},E')}{z-E'} = \int\frac{dE'}{2\pi}\frac{A(\vec{k},E')}{z-E'}$$

$$g^>(\vec{k},E;\vec{r},t) = \sum_{k'}\int dt'\, e^{iEt'+i\vec{k}'\vec{r}}\left\langle c_{\vec{k}+\frac{1}{2}\vec{k}'}(t+\tfrac{1}{2}t')\, c^{\dagger}_{\vec{k}-\frac{1}{2}\vec{k}'}(t-\tfrac{1}{2}t')\right\rangle$$

Shortcut to kinetic equation:

$\Rightarrow$ Quasiparticle approximation (Eq. 74) (not necessary! not necessarily justified!)

$$g^>(\vec{k},E) = 2\pi\,\delta(\omega-E(\vec{k}))\,Z(\vec{k})\,[1-n(\vec{k})]$$

$$g^<(\vec{k},E) = 2\pi\,\delta(\omega-E(\vec{k}))\,Z(\vec{k})\,n(\vec{k})$$

Then: (Eq. 50) $E(\vec{k}) \equiv Z(\vec{k})\,\varepsilon_k$ renormalized dispersion

$$\rightarrow \boxed{\left[\frac{\partial}{\partial t}+\nabla_k E(\vec{k})\cdot\nabla_r-\nabla_r E(\vec{k})\cdot\nabla_k\right]n(\vec{k}) = -Z(\vec{k})\left[\Sigma^>(\hat{k},E(\vec{k}))\,n(\vec{k})-\Sigma^<(\hat{k},E(\vec{k}))\,[1-n(\vec{k})]\right]}$$

For example consider $\Sigma^<_{(k,t,t')} = i \sum_{k'q} |M_q|^2 \delta(k-k'-q)\, D_0^<(q)\, G^<(k')$

$$D_0^<(q) = -i\left((N_B(\omega_q)+1)\, e^{i\omega_q(t-t')} + N_B(\omega_q)\, e^{-i\omega_q(t-t')}\right)$$

$$G^<(k') = i\, n(k')\, z(k')\, e^{i E(k')(t'-t)}$$

$\Rightarrow$ First term only:

$$\Sigma^{<(1)}_{(k,t,t')} = i \sum_{k'q} |M_q|^2 z(k')\, \delta(k-k'-q)\, n(k')\, (N_B(\omega_q)+1)\, e^{i[\omega_q - E(k')](t-t')}$$

Fourier on time: $\int d(t-t')\, e^{iE(k)(t-t')}\, \Sigma^{<(1)}_{(k,t,t')} = \Sigma^{<(1)}_{(k,E(k))}$

$$\Rightarrow \int_0^\infty dt\, e^{i[\omega_q - E(k') + E(k)]t + i\eta t} = -\frac{1}{i(\omega_q - E(k') + E(k) + i\eta)}$$

$$\int_{-\infty}^0 dt\, e^{i[\omega_q - E(k') + E(k)]t - i\eta t} = +\frac{1}{i(\omega_q - E(k') + E(k) - i\eta)}$$

$$\Rightarrow \boxed{\operatorname{Im} \Sigma^{<(1)}(k, E(k)) = \sum_{k'q} 2\pi\, \delta\left(\omega_q - E(k') + E(k)\right) |M_q|^2 z(k')\, \delta(k-k'-q) \times n(k')\left(N_B(\omega_q)+1\right)}$$

Therefore the terms on the r.h.s. (collision terms) ~~which~~ can be written ~~with r.h.s.~~ [Eq(52)] (~~typo in 2nd eq. ?~~)

$$\boxed{\begin{aligned}
Z(\vec{k})\Sigma^{>}(\vec{k},E(\vec{k})) = 2\pi \sum_{k'q} |M_{\vec{q}}|^2 Z(\vec{k})\, Z(\vec{k}')\, \delta(\vec{k}-\vec{k}'-\vec{q}) \;\times \\
\times [1-n(\vec{k}')] \Big\{ \delta(E(\vec{k})-E(\vec{k}')-\omega_{\vec{q}})\,[N(\vec{q})+1] \\
+\delta(E(\vec{k})-E(\vec{k}')+\omega_{\vec{q}})\, N(\vec{q}) \Big\} \\[1em]
Z(\vec{k})\Sigma^{<}(\vec{k},E(\vec{k})) = 2\pi \sum_{k'q} |M_{\vec{q}}|^2 Z(\vec{k})\, Z(\vec{k}')\, \delta(\vec{k}-\vec{k}'-\vec{q}) \times \\
\times\, n(\vec{k}') \Big\{ \delta(E(\vec{k})-E(\vec{k}')-\omega_{\vec{q}})\, N(\vec{q}) \\
+\delta(E(\vec{k})-E(\vec{k}')+\omega_{\vec{q}})\,[1+N(\vec{q})] \Big\}
\end{aligned}}$$

Now we can get to the form in Allen 1988 (PRL 59, 1460) by setting $Z(\vec{k}) = 1$ (unrenormalized dispersion) and thus $E(\vec{k}) = \varepsilon_k$,

see tutorial

In Allen there is also an equation for the phonon occupation to conserve energy; the phonon dynamics will, however, be neglected – "heat bath approximation" – which can be justified when (a) the laser excites mainly the electrons and (b) the lattice has a large heat capacity.