Application: Single particle in dissipative bath

first consider only single particle:

$$H = \frac{1}{2} P^2 + V(X) \qquad [X, P] = i\hbar$$

path integral: see first chapter by M. Seutef:

$$Z = \frac{1}{Z_0} \operatorname{tr}\left[\rho_0 T_c e^{-i\int_c dt\, H(t)}\right] = \frac{1}{Z_0}\int \mathcal{D}(x)\, e^{iS}$$

$$S = \frac{1}{\hbar}\int_c dt \left[\frac{1}{2}\dot{x}^2 - V(x)\right] \qquad \left\{ \hbar = 1 \text{ in the following, later reinserted} \right.$$

remark: as above $\int \mathcal{D}(x)\, e^{iS} \mathrel{\hat{=}}$ continuum notation of discretized action, with normalization $Z = 1$. In the continuum version, one must remember that there are additional initial / boundary conditions (see below) which would be explicitly provided in the discrete version.

Example: harmonic oscillator:

$$V(x) = \frac{\omega_0^2}{2} x^2 \quad \rightsquigarrow \quad \text{quadratic action}$$

$$S = \frac{1}{2} \int_c dt \left(\dot{x}^2 - \omega_0^2 x^2 \right) \overset{\text{part. integration}}{=}$$

$$= \frac{1}{2} \int_c dt \; x(t) \left[-\partial_t^2 - \omega_0^2 \right] x(t)$$

discrete version has form $\underline{x} = (x_1, \dots, x_N)$

$$S = \frac{1}{2} \sum_{kk'=1}^{N} x_k (D^{-1})_{kk'} x_{k'} \equiv \frac{1}{2} \underline{x}^T (\underline{\underline{D}}^{-1}) \underline{x}$$

with some real, symmetric matrix (D^{-1}).

add source fields $\underline{J} = (J_1, \dots J_N)$

$$Z[J] = \frac{1}{Z_0} \int \mathcal{D}(x) \, e^{iS + i \underline{J}^T \underline{x}} \qquad (1)$$

$$= \sqrt{\frac{(2\pi i)^N}{\det D^{-1}}} \; e^{-\frac{i}{2} \underline{J}^T \underline{\underline{D}} \, \underline{J}} \qquad (2)$$

remark on Gaussian integral below ⊛

$$\Rightarrow \quad \frac{\partial}{\partial J_k}\frac{\partial}{\partial J_{k'}} Z[J]\Big/_{J=0} \overset{(1)}{=} i^2 \langle X_k X_{k'} \rangle_S$$

$$\overset{(2)}{=} -i \; D_{kk'}$$

((1): $\langle \dots \rangle_S = \frac{1}{Z_0}\int \mathcal{D}(X)\, e^{iS} \dots$, (2): using $Z=1$, D symmetric)

$$\Rightarrow \quad \boxed{D_{kk'} = -i \langle X_k X_{k'} \rangle_S} \quad \text{Green's function}$$

continuum notation: $D^{-1} \cdot D = 1$

$$D(t,t') = -i \langle T_c \, X(t) \, X(t') \rangle$$

$$D^{-1}(t,t') = \delta(t,t') \, (-\partial_t^2 - \omega_0^2)$$

$$(-\partial_t^2 - \omega_0^2) \, D(t,t') = \delta(t,t')$$

+ initial condition! (see below)

Keldysh rotation for real fields

$$X_{cl}(t) = \frac{1}{2}\left(X(t_+) + X(t_-)\right)$$

$$X_q(t) = \frac{1}{2}\left(X(t_+) - X(t_-)\right)$$

$$S = \frac{1}{2}\int_c dt \; X(t)\left[-\partial_t^2 - \omega_0^2\right] X(t)$$

$$= \frac{1}{2}\int_{-\infty}^{+\infty} dt \left\{ (X_{cl}+X_q)\left[-\partial_t^2-\omega_0^2\right](X_{cl}+X_q) \right.$$

$$\left. - (X_{cl}-X_q)\left[-\partial_t^2-\omega_0^2\right](X_{cl}-X_q) \right\}$$

$$= \frac{1}{2}\int_{-\infty}^{+\infty} dt \; (X_{cl} \;\; X_q) \underbrace{\begin{pmatrix} 0 & 2(-\partial_t^2-\omega_0^2) \\ 2(-\partial_t^2-\omega_0^2) & 0 \end{pmatrix}}_{\mathcal{D}^{-1}} \begin{pmatrix} X_{cl} \\ X_q \end{pmatrix}$$

$\rightsquigarrow \quad \mathcal{D}^{-1}\mathcal{D} = \mathcal{D}\mathcal{D}^{-1} = 1$ with

$$\mathcal{D} = \begin{pmatrix} -i\langle X_{cl}(t)\, X_{cl}(t')\rangle & -i\langle X_{cl}\, X_q\rangle \\ -i\langle X_q\, X_{cl}\rangle & -i\langle X_q X_q\rangle \end{pmatrix} \equiv \begin{pmatrix} D^K(t,t') & D^{ret}(t,t') \\ D^{adv}(t,t') & 0 \end{pmatrix}$$

one typically writes $\mathcal{D}^{-1} = \begin{pmatrix} 0 & (\mathcal{D}^{-1})^{ret} \\ (\mathcal{D}^{-1})^{adv} & (\mathcal{D}^{-1})^K \end{pmatrix}$,

having again in mind that the equation $\mathcal{D}^{-1}\mathcal{D} = 1$ needs an initial condition. In particular, the equation $(-\partial_t^2 - \omega_0^2)\, D^K(t,t') = 0$ for the Keldysh component is an <u>homogeneous equation</u>!

- initial condition in terms of D^{ret}, D^{adv}, D^{K}:

1) $D^{ret}(t,t') = 0$ for $t' > t$
2) $D^{adv}(t,t') = 0$ for $t > t'$

... same as page 167

3) choice of distribution function $F(\omega)$ in

$$D^{K}(t,t') \xrightarrow[t,t' \to -\infty]{} D^{K}(t-t') \qquad D^{K}(\omega) = \left(D^{ret}(\omega+i0) - D^{a}(\omega-i0)\right) F(\omega).$$

(equlib. $F(\omega) = \coth\left(\frac{\beta\omega}{2}\right)$)

For Boson in equilibrium we get:

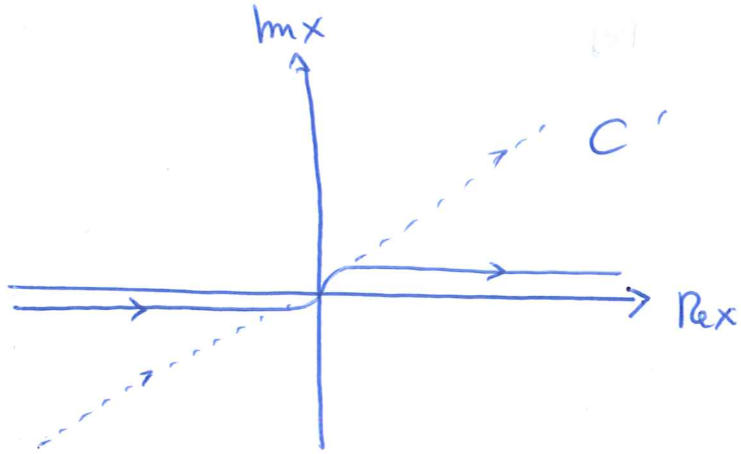
$$\boxed{\begin{aligned} D^{ret}(\omega+i0) &= \frac{1}{2}\,\frac{1}{(\omega+i0)^2-\omega_0^2} \\ D^{adv}(\omega-i0) &= \frac{1}{2}\,\frac{1}{(\omega-i0)^2-\omega_0^2} \\ D^{K}(\omega) &= \left[D^{ret}(\omega+i0) - D^{adv}(\omega-i0)\right]\coth\left(\frac{\beta\omega}{2}\right) \end{aligned}}$$

"fun" task: confirm that this really comes out from a discretized action ...

Remark on Gaussian integral ⊛:

formally non-convergent integral $\int dx\, e^{i \frac{a}{2} x^2}$

$(a>0)$ interpreted with slightly shifted contour



Integration contour may then be deformed to C'

$\hat{=} \quad e^{i\pi/4} x \quad , \quad x \in (-\infty, +\infty)$, and the

integral gives $\quad \int dx\, e^{i \, a/2 \, x^2} = \sqrt{\frac{2\pi i}{a}}$.

Analogous, for real, symmetric, positive $\underline{\underline{A}} \in {}^{(N \times N)}$ we have

$$\int dx\, e^{\frac{i}{2} \underline{x}^T \underline{\underline{A}}\, \underline{x} + i \underline{J}^T \underline{x}} = \sqrt{\frac{(2\pi i)^N}{\det A}}\; e^{-\frac{i}{2} \underline{J}^T \underline{\underline{A}}^{-1} \underline{J}} .$$

particle in contact with environment

$$H = H_{sys} + H_{bath} + H_{sys-bath}$$

$$H_{sys} = \frac{1}{2} P^2 + V(X)$$

set of harm. oscilators

$$H_{bath} = \sum_\alpha \frac{p_\alpha^2}{2} + \frac{\omega_\alpha^2}{2} \varphi_\alpha^2 \qquad [\varphi_\alpha, p_\alpha] = i\hbar$$

$$H_{sys-bath} = \sum_\alpha g_\alpha X \varphi_\alpha$$

$$S = S_{sys} + S_{bath} + S_{sys-bath}$$

$$S_{bath} = \frac{1}{2} \int_{-\infty}^{+\infty} dt\, dt' \sum_\alpha \hat{\varphi}_\alpha^T(t)\, \mathcal{D}_\alpha^{-1}(t,t')\, \hat{\varphi}_\alpha(t') \qquad \hat{\varphi} = \begin{pmatrix} \varphi_{cl} \\ \varphi_q \end{pmatrix}$$

$$S_{sys-bath} = \sum_\alpha \int_c dt\, g_\alpha X \cdot \varphi_\alpha$$

$$= \sum_\alpha g_\alpha \int_{-\infty}^{+\infty} \{(X_{cl} + X_q)(\varphi_{\alpha cl} + \varphi_{\alpha q}) + (X_{cl} - X_q)(\varphi_{cl} - \varphi_q)\}$$

$$= \sum_\alpha \int_{-\infty}^{+\infty} dt\, \hat{X}(t)^T \underbrace{\hat{\sigma}_x}_{\begin{pmatrix} 0 & +1 \\ +1 & 0 \end{pmatrix}} \hat{\varphi}_\alpha(t)$$

effective action for system:

$$Z = \frac{1}{Z_0} \int \mathcal{D}(x)\,\mathcal{D}(\varphi)\, e^{\,i S[x,\varphi]}$$

do integral over φ:

$$\int \mathcal{D}(\varphi)\, e^{\,i(S_{\text{bath}} + S_{\text{sys, bath}})}$$

$$= \prod_\alpha \int \mathcal{D}(\varphi_\alpha)\, e^{\frac{i}{2}\int dt\,dt'\, \hat{\varphi}_\alpha^T \mathcal{D}_\alpha^{-1}(t,t')\hat{\varphi}_\alpha - i\int dt\, g_\alpha \hat{X}^T \hat{\sigma}_x \hat{\varphi}_\alpha}$$

$$\underset{\text{Gauss}}{=} \prod_\alpha Z_\alpha\, e^{-\frac{i}{2}\int dt\,dt'\, g_\alpha \hat{X}(t)^T \hat{\sigma}_x \mathcal{D}_\alpha(t,t')\hat{\sigma}_x g_\alpha \hat{X}(t')}$$

Z_α: normalizati. $\int \mathcal{D}(\varphi_\alpha)\, e^{\,i S_{\text{bath},\alpha}}$

$\mathcal{D}_\alpha(t,t') = \begin{pmatrix} D_\alpha^K(t,t') & D_\alpha^{\text{ret}}(t,t') \\ D_\alpha^{\text{adv}}(t,t') & 0 \end{pmatrix}$

(here we have fixed initial condition!)

$\Longrightarrow$

$$\boxed{\begin{aligned} &Z = \frac{1}{Z_{\text{sys}}}\int \mathcal{D}(x)\, e^{\,i S_{\text{eff}}} \qquad Z_{\text{sys}}: \text{normalization} \\ &S_{\text{eff}} = S_{\text{sys}} + S_{\text{diss}}, \quad S_{\text{diss}} = -\frac{1}{2}\int dt\,dt'\, \hat{X}(t)^T \hat{\Sigma}_{\text{diss}}(t,t')\,\hat{X}(t') \\ &\hat{\Sigma}_{\text{diss}}(t,t') = \sum_\alpha g_\alpha^2\, \hat{\sigma}_x\, \mathcal{D}_\alpha(t,t')\, \hat{\sigma}_x \end{aligned}}$$

where $\hat{\Sigma}_{diss} = \begin{pmatrix} 0 & \Sigma_{diss}^{adv} \\ \Sigma_{diss}^{ret} & \Sigma_{diss}^{K} \end{pmatrix}$

$$\Sigma_{diss}^{adv,ret,K}(t,t') = \sum_{\alpha} g_s^2 \, D_{\alpha}^{adv,ret,K}(t,t')$$

<u>Note:</u> In contrast to the isolated particle, the (q,q) component of $\Sigma_{diss}^{K} \neq 0$, i.e. the equation for D^K is <u>not</u> homogeneous: $\Leftrightarrow$ unique solution!

e.g. $\mathcal{D}^{-1}\mathcal{D} = 1$ for a harm. oscilator $V(x) = \frac{\omega_0^2}{2} x^2$

$$\begin{pmatrix} 0 & -\partial_t^2 - \omega_0^2 - \Sigma_{dis}^{adv} \\ -\partial_t^2 - \omega_0^2 - \Sigma_{diss}^{ret} & -\Sigma_{diss}^{K} \end{pmatrix} \begin{pmatrix} D^K(t,t') & D^{ret}(t,t') \\ D^{adv}(t,t') & 0 \end{pmatrix} = \mathbb{1}$$

$\rightsquigarrow$ Keldysh component:

$$\underbrace{\left(-\partial_t^2 - \omega_0 - \Sigma_{diss}^{ret}\right)}_{(D^R)^{-1}} * D^K - \Sigma_{diss}^{K} * D^{adv} = 0$$

$$\boxed{D^K(t,t') = \int dt_1 dt_2 \, D^{ret}(t,t_1) \, \Sigma_{diss}^{K}(t_1,t_2) \, D^{adv}(t_2,t')}$$

(The initial condition has been fixed by choosing the D_{α}^{K}-component ($\hat{=}$ distribution function) of the bath!)

Ohmic bath

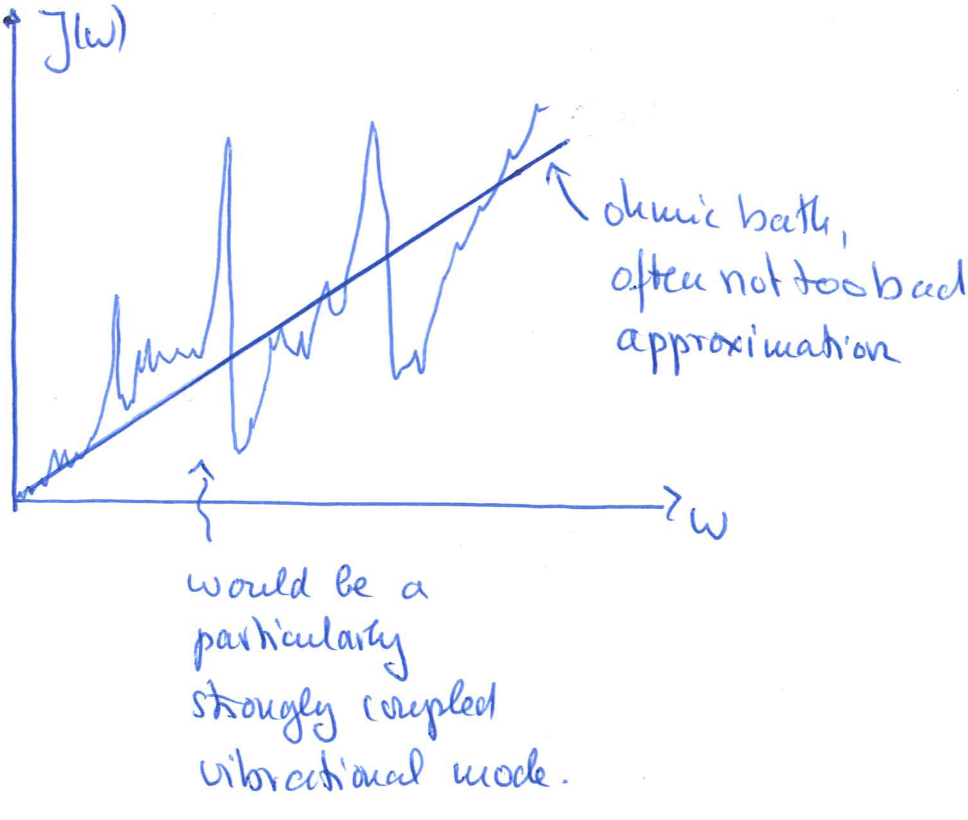
To proceed, we need to specify the properties of the bath. We define the

bath spectral density

$$J(\omega) = \pi \sum_\alpha \frac{g_\alpha^2}{\omega_\alpha} \delta(\omega-\omega_\alpha)$$

$$\Rightarrow \Sigma^{ret}_{diss}(\omega+i0) = \frac{1}{2}\sum_\alpha \frac{g_\alpha^2}{(\omega+i0)^2-\omega^2} = \int_0^\infty \frac{d\omega'}{2\pi}\frac{\omega' J(\omega')}{(\omega+i0)^2-\omega'^2}$$

typical choice: $J(\omega) = 4\gamma\omega$ "Ohmic bath"



$$\Rightarrow \Sigma^r_{diss}(\omega+i0) = 4\gamma \int_0^\infty \frac{d\omega'}{2\pi} \frac{\omega'^2}{(\omega+i0)^2-\omega'^2} = \text{const} - 2i\gamma\omega$$

$$\Sigma^{adv}_{\cancel{diss}}(\omega+i0) = \dots = \text{const} + 2i\gamma\omega$$

$$\Sigma^K_{diss}(\omega) = -4i\gamma\omega \coth\left(\frac{\beta\omega}{2}\right)$$

Back-transformation in time:

$$\Sigma^r_{diss}(t,t') = \Theta(t-t')[\overset{\text{const}+}{\vee} 2\gamma\partial_t]\delta(t-t'-0^+)$$

$$\Sigma^{adv}_{diss}(t,t') = \Theta(t'-t)[\text{const} - 2\gamma\partial_t)\delta(t-t'+0^+)$$

const. will be omitted (corresponds to a redefinition of the potential $V(x) \to V(x) + \text{const}.\, x^2$

$$\Sigma^K_{diss}(t-t') = -4i\gamma\left[(2T+C)\delta(t-t') - \frac{\pi T^2}{\sinh^2(\pi T(t-t'))}\right]$$

[$C = \pi T^2 \int dt \frac{1}{\sinh^2(\pi T t)}$ (so that $\int dt\, \Sigma^K(t) = \Sigma^K(\omega\to 0) = -8i\gamma T$)
infinite constant]

$$S_{diss} = -\frac{1}{2}\int_{-\infty}^{+\infty} dt\,dt'\; \hat{X}(t) \begin{pmatrix} 0 & -2\gamma\partial_t\delta(t-t') \\ 2\gamma\partial_t\delta(t-t') & \Sigma^K(t-t') \end{pmatrix} \begin{pmatrix} X_{cl} \\ X_q \end{pmatrix}$$

$$= -\frac{1}{2}\int_{-\infty}^{+\infty} dt \Big(X_{cl}\big(\overset{\text{part. integration}}{-2\gamma\partial_t}\big) X_q + X_q\, 2\gamma\partial_t X_{cl}\Big) - \frac{i}{2}\int dt\,dt'\; X_q \Sigma^K X_q$$

final action, using

$$S_{sys} = \int_C dt \left[\frac{1}{2}\dot{X}^2 - V(X)\right]$$

$$= \frac{1}{2}\int_{-\infty}^{+\infty} dt \Big(2\dot{X}_{cl}\dot{X}_q + 2\dot{X}_q\dot{X}_{cl} - V(\overbrace{X_{cl}+X_q}^{X(t_+)}) + V(\overbrace{X_{cl}-X_q}^{X(t_-)}) \Big)$$

$$\boxed{\begin{aligned} S_{eff} = &\int_{-\infty}^{+\infty} dt \left\{ -2X_q\left(\ddot{X}_{cl} + \gamma\dot{X}_{cl}\right) - V(X_{cl}+X_q) + V(X_{cl}-X_q)\right\} \\ &+ 2i\gamma \int_{-\infty}^{+\infty} dt \left[2T\, X_q(t)^2 + \frac{\pi T^2}{2}\int_{-\infty}^{+\infty} dt'\; \frac{\left(X_q(t) - X_q(t')\right)^2}{\sinh^2(\pi T(t-t'))}\right] \end{aligned}}$$