Keldysh path integral

goal: path integral for real-time evolution

$$\langle O(t)\rangle = \mathrm{tr}\left(\rho_0\, \mathcal{U}(t,t_0)^\dagger\, O\, \mathcal{U}(t,t_0)\right)$$

ρ_0: initial state $\qquad$ $\mathcal{U}(t,t_0)$: time evolution

(... largely equivalent to imaginary-time path integral introduced in first part of the lecture)

generalized partition function

$$Z = \frac{1}{\mathrm{tr}\,\rho_0}\, \mathrm{tr}\left\{ \rho_0\, T_c\, e^{-i\int d\bar{t}\, H(\bar{t})} \right\}$$

C : contour, see below.

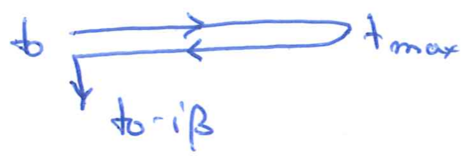
... later: H includes external "source" fields such as $\lambda(t)\, O$, so that $Z[\lambda]$ is the generating function for $\langle O(t)\rangle$.

Note: Keldysh vs. L-shaped contour

1) initial state $\rho \sim e^{-\beta H}$ at $t = t_0$:

$$Z \sim \operatorname{tr}\left\{ T_c \, e^{-i\int_c d\bar{t}\, H(\bar{t})} \right\}$$ for L-shaped

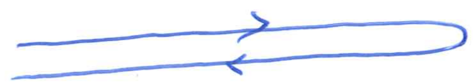
contour $t_0 \to t_{max} \to t_0 \to t_0 - i\beta$



2) initial state $\rho_0 \sim$ <u>non-interacting</u>

density matrix, typically $t_0 \to -\infty$.

$$\rightsquigarrow \quad Z = \operatorname{tr}\left\{ \rho_0 \, T_c \, e^{-i\int_c d\bar{t}\, H(\bar{t})} \right\}$$

$C \,\hat{=}\, t_0$ 

- treating 1) & 2) has many similarities formally. We focus on 2) and just point out differences.
- 2) is more favorable for analytical arguments.
- 2) can describe same situation like 1) if we resort to <u>adiabatic switch-on of interaction</u>

Example: $H = \omega_0 b^\dagger b$, $\tilde{\rho}_0 \sim e^{-\beta\omega_0 b^\dagger b}$

$M=3$: quadratic action.

$$S = \sum_{kk'=1}^{2M} \bar{\Phi}_k A_{kk'} \Phi_k$$

$$i\underline{\underline{A}} = \left(\begin{array}{ccc|ccc} -1 & & & & & \rho(\omega_0) \\ h_- & -1 & & & & \\ & h_- & -1 & & & \\ \hline & & +1 & -1 & & \\ & & & h_+ & -1 & \\ & & & & h_+ & -1 \end{array}\right)$$

$\rho(\omega_\alpha) = e^{-\beta\omega_\alpha}$

$h_\pm = 1 \pm i\Delta t\omega_0$

Green's functions:

analogous proof as for Z

$$G(t_a, t_b) = -i\langle T_c\, b(t_a)\, b^\dagger(t_b)\rangle$$

$$= -i\frac{1}{Z_0}\int \mathcal{D}[\bar{\Phi}\Phi]\left\{e^{iS[\bar{\Phi},\Phi]}\ \Phi_a \bar{\Phi}_b\right\}$$

omitting orbital indices for simplicity.

$\rightsquigarrow$ introduce source fields $\{J_a, \bar{J}_a\}$

$$\Rightarrow Z[\bar{J},J] = \frac{1}{Z_0}\int \mathcal{D}[\bar{\Phi}\Phi]\, e^{iS[\bar{\Phi}\Phi]}\ e^{\sum_{a=1}^{2M}(\bar{J}_a\Phi_a + \bar{\Phi}_a J_a)}$$

$$G(t_a, t_b) = -i\frac{\partial}{\partial \bar{J}_a}\frac{\partial}{\partial J_b} Z[\bar{J},J]\Big|_{J=0}$$

proof: $f(\alpha) = \langle \phi | e^{\alpha b^+ b} | \phi' \rangle$

$$\partial_\alpha f(\alpha) = \langle \phi | b^+ b \, e^{\alpha b^+ b} | \phi' \rangle$$

$$= \bar{\phi} \langle \phi | b \, e^{\alpha b^+ b} | \phi' \rangle = \dots$$

use $b (b^+ b)^n = [(b^+ b)^n + 1] b$, proven by acting on $|n\rangle$.
and thus $b e^{\alpha b^+ b} = e^{(b^+ b + 1)\alpha} b$

$$\dots = \bar{\phi} \phi' e^{\alpha} f(\alpha)$$

integrate
$\left. f(0) = e^{\bar{\phi}\phi'} \right\} \Rightarrow \quad f(\alpha) = e^{e^{\alpha} \bar{\phi}\phi'}$

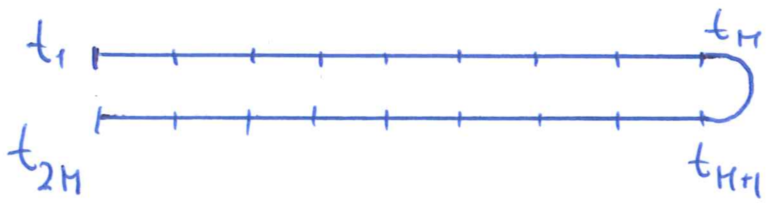
<u>final result</u>

$$\boxed{\begin{aligned} & Z = \frac{1}{Z_0} \int \mathcal{D}[\bar{\phi}\phi] \, e^{iS[\bar{\phi}\phi]} \qquad \text{mind the } i\,! \\ & \qquad\qquad\qquad\qquad\qquad\qquad Z_0 = \mathrm{tr}\, \hat{\rho}_0 \\ & \mathcal{D}[\bar{\phi},\phi] = \int \prod_{j=1}^{2M} \prod_{\alpha} d(\bar{\phi}_{\alpha j} \phi_{\alpha j}) \\ & S[\bar{\phi}\phi] = \sum_{j=2}^{2M} \Delta t_j \left\{ \sum_{\alpha} \bar{\phi}_{j\alpha} \, i \left(\frac{\phi_{j\alpha} - \phi_{j-1\alpha}}{\Delta t_j} \right) - H(\bar{\phi}_j, \phi_{j-1}) \right\} \\ & \qquad + i \sum_{\alpha} \bar{\phi}_{1\alpha} (\phi_{1\alpha} - \underbrace{\rho(\omega_\alpha)}_{e^{-\beta\omega_\alpha}} \phi_{2M\alpha}) \end{aligned}}$$

path integral for Z:

analogous derivation like in imag. time:

Step 1): time-discretization



$$\Delta t_j = t_j - t_{j-1} \qquad |\Delta t_j| \sim 1/M \quad \text{for } M \to \infty$$

Step 2) use complete set of states to represent propagator on timeslice:

Here: written for bosonic coherent states, Fermions below.

full set of creation / annihilation op: b_α, b_α^+

$\longrightarrow$ coherent states $|\underline{\phi}\rangle \equiv |\{\phi_\alpha\}\rangle$ with

$$b_\alpha |\underline{\phi}\rangle = \phi_\alpha |\underline{\phi}\rangle \qquad \langle\phi| b_\alpha^+ = \langle\phi| \bar{\phi}_\alpha$$

$$\langle\phi|\phi'\rangle = \exp\left(\sum_\alpha \bar{\phi}_\alpha \phi_\alpha'\right)$$

$$\mathbb{1} = \int \prod_\alpha \underbrace{d(\bar{\phi}_\alpha \phi_\alpha)}_{\frac{d\,\mathrm{Re}\,\phi\, d\,\mathrm{Im}\,\phi}{\pi}} e^{-\sum_\alpha \bar{\phi}_\alpha \phi_\alpha} |\underline{\phi}\rangle\langle\underline{\phi}|$$

$$\mathrm{tr}(\dots) = \int d(\bar\phi\phi)\, e^{-\bar\phi\phi} \langle\phi|\dots|\phi\rangle$$

$\rightsquigarrow$ insert $\mathbb{1}$ at each time slice

Step 3: normal ordered Hamiltonian (b^+ to the left)

$$\langle\phi|H(b^+,b)|\phi'\rangle = \langle\phi|\phi'\rangle\, H(\phi,\phi')$$

$$\langle\phi_j|\, e^{i\Delta t(H_1 + H_2 \dots)} \,|\phi_{j-1}\rangle$$

typically non-commuting parts of H

$$\simeq \langle\phi_j|\phi_{j-1}\rangle\, e^{i\Delta t[H_1(\bar\phi_j,\phi_{j+1}) + H_2(\dots) \dots]} + \mathcal{O}(1/M^2)$$

neglecting commutators in exponent $\sim \Delta t^2 \sim \frac{1}{M^2}$

Matrix-element of $\hat\rho_0$:

non-interacting state $\Rightarrow$ quadratic operator

$\rho_0 \sim e^{-\sum_{\alpha\beta} b^+_\alpha \rho_{\alpha\beta} b_\beta}$ $\Rightarrow$ can assume

that $|\alpha\rangle$ diagonalizes initial state $\rho = \prod_\alpha e^{-\beta\omega_\alpha b^+_\alpha b_\alpha}$

use $\langle\phi|e^{-\beta\omega_\alpha b^+ b}|\phi'\rangle = e^{e^{-\beta\omega_\alpha}\bar\phi\phi'}$

<u>Note:</u> For the Keldysh action we have

$$Z[J,\bar{J}] = 1 \quad \text{at } J = \bar{J} = 0$$

$\underbrace{\mathrm{tr}\left[\rho_0 \, T_c \, e^{-i\int d\tilde{t}\, H(\tilde{t})}\right]}_{=1}$

by construction.

By contrast, for Matsubara (imag. time) or L-shaped contour, we would have

$$G = -i \frac{1}{Z} \int \mathcal{D}(\bar{\phi}\phi)\, e^{iS}\, \phi \bar{\phi} = -i \frac{\partial}{\partial J} \frac{\partial}{\partial \bar{J}} \ln Z[J,\bar{J}]$$

Noninteracting Green's function

For a quadratic action $S = \sum_{kk'} \bar{\phi}_k A_{kk'} \phi_{k'}$,

$Z[\bar{J},J]$ can be computed by Gaussian integral,

using

$$Z[J\bar{J}] = \int \mathcal{D}(\bar{\phi}\phi)\, e^{-\sum_{kk'} \bar{\phi}_k (-i A_{kk'}) \phi_k + \sum_k (\bar{J}_k \phi_k + \bar{\phi}_k J_k)} / Z_0$$

$$= \frac{1}{Z_0 \det(-i\underline{A})} \exp\left(\sum_{kk'} \bar{J}_k (i A^{-1})_{kk'} J_{k'} \right)$$

provided that $-i\underline{A}$ <u>has positive hermitian part</u>. see below

(see Gaussian integrals in first part of lecture) using the normalization,

$$\frac{1}{Z_0}\int \mathcal{D}[\bar{\Phi}\phi] = \frac{1}{Z_0 \det(-i\underline{\underline{A}})} = 1$$

and taking derivatives, we get.

$$\boxed{G(t_a, t_b) = -i \quad i\,(A^{-1})_{ab} = (A^{-1})_{ab}}$$

(in turn, the non-interacting action may be written as

$$\boxed{S = \sum_{kk'} \bar{\Phi}_k \,(G^{-1})_{kk'}\, \phi_k}$$

<u>continuum notation:</u>

$$S = \sum_{kk'} \bar{\Phi}_k\,(G^{-1})_{kk'}\,\phi_{k'} = \sum_{kk'} \bar{\Phi}_k\,\Delta t_k\,\underbrace{M_{kk'}}_{\frac{1}{\Delta t_k}(G^{-1})_{kk'}\frac{1}{\Delta t_{k'}}}\,\Delta t_{k'}\,\phi_{k'}$$

where $M(t,t') = G^{-1}(t,t') = [i\partial_t - h(t)]\,\delta(t,t')$

$M_{kk'} \equiv M(t_k, t_{k'})$

$$S = \int dt_1 dt_2\, \bar{\Phi}(t_1)\, G^{-1}(t_1 t_2)\,\phi(t_2)$$

$$= \underbrace{\int dt_1\, \bar{\phi}(t_1)\,[i\partial_{t_1} - h(t_1)]\,\phi(t_1)}_{\text{cont. version of } \sum_k \left(\bar{\Phi}_k\, i\,\frac{\phi_k - \phi_{k-1}}{\Delta t_k} - h_k \bar{\Phi}_k \phi_{k-1}\right)\Delta t_k}$$

OK: check $G^{-1} \cdot G = 1$:

discrete: $$\sum_{e} \underbrace{(G^{-1})_{ke}} G_{ek'} = \delta_{kk'}$$

$$\Delta t_k \left(i \frac{\delta_{ke} - \delta_{e,k-1}}{\Delta t_k} - h(t_k)\, \delta_{k-1,e} \right)$$

$$\rightsquigarrow \quad \left(i \frac{\delta_{ke} - \delta_{e,k-1}}{\Delta t_k} - h(t_k)\, \delta_{e,k-1} \right) G_{e,k'} = \frac{\delta_{kk'}}{\Delta t_k}$$

$$\underset{\text{cont.}}{\rightsquigarrow} \quad \left[i\, \partial_t - h(t) \right] G(t,t') = \delta(t,t')$$

$\Rightarrow$ same equation of motion as derived before!

<u>Remark:</u>

discrete matrix $(G^{-1})_{ij}$ has definite inverse, but equation of motion $[i\partial_t - h(t)]G = \delta(t,t')$ needs initial or boundary condition $\Rightarrow$ where is i.c. in the discrete version ?? Correct initial condition in the discrete case ensured by the "harmless" single element $\rho(\omega)$ in the upper right corner! Also, positive definiteness of $-i(G^{-1})$, needed for Gaussian integral ensured by this element. Detailed discussion of this issue in Kamenev, Chapter 2.

Keldysh rotation

to deal with the two-branch contour, we introduce the parametrization:

$$\boxed{\begin{aligned} \phi_{cl}(t) &= \frac{1}{\sqrt{2}}\left(\phi(t_+) + \phi(t_-)\right) \\ \phi_q(t) &= \frac{1}{\sqrt{2}}\left(\phi(t_+) - \phi(t_-)\right) \end{aligned}}$$

$t_\pm$: upper / lower contour

cl: "classical", q: "quantum": meaning clear below

(cl, q) Green's functions:

e.g. $-i\langle \phi_{cl}(t)\,\phi_q(t')\rangle \equiv -i\int \mathcal{D}(\bar\phi\phi)\, e^{iS}\, \phi_{cl}(t)\,\phi_q(t')$

$= \frac{1}{2}\left[G(t_+,t'_+) + G(t_-,t'_+) - G(t_+,t'_-) - G(t_-,t'_-)\right]$

$= G^{ret}(t,t') \equiv G_{cl,q}(t,t')$

analogous: check!

$G_{q,cl}(t,t') = G^{adv}(t,t')$

$G_{cl,cl}(t,t') = G^{K}(t,t')$

$G_{qq}(t,t') = 0$

- the (cl,q) representation nicely clarifies the causal structure of the Keldysh formalism:

Consider a Hamiltonian with a particle creation / annihilation field:

$$H \to H_J = H_0 + (\bar{J}(t)\, b + h.c.)$$

$$i S_J = i S_0 - i \int_c dt \left(\bar{J}(t)\, \phi(t) + h.c. \right)$$

here physical field takes same value on upper / lower branch
$J(t_+) = J(t_-) = J(t)$

$$= i S_0 - i \int_{-\infty}^{+\infty} dt \left(\bar{J}(t)\, \phi(t_+) - \bar{J}(t)\, \phi(t_-) + h.c. \right)$$

$$= i S_0 - i\sqrt{2} \int_{-\infty}^{+\infty} dt \left(\bar{J}(t)\, \phi_q(t) + h.c. \right)$$

- response (using notation $\langle \cdots \rangle_J = \int \mathcal{D}(\bar{\phi}\phi)\, e^{i S_J}$

$$\frac{\delta}{\delta J(t)} \langle O(t_1) \cdots O(t_n) \rangle_J = -i\sqrt{2} \langle O(t_1) \cdots O(t_n)\, \phi_q(t) \rangle_J$$

physical response must vanish for $t > t_1, \dots t_n$

$$\Rightarrow \quad G^{ret}(t,t') = G^{cl,q} = -i \langle \phi_{cl}(t)\, \phi_q(t') \rangle = 0 \quad \text{for } t' > t$$

$$G^{adv}(t,t') = G^{q,cl} = -i \langle \phi_q\, \phi_{cl} \rangle = 0 \quad \text{for } t' < t$$

$$G^{qq}(t,t') = 0 \quad \text{for all times } t, t'$$

<u>non-interacting action in cl,q-notation:</u>

$$\phi(t_\pm) = \frac{\phi_{cl} \pm \phi_q}{\sqrt{2}}$$

$$S = \int_C dt \; \bar{\phi}(t)\,(i\partial_t - h)\,\phi(t)$$

$$= \int_{-\infty}^{+\infty} dt \left[\bar{\phi}(t_+)\,(i\partial_t - h)\,\phi(t_+) - \bar{\phi}(t_-)\,(i\partial_t - h)\,\phi(t_-) \right] =$$

$$= \ldots \int_{-\infty}^{+\infty} dt \; (\bar{\phi}_{cl} \;\; \bar{\phi}_q) \underbrace{\begin{pmatrix} 0 & i\partial_t - h(t) \\ i\partial_t - h(t) & 0 \end{pmatrix}}_{\#} \begin{pmatrix} \phi_{cl} \\ \phi_q \end{pmatrix}$$

as above, naive Gaussian integration would imply that the operator (#) is the inverse of

$$\begin{pmatrix} -i\langle \phi^{cl}(t)\,\phi^{cl}(t')\rangle & -i\langle \phi^{cl}(t)\,\phi^{q}(t')\rangle \\ -i\langle \phi^{q}(t)\,\phi^{cl}(t')\rangle & -i\langle \phi^{q}(t)\,\phi^{q}(t')\rangle \end{pmatrix} = \hat{G},$$

i.e. the equation of motion

$$\begin{pmatrix} 0 & i\partial_t - h \\ i\partial_t - h & 0 \end{pmatrix} \begin{pmatrix} G^K(t,t') & G^R(t,t') \\ G^{adv}(t,t') & 0 \end{pmatrix} = \begin{pmatrix} \delta(t-t') & 0 \\ 0 & \delta(t-t') \end{pmatrix}$$

however, equation of motion ~~is~~ only unique if <u>initial condition</u> is specified ...

↑ in particular, equation for G^K is homogeneous!

1) $D^r(t,t') \stackrel{!}{=} 0$ for $t<t'$

2) $D^{adv}(t,t') \stackrel{!}{=} 0$ for $t>t'$

3) choice of initial distribution function F

$$G^K(t,t') \xrightarrow[t,t'\to-\infty]{} G^K_{eq}(t-t') \quad \text{with}$$

$$G^K_{eq}(\omega) = F(\omega)\left[G^{ret}(\omega+i0) - G^{adv}(\omega-i0)\right]$$

$$F(\omega) = \coth\frac{\beta\omega}{2}$$

[check ... that this follows from the discrete action?]

⇝ <u>One often writes</u>

$$G^{-1} = \begin{pmatrix} 0 & (G^{-1})^{adv} \\ (G^{-1})^{ret} & (G^{-1})^K \end{pmatrix}$$

$$(G^{ret})^{-1} = (G^{-1})^{adv} = \delta(t-t')\,(i\partial_t - h)$$

and $(G^K)^{-1}$ is an infinitesimal that fixes the initial condition. This is also motivated because as soon as there is interaction or a coupling to a bath, $(G^{-1})^K$ is replaced by some finite $\Sigma^K(t,t')$, and then the inverse is already unique. (see next chapter!)