Analysis of the collision term

specific form depends on functional form

$\underline{\underline{\Sigma}}[\underline{G}]$. Here will first consider el.-el.

interaction

$$H_{int} = \frac{1}{2} \sum_{\sigma\sigma'} \int d^3r \; \psi_\sigma^+(\bar{r}) \; \psi_{\sigma'}^+(\bar{r}') \, V(\bar{r}-\bar{r}') \, \psi_{\sigma'}(\bar{r}') \, \psi_\sigma(\bar{r})$$

[Later: electron-phonon interaction $\Rightarrow$ relaxation of laser excited electron distributions in the solid $\Rightarrow$ see following lectures by M. Sentef.]

- For simplicity, we will also assume short-range interaction $V(\bar{r}-\bar{r}) = U\,\delta(\bar{r}-\bar{r}')$

$\rightsquigarrow$ $$H_{int} = U \int d^3r \; \psi_\uparrow^+(\vec{r})\psi_\uparrow(\vec{r}) \; \psi_\downarrow^+(\vec{r})\psi_\downarrow(\vec{r})$$

motivation:

⊛ good for systems like neutral gases with only van der Waals or contact interaction ($\Rightarrow$ ^{3}He, "cold atoms")

⊛ <u>in solids</u>: Coulomb interactions short range due to "<u>screening</u>". If we take H_{int} with long-range Coulomb, should also include terms in diagrammatic expansion which describe screening $\Rightarrow$ "<u>RPA</u>"

For kinetic equation based on RPA self energies, see Kamenev, Chapter 9.

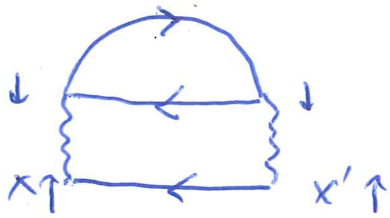
$\rightsquigarrow$ very similar final expressions and analogous steps in derivation as for the simpler short range interaction discussed here.

• <u>Self-energy</u>: (see also exercise 5!)

arguments $(x, \underset{\uparrow,\downarrow}{\sigma})$, but diagonal in spin:

$$\Sigma(x,\sigma; x'\sigma') \equiv \Sigma_{\sigma'}(x,x') \equiv \Sigma(x,x')$$

$\Sigma_\uparrow(x,x') \equiv$



$=$

$$= \underbrace{i^2(-1)}_{\text{Loop}} U^2\, G_\uparrow(x,x')\, G_\downarrow(x,x')\, G_\downarrow(x',x)$$

Note: here: self-consistent perturbation theory (skeleton expansion). Comments on using bare perturbation expansion ($G \leftrightarrow G_0$) see below.

Explicit expressions for Σ^K, Σ^R

use:
$$\Sigma^K = \Sigma^> + \Sigma^<$$
$$\Sigma^R = \Theta(t-t')\left(\Sigma^> - \Sigma^<\right)$$

$$\Sigma^>(x,x') = \Sigma(x_-, x'_+)$$

$x_\pm$: t on upper / lower branch of $\mathcal{C}$

$$= U^2\, G(x_-, x'_+)\, G(x_-, x'_+)\, G(x'_+, x_-)$$

$$= U^2\, G^>(x,x')\, G^>(x,x')\, G^<(x',x)$$

analog:

$$\Sigma^<(x,x') = U^2\, G^<(x,x')\, G^<(x,x')\, G^>(x',x)$$

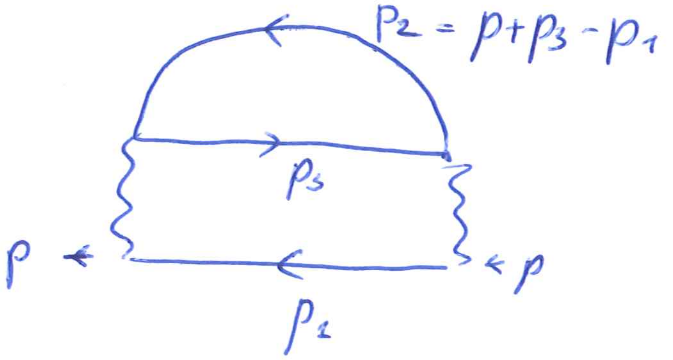
Wigner transform of $\Sigma^{\gtrless}$

$$\Sigma^{>}(x,x') = \sum_{p_1 p_2 p_3} \tilde{G}^{>}\left(\tfrac{x+x'}{2}, p_1\right) \tilde{G}^{>}\left(\tfrac{x+x'}{2}, p_2\right) \tilde{G}^{<}\left(\tfrac{x+x'}{2}, p_3\right)$$

$$\exp\left\{ip_1(x-x') + ip_2(x-x') + ip_3(x'-x)\right\}$$

$\Longrightarrow$

$$\tilde{\Sigma}^{>}(x_{av}, p) = \sum_{p_1 p_2 p_3} \tilde{G}^{>}(x_{av}, p_1)\, \tilde{G}^{>}(x_{av}, p_2)\, \tilde{G}^{<}(x_{av}, p_3) \times \underbrace{\delta_{p+p_3,\, p_1+p_2}}_{\text{energy \& momentum conservation in rel. time variables}}$$



- analogous for $\tilde{\Sigma}^{<}$.

express $G^{\gtrless}$ throug ansatz for G^K:

use
$$G^R - G^A = G^> - G^<$$
$$G^K = G^> + G^<$$

$$\Rightarrow \quad G^> = \frac{1}{2}\left(G^K + G^R - G^A\right)$$
$$\underset{\text{ansatz}}{=} \frac{1}{2}\left(G^R * F - F * G^A + G^R - G^A\right)$$

$\rightsquigarrow$ Wigner transform + gradient expansion to lowest (0th) order*:

$$\tilde{G}^>(x_{av}, p) = \left[G^R(x_{av}, p) - G^A(x_{av}, p)\right] \frac{F(x_{av}, p)+1}{2}$$

$$G^<(x_{av}, p) \overset{\text{analog}}{=} \left[G^R(x_{av}, p) - G^A(x_{av}, p)\right] \frac{F(x_{av}, p)-1}{2}$$

below we use the "parametrization"

$$(F+1)/2 = 1-f \quad , \quad (F-1)/2 = -f$$

(in equilibrium, when $F = \tanh\frac{\beta\omega}{2}$, $f = \frac{1}{e^{\beta\omega}+1}$)

* Why 0th order gradient expansion ??

in kinetic equation: gradient expansion was applied to integral $\Sigma * G$, gradient corrections $(\partial_\omega \Sigma, \partial_k \Sigma, \dots)$ have already been included in kinetic term, see page 136.

Furthermore, we [redacted] use

$$\left[G^R(x_{av}, p) - \underbrace{G^A(x_{av}, p)}_{G^R(x_{av},\, p)^*}\right] = 2i\,\mathrm{Im}\, G^R(x_{av}, p)$$

$$= -2\pi i \; A(x_{av}, p)$$

so that we have $G^> = -2\pi i A (1-f)$, $G^< = 2\pi i A f$

note: looks like in equilibrium, but $f = f(\cancel{x}, t, k, \omega)$!

$$\boxed{\begin{aligned} \Rightarrow \tilde{\Sigma}^>(x, p) &= U^2 \sum_{p_1 p_2 p_3} A(x_{av}\, p_1)\, A(x_{av}, p_2)\, A(x_{av}; p_3)\, \times \\ &\times \delta_{p+p_3,\, p_1+p_2}\, (2\pi)^3 (-i)\, \bar{f}(x_{av}, p_1)\, \bar{f}(x_{av}, p_2)\, f(x, p_3) \\ \Sigma^<(x_{av}, p) &= U^2 \sum_{p_1 p_2 p_3} A A A\, \delta_{p_1+p_2,\, p+p_3}\, (2\pi)^3 (+i)\, f_1 f_2 \bar{f}_3 \\ &\qquad \bar{f} = 1 - f \end{aligned}}$$

<u>collision term</u>

$$i\tilde{\Sigma}^K + \tilde{F}\,\overbrace{2\,\mathrm{Im}\,\tilde{\Sigma}^R}^{-i(\Sigma^> - \Sigma^<)}$$

$$= i\left[\tilde{\Sigma}^> + \tilde{\Sigma}^< \mp \tilde{F}\left(\tilde{\Sigma}^> - \tilde{\Sigma}^<\right)\right]$$

$$= i\left[\tilde{\Sigma}^>(1 \mp \tilde{F}) + \tilde{\Sigma}^<(1 + \tilde{F})\right]$$

$$= 2i\left[\tilde{\Sigma}^> f + \tilde{\Sigma}^< \bar{f}\right]$$

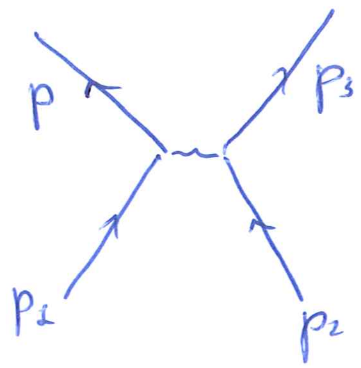
$$= 2(2\pi)^3 U^2 \sum_{p_1 p_2 p_3} \delta_{p+p_3,\,p_1+p_2}\, A(x_{av}, p_1)\, A(x_{av}, p_2)\, A(x, p_3)$$

$$\Big\{ f(x_{av}, p)\, f(x_{av}, p_3)\, \bar{f}(x_{av}, p_1)\, \bar{f}(x_{av}, p_2)$$

$$- \bar{f}(x_{av}, p)\, \bar{f}(x_{av}, p_3)\, f(x_{av}, p_1)\, f(x_{av}, p_2) \Big\}$$

<u>interpretation:</u>

$f_p f_{p_3} \bar{f}_{p_1} \bar{f}_{p_2}$: "gain" scattering into p, under energy and momentum conservation



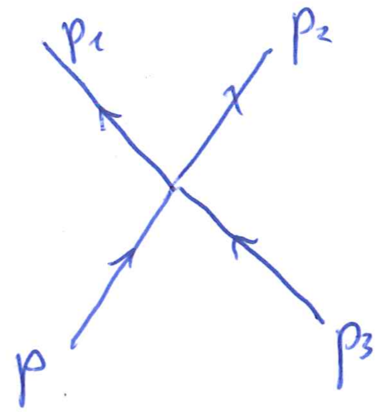
$f, \bar{f}$: phase-space factors

$p_1\ p_2$ must be occupied, $f \neq 0$

p, p_3 unoccupied $1-f \neq 0$

<u>analogous:</u>

$\bar{f}_p \bar{f}_3 f_{p_1} f_{p_2}$: loss of occupation at p.



<u>but still</u>:
f not occupation of single particle levels, but <u>occupation function</u> $f(x, \vec{k}, \omega)$

kinetic equation in this form also called <u>Quantum Boltzmann equation</u>, with distribution that still depends on energy ω <u>and</u> time:

Steady state solutions of the QBE

assume all time-dependent perturbations are over after some time:

$V(x) = V(\vec{r})$ for $t > t_0$

$\rightsquigarrow$ possible t_{av}-independent solutions of QBE ?

$$\left[(1-\partial_\omega \mathrm{Re}\Sigma)\partial_t + \cancel{(\partial_t V)}\partial_\omega + \vec{v}_k \cdot \vec{\nabla}_r - (\nabla_r V)\vec{\nabla}_k \right] \tilde{F} = I^{coll}[\tilde{F}]$$

1) left-hand side solved by <u>any</u> function

$$\underline{\underline{\tilde{F}(\vec{r}, t, \vec{k}, \omega) = F(\omega)}}$$

$\Rightarrow$ in <u>collisionless approximation</u> ($I^{coll} = 0$) $\Rightarrow$ ∞ many steady states ?, not necessarily relaxation to equilibrium form $F(\omega) = \tanh \frac{\beta\omega}{2}$

<u>Interpretation</u>: collisionless $\Rightarrow$ only $\mathrm{Re}\Sigma \neq 0$

$\Leftrightarrow$ Dynamics of independent particles with some quadratic $H = \sum_{\alpha\beta} c_\alpha^\dagger h_{\alpha\beta} c_\beta$

<u>Note</u>: collisionless dynamics can be non-trivial because of the possible non-linear feedback of the distribution on $\mathrm{Re}\,\Sigma^R$.

Here I illustrate the collisionless relaxation to a "non-thermal" distribution function $F(\omega) \neq \tanh\frac{\beta\omega}{2}$ for non-interacting particles.

<u>assume</u>: • after some perturbation, the Hamiltonian is given by $H = \sum_\alpha \varepsilon_\alpha c_\alpha^+ c_\alpha$ for $t > t_0$, $\alpha \hat{=}$ some single particle basis.

$$G^>\left(X_{av} + \frac{X_{rel}}{2}, X_{av} - \frac{X_{rel}}{2}\right) =$$

use $\psi(\vec{r})^+ = \sum_\alpha c_\alpha^+ \times \langle\alpha|\vec{r}\rangle$

$$= -i \left\langle \psi\left(X_{av} + \frac{X_{rel}}{2}\right) \psi\left(X_{av} - \frac{X_{rel}}{2}\right)^+ \right\rangle =$$

$$= -i \sum_{\alpha\alpha'} \left\langle c_\alpha\left(t_{av} + \frac{t_{rel}}{2}\right) c_{\alpha'}\left(t_{av} - \frac{t_{rel}}{2}\right)^+ \right\rangle \langle\vec{r}|\alpha\rangle\langle\alpha'|\vec{r}'\rangle$$

$$= -i \sum_{\alpha\alpha'} \langle c_\alpha c_{\alpha'}^+ \rangle_{t=t_0} e^{-i(t_{av}-t_0)(\varepsilon_\alpha - \varepsilon_{\alpha'})} \times \langle\vec{r}|\alpha\rangle\langle\alpha'|\vec{r}'\rangle \times e^{-\frac{i}{2} t_{rel}(\varepsilon_\alpha + \varepsilon_{\alpha'})}$$

$c_\alpha(t) = c_\alpha e^{-i\varepsilon_\alpha(t-t_0)}$

$t_{av} \to \infty \Rightarrow$ rapidly oscillating terms $e^{-i t_{av}(\varepsilon_\alpha - \varepsilon_{\alpha'})}$

cancel apart from $\varepsilon_\alpha = \varepsilon_{\alpha'}$ ("dephasing")

$\Rightarrow$ G becomes stationary t_{av}-independent

$$G^{\gtrless}(x, x') \xrightarrow[\substack{t_{av} \to \infty \\ t_{rel}\ \text{fix}}]{} G_\infty^{\gtrless}(\vec{r}, \vec{r}\,', t_{av})$$

(assume non-degenerate spectrum (e.g. degen. lifted by small disorder) $\Rightarrow$ only $\alpha = \alpha'$ terms

$$G_\infty^{<}(t_{rel}) = \sum_\alpha \langle \vec{r}|\alpha\rangle\langle\alpha|\vec{r}\,'\rangle e^{-i t_{rel} \varepsilon_\alpha} \langle c_\alpha^\dagger c_\alpha\rangle_{t=t_0}$$

... handwaving argument ...

$\hookrightarrow$ assume $h(\varepsilon_\alpha) \equiv \langle c_\alpha^\dagger c_\alpha\rangle_{t=t_0}$ is smooth function of ε in thermodynamic limit (i.e. replace $h(\varepsilon_\alpha)$ by average over ~~small~~ α in small ε_α interval)

problematic case: disorder with localized states ??

$\Rightarrow$ check!

$$G_\infty^{<}(t_{rel}) = \int d\omega\, e^{-i\omega t_{rel}} A_{rr'}(\omega)\, h(\omega)$$

$$A_{rr'}(\omega) = -\frac{1}{\pi} \operatorname{Im} G^R = \sum_\alpha \delta(\omega - \varepsilon_\alpha) \langle r|\alpha\rangle\langle\alpha|r'\rangle$$

$\leadsto \quad f(t_{av}, \vec{r}_{av}, \vec{k}, \omega) \xrightarrow[t_{av}\to\infty]{} h(\omega)$

$h(\omega)$: depends on <u>initial</u> ($t=t_0$) occupation of single particle levels, can have any form $0 \leq h \leq 1$

2) <u>time-independent solutions in presence of collision term</u>

$$\underbrace{\partial_{t_{av}} \tilde{F} + \{\ldots\} \tilde{F}}_{\text{"kinetic term"}} = I^{coll}[\tilde{F}]$$

<u>so far</u>: kinetic term allows $\partial_{t_{av}} \tilde{F} = 0$ for any $F(\omega)$ $\leadsto$ for which $\tilde{F}(\omega)$ is $I^{coll}[\tilde{F}] = 0$ (time-independent solutions with collisions)

page 143) $\leadsto$

$$I^{coll} = \sum_{p_1 p_2 p_3} \delta_{p+p_3,\, p_1+p_2}\, A\, A A \left\{ f_0 f_3 \bar{f}_1 \bar{f}_2 - \bar{f}_0 \bar{f}_3 f_1 f_2 \right\}$$

$f_j = f(\omega_j) \quad \omega_0 = \omega$

A : positive $\Longrightarrow$

$I^{coll} = 0$ requires

$$\boxed{f_0 f_3 \bar{f}_1 \bar{f}_2 \overset{!}{=} \bar{f}_0 \bar{f}_3 f_1 f_2 \quad \text{for } \omega + \omega_3 = \omega_1 + \omega_2}$$

Because $f, \bar{f} \geq 0$ (positivity of $G^{\gtrless}(t,t')$)

$\Longleftrightarrow$

$$\ln \frac{f_0 f_3 \bar{f}_1 \bar{f}_2}{\bar{f}_0 \bar{f}_3 f_1 f_2} \overset{!}{=} 0 \quad \forall \; \omega + \omega_3 = \omega_1 + \omega_2$$

with $k(\omega) \equiv \ln \dfrac{\bar{f}(\omega)}{f(\omega)}$

$$k(\omega_1) + k(\omega_2) - k(\omega) - k(\omega_3) \overset{!}{=} 0 \quad \forall_{\substack{\omega + \omega_3 \\ = \omega_2 + \omega_4}}$$

$\Longrightarrow$ $k(\omega)$ must be linear function of ω

$$k(\omega) = a\omega + b$$

$\Longrightarrow$ $\dfrac{1-f}{f} \overset{!}{=} e^{a\omega + b}$

$$\boxed{f = \frac{1}{e^{a\omega + b} + 1}}$$

$\Longrightarrow$ <u>only fermi function</u> (with some chemical potential $\mu = b/a$, $\beta = a$

inverse temperature $1/\beta$ and μ in final state determined by energy conservation

Remarks: assume we had used G_0 instead of G in the Σ-diagram

$\Rightarrow$ 3 distribution functions in I^{coll} would be fixed to $f_0(\omega) = \frac{1}{e^{\beta_0\omega}+1}$

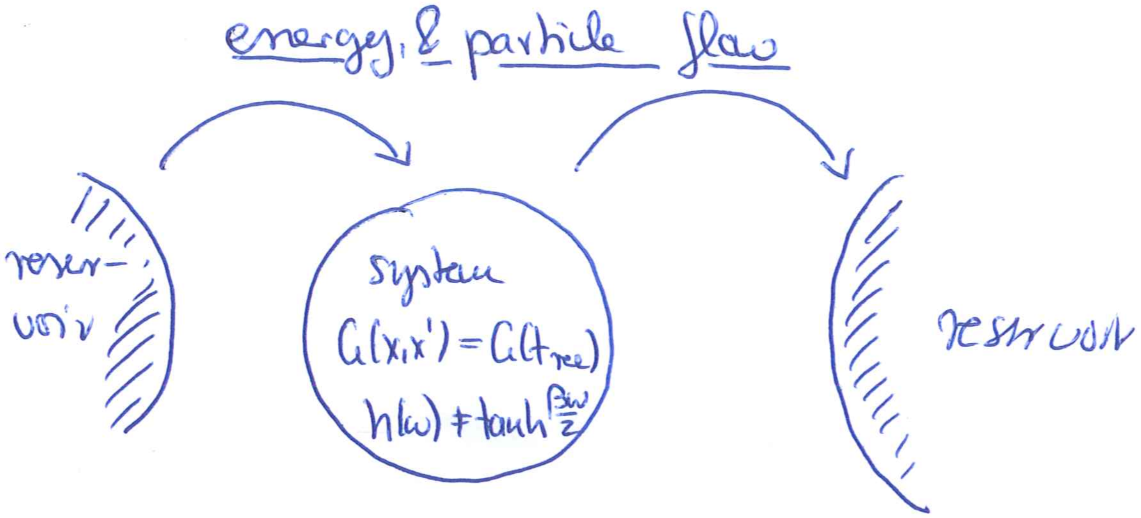
($\hat{=}$ initial state) $\Rightarrow$

only possible final stationary state $\beta = \beta_0$.

temperature fixed by initial state temperature

($\Leftrightarrow$ no energy conservation for bare perturbation expansion !)

<u>Remark 2:</u> system coupled to reservoir ("external degrees of freedom) $\Rightarrow$ non-equilibrium steady state possible



even parts of system can act as reservoir. E.g. <u>turbulence</u> $\hat{=}$ flow of energy / particles between high and low momenta.

STEP - 3 Quasi-particle approximation

remove ω-dependence from QBE ??

variable shift

$$\tilde{F}(\vec{r},t,\vec{k},\omega) \equiv F'(\vec{r},t,\vec{k},\omega-\varepsilon_k-\tilde{V}(\quad))$$

$$\tilde{V}(x,p) = V(x) + \mathrm{Re}\,\Sigma(x,p)$$ see page 136

in QBE: term $(\partial_t\tilde{V})\partial_\omega$ transformed away:

$$\left[\left(1-\partial_\omega \mathrm{Re}\,\Sigma\right)\partial_t + \vec{\tilde{V}}_k\cdot\vec{\nabla}_r - (\vec{\nabla}_r\tilde{V})\vec{\nabla}_k\right]F' = I^{coll}[F']$$

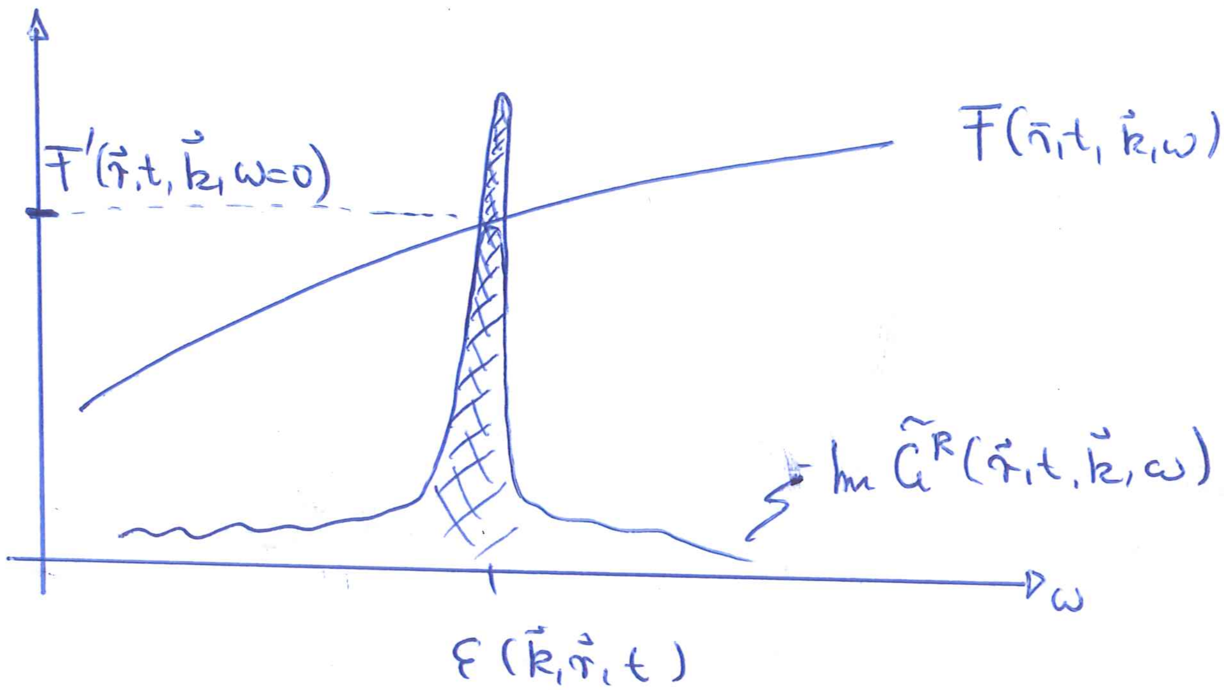
F' appears only in combination with spectra $\mathrm{Im}\,G^R(x,p)$ (in I^{coll}, and in evalutation of occupation function

$\mathrm{Im}\,G^R(x,p)$: sharply peaked at approx pole (quasiparticle energy)

$$\boxed{\omega - \varepsilon_k - V(\vec{r},t) - \mathrm{Re}\,\Sigma(\vec{r},t,k,\omega) = 0}$$

$\rightsquigarrow$ local quasiparticle dispersion $\underline{\underline{\tilde{\varepsilon}_k(\vec{k},\vec{r},t)}}$

(analogous to QP $\tilde{\varepsilon}_k$ on page 119)



$$\boxed{\begin{array}{l} \Rightarrow \text{mass-shell (Quasiparticle) approx} \\ F'(\vec{r},t,\vec{k},\omega) \cong F'(\vec{r},t,\vec{k},\omega=0) \end{array}}$$

QBE on mass shell:

$$\left(\tilde{Z}^{-1}\partial_t + \vec{V}_k\cdot\vec{\nabla}_r - (\vec{\nabla}_r\tilde{V})\,\vec{\nabla}_k\right) F(\vec{r},\vec{k},t) = I^{coll}$$

$F(\vec{r},\vec{k},t) \hat{=}$ phase space distribution

$Z^{-1}(\vec{r},k,t) \equiv (1-\partial_\omega\Sigma')_{\omega=\tilde{\varepsilon}(k,\vec{r},t)}$ QP weight

$\tilde{\varepsilon}(\vec{r},k,t) \equiv \varepsilon_k + \Sigma'(\vec{r},t,\vec{k},\tilde{\varepsilon}_k)$ QP dispersion

$\vec{V}_k \equiv \vec{\nabla}_k\tilde{\varepsilon}_k$ QP velocity

$\tilde{V} = V + \mathrm{Re}\,\Sigma|_{\omega=\tilde{\varepsilon}_k}$

$I^{coll} = I^{coll}[F(\vec{r},t,\vec{k},\omega) \longrightarrow F(\vec{r},t,\vec{k},\tilde{\varepsilon}_k)]$