V.2 Spectroscopic signature of quasi-particles

(≙ physical meaning of the Green's function)

consider some single-particle propagator of a system of free particles: (at $T=0$)

$H = \sum_k \varepsilon_k c_k^+ c_k \qquad |k\rangle$: single-particle basis

$$\langle \psi_0 | \underbrace{e^{iHt} c_k e^{-iHt} c_k^+ \underbrace{|\psi_0\rangle}_{GS,\ E=E_0}} = \ldots$$

$c_k^+|\psi_0\rangle$: adding particle in eigenstate $|k\rangle$ of single-particle Hamiltonian $\Rightarrow$ new state is also many-body eigenstate, because particles are independent

$$\ldots = \langle \psi_0 | e^{iE_0 t} c_k e^{-i(E_0+\varepsilon_k)t} c_k^+ |\psi_0\rangle$$

$$= \underbrace{e^{-i\varepsilon_k t}} \underbrace{\langle \psi_0 | c_k c_k^+ | \psi_0 \rangle}_{\text{occupation factor}}$$

spectral information in time-dependence (≙ frequency dependence) of propagator.

expected for weakly interacting system:

same decay $\sim e^{-i\varepsilon_k t} e^{-\gamma t}$

$\hat{=}$ pretty well defined particles if $\gamma \ll \varepsilon_k$.

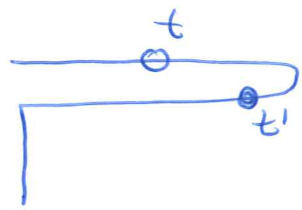
We will now formalize this. In particular, we will formalize, how to separate spectral information and "occupation" in the one-particle Greens function.

retarded / lesser / greater propagators

for each Green's function with time-arguments t, t' on the Keldysh contour, we define "physical meaningful components" as follows ($t_\pm$: lower / upper branch

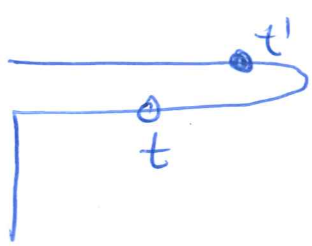
"lesser-component":

$$G^<(t,t') := G(t_-, t'_+)$$



"greater-component"

$$G^>(t,t') \equiv G(t_+, t'_-)$$



"retarded component"

$$G^R(t,t') = \Theta(t-t')\left(G^>(t,t') - G^<(t,t')\right)$$

"advanced component"

$$G^A(t,t') = \Theta(t'-t)\left(G^<(t,t') - G^>(t,t')\right)$$

"Keldysh - component"

$$G^K(t,t') = G^<(t,t') + G^>(t,t')$$

(... and some components with mixed imaginary-time / real-time arguments)

Note: these definitions will be used for any ~~Greens~~ function $A(t,t')$ on C, in particular also the self-energy.

For $G(1,1') = -i\,\langle T_c\, \psi(1)\, \psi(1')^+\rangle$

we have:

$$G^>(1,1') = -i\,\langle \psi(1)\,\psi^+(1')\rangle$$
$$G^<(1,1') = \mp i\,\langle \psi^+(1')\,\psi(1)\rangle$$
$$G^R(1,1') = -i\,\Theta(t-t')\,\langle [\psi(1), \psi^+(1')]_{\mp}\rangle$$

Remarks:

- physical meaning of $G^>$, $G^<$ $\hat{=}$ particle and hole propagator, as on page 96.

- G^R, G^K seem to have mathematical analogy to response / correlation functions defined in III.2. But why is there, eg. an anticommutator $[\,,\,]_+$ in G^R for Fermions. ⇝ more clear in path integral formulation, see later

General properties of GF components:

(1) Hermitian:

$$G^{\gtrless}(1,1') = -G^{\gtrless}(1',1)^*$$

(proof: $\left(\pm i \operatorname{tr}\left(e^{-\beta H} \psi(1) \psi(1')^+\right)\right)^* =$

$= \mp i \operatorname{tr}\left(\psi(1') \psi^+(1) e^{-\beta H}\right)$ etc.

(2) in equilibrium (H time-independent, commutes with initial state density matrix)

$\Rightarrow$ time-translational invariance.

$$G^{R,<,>,K}(t,t') = G^{R,>,<,K}(t-t')$$

proof... use cyclic permutation of operators under trace ...

(3) For the time-translationally invariant case, we define the Fourier transforms.

$$G^R(\omega) = \int_0^\infty dt\, e^{i\omega t}\, G^R(t)$$

$$G^{\gtrless}(\omega) = \int dt\, e^{+i\omega t}\, G^{\gtrless}(t)$$

$G^R(\omega)$ is an analytical function of ω in the upper complex plane ($\mathrm{Im}\,\omega > 0$), and $G^R(\omega) \sim 1/\omega$ for $|\omega| \to \infty$

(4) $G^R(\omega)$ has <u>spectral representation</u>

$$A(\omega) \stackrel{\omega\in\mathbb{R}}{:=} -\frac{1}{\pi}\,\mathrm{Im}\, G^R(\omega+i0) \equiv \lim_{\eta\to 0^+}\left(-\frac{1}{\pi}\right)\mathrm{Im}\, G(\omega+i\eta)$$

$$G(\omega) = \int d\omega' \,\frac{A(\omega')}{\omega-\omega'}$$

$$G(t) \stackrel{t>0}{=} -i \int d\omega\, A(\omega)\, e^{-i\omega t}$$

$A \equiv$ "spectral function"

proof: can be prooven directly from analytical properties (3), but we follow more simple strategy, ($\rightsquigarrow$ Lehmann representation) analogous to Ch. III, lin response.

$$G^R(t) \stackrel{t>0}{=} -i\,\frac{1}{Z}\,\mathrm{tr}\left(e^{-\beta H}\,[\psi(t),\psi^+(0)]_\mp\right)$$

omitting spatial indices for simplicity

many-body eigenbasis $H|m\rangle = E_m|m\rangle$

$$= -i\frac{1}{Z}\sum_{nm} e^{-\beta E_n}\Big(\langle n|\psi|m\rangle\langle m|\psi^+|n\rangle\, e^{i(E_n-E_m)t}$$

$$\mp \langle n|\psi^+|m\rangle\langle m|\psi|n\rangle\, e^{i(E_m-E_n)t}\Big)$$

$n \leftrightarrow m$ in second term

$$= -i \frac{1}{Z} \sum_{nm} |\langle n|\psi|m\rangle|^2 e^{i(E_n-E_m)t} \left(e^{-\beta E_n} \mp e^{-\beta E_m}\right)$$

FT: $\int_0^\infty e^{i(\omega+i0)t} e^{i(E_n-E_m)t} =$

$$= \left. \frac{e^{i(\omega+i0^+ + E_n - E_m)t}}{i(\omega+i0+E_n-E_m)} \right|_0^\infty$$

$\infty \to 0$ ∞, because $i0^+$ factor

$$= \frac{i}{\omega+i0+E_n-E_m}$$

$$\boxed{G^R(\omega) = \sum_{nm} \frac{e^{-\beta E_n} \mp e^{-\beta E_m}}{Z} \frac{|\langle n|\psi|m\rangle|^2}{\omega+i0+E_n-E_m}}$$

mit $\frac{1}{\omega+i0} = -i\pi\,\delta(\omega) + \mathcal{P}\frac{1}{\omega}$ (Ch. III)

$$\Rightarrow -\frac{1}{\pi} \operatorname{Im} G^R(\omega+i0) = \sum_{mn} \frac{e^{-\beta E_n} \mp e^{-\beta E_m}}{Z} |\langle n|\psi|m\rangle|^2 \, \delta(\omega+E_n-E_m)$$

$$\equiv A(\omega)$$

with this expression for $A(\omega)$, $G(\omega) = \int d\omega' \frac{A(\omega')}{\omega-\omega'}$ and $G(t) \overset{t>0}{=} \int d\omega\, A(\omega)\, e^{-i\omega t}$ can be verified ~~by~~ directly ∎

(5) "<u>Fluctuation - dissipation relation</u>" :

$G^>, G^<, G^R$ satisfy a relation analogous to the FDT :

$$G^>(\omega) = -i \; A(\omega) \; h(\omega) \; 2\pi$$

$$G^<(\omega) = \mp i \; A(\omega) \; h(-\omega) \; 2\pi$$

$$h(\omega) = \frac{1}{e^{\beta\omega} \pm 1}$$ Fermi / Bose function

(<u>Note</u> $h(-\omega) = 1 \pm h(\omega)$)

<u>proof</u>: Lehmann - representation.

(6) <u>positive definiteness</u>

$i\, G^>(1,1')$ and $\pm i\, G^<(1,1')$ are positive definite matrices in the sense

$$\boxed{\int d1\, d1' \; s(1)\, M(1,1')\, s(1')^* \geq 0 \quad \forall \text{ functions } s(1)}$$

proof: Lehmann - representation (here for $G^>$)

$$M(1,1') = i\, G^>(1,1') = \langle \psi(1)\, \psi^+(1') \rangle$$

$$= \sum_{nm} \underbrace{\langle n|\, e^{-\beta H}/Z}_{e^{-\beta E_n}/Z\, \langle n|}\, \psi(1)\, |m\rangle \underbrace{\langle m|\, \psi(1')^+ |n\rangle}_{\langle n|\, \psi(1')\, |m\rangle^*}$$

$$\int d1\, d1'\; s(1)\, M(1,1')\, s(1') =$$

$$= \sum_{nm} \left| \int d1\; s(1) \sqrt{\frac{e^{-\beta E_n}}{Z}}\, \langle n|\psi(1)|m\rangle \right|^2 \geq 0$$

Note: together with the hermitian property (1), positivity implies

$$i\, G^>(\omega) \quad \text{real}, \geq 0$$

$$\pm i\, G^<(\omega) \quad \text{real} \geq 0$$

for any stationary state ($\hat{=}\; G(t,t') = G(t-t')$)

(proof: take "$s(t) \to e^{i\omega t}$" above)

$\rightsquigarrow$ positive $\Leftrightarrow$ related to probability of emitting ($G^<$) or absorbing ($G^>$) particle from perturbation, with energy transfer ω

Note: if we define $\Sigma^R(t,t') = \Theta(t-t')\,(\Sigma^> - \Sigma^<)$, $\Sigma^>(t,t') = \Sigma(t_-,t_+)$ etc. analogous to Green's function, all properties (1)... (7) above hold for Σ.

Green's function of non-interacting system, equil.

$$H = \sum_k \varepsilon_k c_k^+ c_k \qquad \text{(no interaction, transl. invariance)}$$

$$(\leadsto\ G(t,k,t'k') \equiv G_k(t,t')\ )$$

Note: For simplicity, only Fermion in this section

$$G_k^R(t) = -i\,\Theta(t)\,\langle [\underbrace{c_k(t)}_{e^{-i\varepsilon_k t} c_k},\ c_k^+(0)]_+ \rangle$$

$$= -i\,\Theta(t)\,e^{-i\varepsilon_k t}$$

$$\Rightarrow G_k^R(\omega + i0) = \int_0^\infty dt\, e^{i(\omega+i0)t}\, G_k^R(t)$$

$$= \frac{1}{\omega + i0 - \varepsilon_k}$$

$$A_k(\omega) = -\frac{1}{\pi}\,\mathrm{Im}\,\frac{1}{\omega + i0 - \varepsilon_k} = \delta(\omega - \varepsilon_k)$$

$$G^>_k(\omega) = -2\pi i \underbrace{A_k(\omega)}^{\delta(\omega-\varepsilon_k)} \bar{f}(\omega) \qquad \bar{f} = 1-f$$

$$= -2\pi i \, \delta(\omega-\varepsilon_k) \bar{f}(\varepsilon_k)$$

$$G^<_k(\omega) = 2\pi i \, \delta(\omega-\varepsilon_k) f(\varepsilon_k)$$

general: $G^{\gtrless}$ = spectrum $\times$ occupation

Dyson equation for $G^R, G^A, G^{\lessgtr}, \ldots$

<u>goal</u>: write $G = G_0 + G_0 \Sigma G$

in terms of $G^{R,A,\lessgtr}$, $\Sigma^{R,A,\gtrless}, \ldots$

Intermezzo: L-shaped vs. Keldysh contour

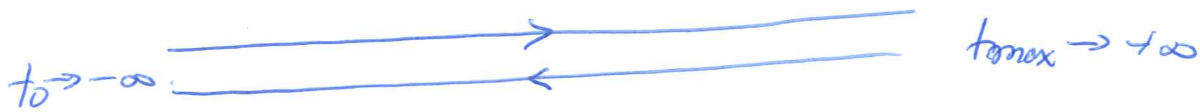
so far: G determined by $\Sigma[G]$, and equation of motion on contour C:

+ KMS boundary condition

$\rightsquigarrow$ good for numerics, often inconvenient for analytical arguments

- for analytical argument, we shift $t_0 \to -\infty$ and omit vertical part of $\mathcal{C}$.

$\rightsquigarrow$ Keldysh contour $\mathcal{C}_K$



- KMS boundary condition would have to be replaced by <u>asymptotic initial condition</u>

$$G_0(t,t') \xrightarrow[\substack{t,t' \to -\infty \\ t-t'=s \text{ fix}}]{} G_0^{\text{equilib}}(s)$$

- all derivations below can be repeated easily for either L-shaped contour or Keldysh contour, just for L-contour there are additional terms involving $G(t,\tau)$

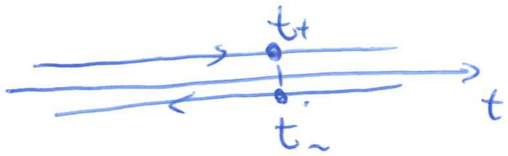
real branch imag branch

- in perturbative description, ok for adiabatic switch-on of interaction from $t_0 = -\infty$

- <u>real-time convolutions</u> $\int_C A(t,\bar{t})\, B(\bar{t},t')\, d\bar{t}$

parametrization $\hat{A}(t,t') = \begin{pmatrix} A_{++}(t,t') & A_{+-}(t,t') \\ A_{-+}(t,t') & A_{--}(t,t') \end{pmatrix}$

$A_{+-}(t,t') = A(t_+, t'_-)$



$t_\pm$: upper / lower branch

$A * B = C$ $\quad * :$ convolution

$$\Rightarrow \hat{C}(t,t') = \underbrace{\int_{-\infty}^{+\infty}}_{\text{only real time integral}} d\bar{t}\ \hat{A}(t,\bar{t})\ \underbrace{\hat{\tau}_3}_{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}\ \hat{B}(\bar{t},t')$$

$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ takes care of direction of $\int_{C_K} d\bar{t}$

- rotation $\underline{A} = L\, \hat{\tau}_3\, \hat{A}\, L^+$ $\quad$ unitary L

$$\Rightarrow \underline{C}(t,t') = L\, \tau_3 \int d\bar{t}\ \hat{A}(t,\bar{t}) \underset{L^+L}{\wedge} \hat{\tau}_3\, \hat{B}(\bar{t},t')\, L^+$$

$$= \int d\bar{t}\ \underline{A}(t,\bar{t})\, \underline{B}(\bar{t},t')$$

standard real-time convolution!

choice of L ?

$$L = \frac{1}{\sqrt{2}} \begin{pmatrix} +1 & -1 \\ +1 & +1 \end{pmatrix} \Rightarrow$$

$$\boxed{\underline{A}(t,t') = \begin{pmatrix} A^R(t,t') & A^K(t,t') \\ 0 & A^A(t,t') \end{pmatrix}}$$

Keldysh matrix

proof:

$$L \hat{\tau}_3 \hat{A} L^+ = \frac{1}{2} \begin{pmatrix} 1 & -1 \\ 1 & +1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} A_{++} & A_{+-} \\ A_{-+} & A_{--} \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & +1 \end{pmatrix} =$$

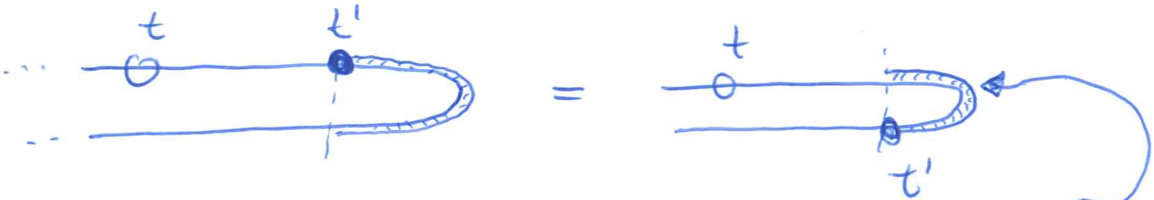
$$= \frac{1}{2} \begin{pmatrix} A_{++} \mp A_{+-} + A_{-+} - A_{--} & A_{++} + A_{+-} + A_{-+} + A_{--} \\ A_{++} \mp A_{+-} \mp A_{-+} + A_{+-} & A_{+-} + A_{+-} \mp A_{-+} - A_{--} \end{pmatrix}$$

- (1,1) - component:

$\underline{t > t'}$: $\left[A_{++}(t,t') \mp A_{+-}(t,t') + A_{-+}(t,t') - A_{--}(t,t')\right]/2$

$= \#$

<u>Note</u>: larger time-argument can be shifted between upper and lower contour!



(causality ... time-propagation on this part of contour cancels anyway!)

$$\# = [A_{-+}(t,t') - A_{+-}(t,t') + A_{-+}(t,t') - A_{+-}(t,t')]/2$$

$$= A_{-+}(t,t') - A_{+-}(t,t') = A^{>}(t,t') - A^{<}(t,t')$$

<u>$t < t'$</u>: shift t' argument $t'_- \leftrightarrow t'_+$:

$$\# = [A_{-+}(t,t') - A_{+-}(t,t') + A_{+-}(t,t') - A_{-+}(t,t')]$$

$$= 0$$

$\Rightarrow$ (1,1)-component : $= (A^{>}(t,t') - A^{<}(t,t'))\,\Theta(t-t')$

$$= A^{R}(t,t') \quad ✓$$

analogous for other components:

<u>Langreth rules</u>: $\underline{C} = \underline{A} \underbrace{*}_{\text{real-time conv.}} \underline{B}$

$$\begin{pmatrix} C^R & C^K \\ 0 & C^A \end{pmatrix} = \begin{pmatrix} A & A \\ & A \end{pmatrix} * \begin{pmatrix} B & B \\ & B \end{pmatrix}$$

$$= \begin{pmatrix} A^R * B^R & A^R * B^K + A^K * B^A \\ & A^A * B^A \end{pmatrix}$$

analogous: $C^{\gtrless} = A^R B^{\gtrless} + A^{\gtrless} B^A$

Representation of Dyson equation

omitting space indices for simplicity ...

$$[i\partial_t - h(t)]\, G(t,t') - \int_C d\bar{t}\, \Sigma(t,\bar{t})\, G(\bar{t},t') = \delta(t,t')$$

note: $\delta(t_+, t'_-) = \delta(t_-, t'_+) = 0$

$\delta(t_+, t'_+) = -\delta(t_-, t'_-) = \delta(t-t')$

in matrix-form: $\underline{t, t' \in \mathbb{R}}$

$$\begin{pmatrix} i\partial_t - h(t) & 0 \\ 0 & i\partial_t - h(t) \end{pmatrix} \hat{G}(t,t') - \overbrace{\hat{\Sigma} * \hat{G}(t,t')}^{\int \hat{\Sigma}\, \tau_3\, \hat{G}\, d\bar{t}\text{, see above}} = \hat{\tau}_3\, \delta(t-t')$$

... multiply with $L\, \hat{\tau}_3 \{\ldots\} L^+$:
(Keldysh rotation)

$$[i\partial_t - h(t)]\, \underline{G}(t,t') - \underline{\Sigma} * \underline{G} = \delta(t-t')\, \mathbb{1}$$

<u>general notation</u>:

arguments of $\underline{A}$ matrices:

$$1 \hat{=} t_1, \xi_1 \quad ; t \in (-\infty, +\infty)$$

$\xi \hat{=}$ orbital, spin, space, ...

real-time convolution:

$$\underline{A} * \underline{B} = \underline{C}, \quad \underline{C}(1,1') = \int d\bar{1}\, A(1,\bar{1})\, B(\bar{1},1')$$

$$\int d\bar{1} = \int_{-\infty}^{+\infty} d\bar{t}_1 \int d\bar{\xi}_1$$

$\Longrightarrow$

$$\boxed{\begin{aligned} \left(\underline{G}_0^{-1} - \underline{\Sigma}\right) * \underline{G} &= \mathbb{1} \\ \underline{G} * \left(\underline{G}_0^{-1} - \underline{\Sigma}\right) &= \mathbb{1} \end{aligned}}$$

$$\mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \delta(t-t')\,\delta(\xi,\xi')$$

$$\underline{G}_0^{-1} \equiv \begin{pmatrix} (G_0^R)^{-1} & 0 \\ 0 & (G_0^A)^{-1} \end{pmatrix}$$

$$(G_0^R)^{-1}(1,1') = \delta(t-t')\left[i\partial_t - h(\xi,\xi';t)\right].$$

if $H_0 = \int d\xi d\xi'\, \Psi(\xi)^+\, h(\xi,\xi';t)\, \Psi(\xi')$

example: $h(\vec{r},\sigma,\vec{r}'\sigma') = \delta_{\sigma\sigma'}\,\delta(\vec{r}-\vec{r}')\left[-\frac{\hbar^2}{2m}\vec{\nabla}_r^2 + V(\vec{r})\right]$

Note: multiplication with $(G_0^R)^{-1}$ from right:

$$\int d\bar{t}\; A(t,\bar{t})\; i\partial_{\bar{t}}\,\delta(\bar{t}-t') \underset{\substack{\uparrow \\ \text{part.} \\ \text{integration}}}{=} -i\partial_{t'} A(t,t')$$

Equation for noninteracting GF, $\underline{G_0}^{-1}\underline{G_0} = \underline{G_0}\,\underline{G_0}^{-1} = 1$

again, omitting space indices for simplicity

$$\begin{pmatrix} (G_0^R)^{-1} & 0 \\ 0 & (G_0^A)^{-1} \end{pmatrix} * \begin{pmatrix} G_0^R & G_0^K \\ 0 & G_0^A \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \delta(t-t')$$

I) $(i\partial_t - h)\, G_0^R(t,t') = \delta(t-t')$

II) $(i\partial_t - h)\, G_0^A(t,t') = \delta(t-t')$

III) $(i\partial_t - h)\, G_0^K(t,t') = 0$

IIIa) $(-i\partial_{t'} - h)\, G_0^K(t,t') = 0$

<u>solution ?</u> ... not unique ... homogeneous equation

$\Rightarrow$ initial condition needed ($\rightsquigarrow$ see comments on L-shaped contour vs. Keldysh contour.

<u>initial condition</u>

- $G^R(t,t') = 0$ for $t < t'$

  integration of $i\partial_t G(t,t') - \ldots = \delta(t-t')$ $[t'-\varepsilon, t'+\varepsilon]$:

  $\rightsquigarrow$ $G_0^R(t,t') \xrightarrow[t \downarrow t']{} -i$ $\quad$ ("$G_0^R(t,t) = -i$")

  (consistent with $G^R(t,t') = -i\,\Theta(t-t')\,\langle [\psi(t), \psi^+(t')]_\mp \rangle$)

- $G^A(t,t) = +i$ $\quad$ (analogous)

- $(i\partial_t - h)\,G^K = 0$, $(-i\partial_{t'} - h)\,G^K = 0$:

  use asymptotic initial condition

  $t, t' \to -\infty$, $t - t' = s$ $\Rightarrow$

  $G_0^K(t,t') \longrightarrow \tilde{G}^K_{0,eq}(s) = \int d\omega\, e^{-i\omega s}\, G_0^K(\omega)$

  $G^K_{0,eq}(\omega) = G^>_{0,eq}(\omega) + G^<_{0,eq}(\omega)$

  $= -2\pi i\,[\bar{f}(\omega) - f(\omega)]\,A_{eq}(\omega)$

  $= -2\pi i \tanh\left(\frac{\beta\omega}{2}\right) A_{eq}(\omega)$ $\qquad$ <u>FDT for Keldysh component</u>

  $[= -2\pi i \coth\frac{\beta\omega}{2}\, A_{eq}(\omega)$ for Bosons$]$

Causal structure of Dyson equation

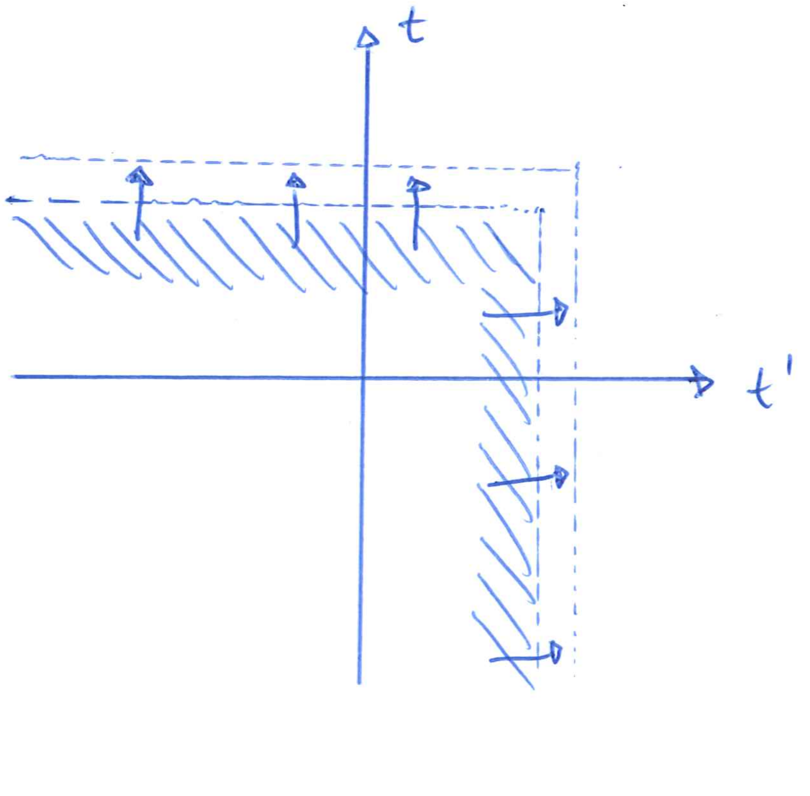
Langreth rules imply that $\partial_t G^{R,A,K}(t,t')$, $\partial_{t'} \underline{G}(t,t')$ are determined by $\underline{G}(t_1,t_2)$, $\underline{\Sigma}(t_1,t_2)$

for $t_1, t_2 \le \max(t, t')$.

Example: $\cancel{G}\,[\Sigma^R * G^R](t,t') =$

$$= \int_{-\infty}^{+\infty} d\bar{t}\; \underbrace{\Sigma^R(t,\bar{t})}_{\sim\Theta(t-\bar{t})}\; \underbrace{G^R(\bar{t},t')}_{\sim\Theta(\bar{t}-t')} = \int_{t'}^{t} d\bar{t}\; \Sigma^R(t,\bar{t}) \cdot G^R(\bar{t},t')$$

$\Rightarrow$ forward time-propagation in (t,t')-plane



explicit:

$$[i\partial_t - h(t)]\, G^R(t,t') = \delta(t-t') + \int_{t'}^{t} d\bar{t}\; \Sigma^R(t,\bar{t})\, G^R(\bar{t},t')$$

$$[i\partial_t - h(t)]\, G^A(t,t') = \delta(t-t') + \int_{t}^{t'} d\bar{t}\; \Sigma^A(t,\bar{t})\, G^A(\bar{t},t')$$

$$[i\partial_t - h(t)]\, G^{K,\gtrless}(t,t') = \int_{-\infty}^{t} d\bar{t}\; \Sigma^R(t,\bar{t})\, G^{K,\gtrless}(\bar{t},t')$$

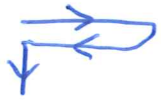
$$+ \int_{-\infty}^{t'} d\bar{t}\; \Sigma^{K,\gtrless}(t,\bar{t})\, G^A(\bar{t},t')$$

(analogous for equivalent equation $\underline{G}\cdot(\underline{G_0}^{-1} - \underline{\Sigma}) = \underline{1}$

$\underline{\Sigma}[\underline{G}]$ is a functional of $\underline{G}$ (e.g. perturbative!)

$\Rightarrow$ Dyson equation $\hat{=}$ non-linear integro-differential equation of motion for G, with <u>memory - kernel Σ</u>

kinetic equations (next section) $\rightsquigarrow$ approximate way to truncate memory integral

<u>Note:</u> every-thing can be formulated ~~in~~ on contour  ... just terms with mixed terms $G(t,\tau)$, KMS boundary condition instead of initial condition, still causal propagation for <u>real-time</u> functions $G^{R,A,\gtrless}$.

<u>Self energy and Quasi-particles</u>

$H_0 \overset{(*)}{=} \sum_k \varepsilon_k c_k^\dagger c_k$, consider Dyson equation for $G_k^R(t,t') = -i\,\theta(t-t')\langle[c_k(t), c_k^\dagger(t')]\rangle$

in equilibrium. (Note (*): in this notation, ε_k is measured with respect to the Fermi-energy (≙ chemical potential))

$$(i\partial_t - \varepsilon_k)\, G_k^R(t,t') - \int_{t'}^{t} d\bar{t}\, \Sigma_k^R(t,\bar{t})\, G^R(\bar{t},t') \overset{t>t'}{=} 0$$

$$G_k^R(t,t) = -i$$

equilibrium $t,t' \to t-t'$

$\rightsquigarrow$ Laplace transform $G^R(\omega+i0) = \int_0^\infty dt\, e^{i(\omega+i0)t} G^R(t)$

$$\rightsquigarrow \underbrace{(\omega+i0-\varepsilon_k)}_{(G^R_{k,0})^{-1}(\omega+i0)} G^R_k(\omega+i0) - \Sigma^R_k(\omega+i0)\, G^R_k(\omega+i0) = 1$$

(in the following, omit ~~at~~ "+i0" in ~~ar~~ argument of retarded functions for simplicity, "$G^R(\omega) \equiv G^R(\omega+i0)$")

$$\boxed{G^R_k(\omega) = \frac{1}{\omega+i0-\varepsilon_k-\Sigma^R_k(\omega)}}$$

- Hartree & Fock self-energies:

local in time ($\sim \delta(t-t')$) $\Rightarrow$ ω-independent

$$\Rightarrow A_k(\omega) \equiv -\frac{1}{\pi}\,\mathrm{Im}\, G^R_k(\omega) =$$

$$= \delta(\omega - \varepsilon_k^{HF})$$

$$\varepsilon_k^{HF} = \varepsilon_k + \Sigma_k^{\text{hartree}} + \Sigma_k^{\text{fock}}$$

$\Rightarrow$ just renormalized (mean-field) band structure

- beyond HF: Σ^R has imaginary part (see also exercise) with $\mathrm{Im}\,\Sigma^R(\omega) < 0$ (same positive definit structure like Green's function)

$$\Sigma^R_k(\omega) = \Sigma'_k(\omega) + i\,\Sigma''_k(\omega)$$

- define $\tilde{\varepsilon}_k$ by solution of

$$\omega - \varepsilon_k - \Sigma'_k(\omega) = 0$$

$$\tilde{\varepsilon}_k = \varepsilon_k + \Sigma'_k(\tilde{\varepsilon}_k)$$

$\Rightarrow$ close to $\omega \to \tilde{\varepsilon}_k$: expand Σ:

$$G^R_k(\omega)^{-1} = \omega - \varepsilon_k - \Sigma'_k(\tilde{\varepsilon}_k) - \partial_\omega \Sigma'(\tilde{\varepsilon}_k)(\omega - \tilde{\varepsilon}_k) - i\,\Sigma''_k(\tilde{\varepsilon}_k) + \dots$$

$$= \underbrace{\left(1 - \partial_\omega \Sigma'(\tilde{\varepsilon}_k)\right)}_{\equiv Z_k^{-1}}(\omega - \tilde{\varepsilon}_k) - i\,\underbrace{\Sigma''_k(\tilde{\varepsilon}_k)}_{\equiv Z_k \gamma_k}$$

$$G^R_k(\omega) \xrightarrow[\omega \to \tilde{\varepsilon}_k]{} \frac{Z_k}{\omega - \tilde{\varepsilon}_k + i\gamma_k} + \dots$$

$$\boxed{A_k(\omega) \xrightarrow[\omega \to \tilde{\xi}_k]{} Z_k \cdot \frac{\gamma_k/\pi}{(\omega - \tilde{\xi}_k)^2 + \gamma_k^2} + \dots}$$

- if $\gamma_k \ll \tilde{\xi}_k$, scale on which $\Sigma(\omega)$ changes.

$\Rightarrow$ Lorenz peak width γ_k, weight Z_k

+ "incoherent spectral weight" at other energies. $\Rightarrow$

particle-like excitation:

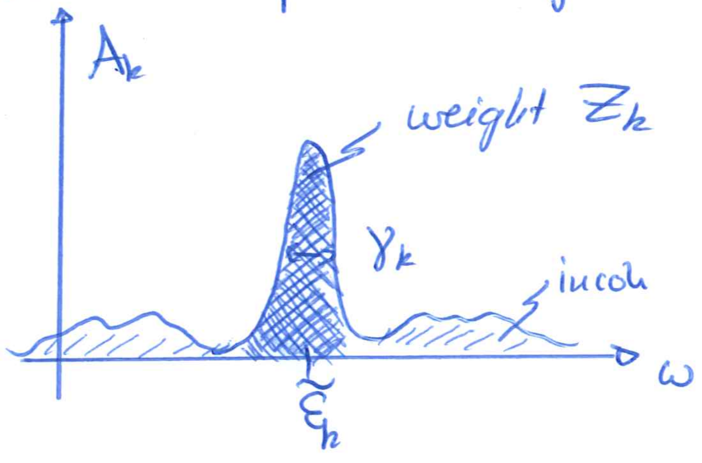
$\tilde{\varepsilon}_k = \varepsilon_k - \Sigma'(\tilde{\varepsilon}_k)$: renormalized dispersion

$\gamma_k \cong 1/\tau_k$: inverse lifetime

Z_k : quasiparticle weight

Note: sum-rule $\int d\omega \, A_k(\omega) = 1 \quad (= i\, G^R(t,t))$

$\Rightarrow$ incoherent spectral weight $\to$ finite weight $1 - Z_k$



signature of a quasi particle

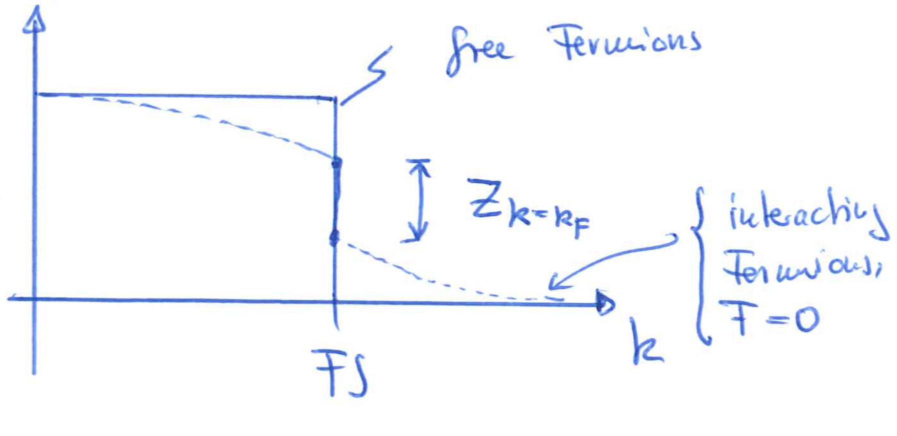
microscopic manifestation of Fermi liquid

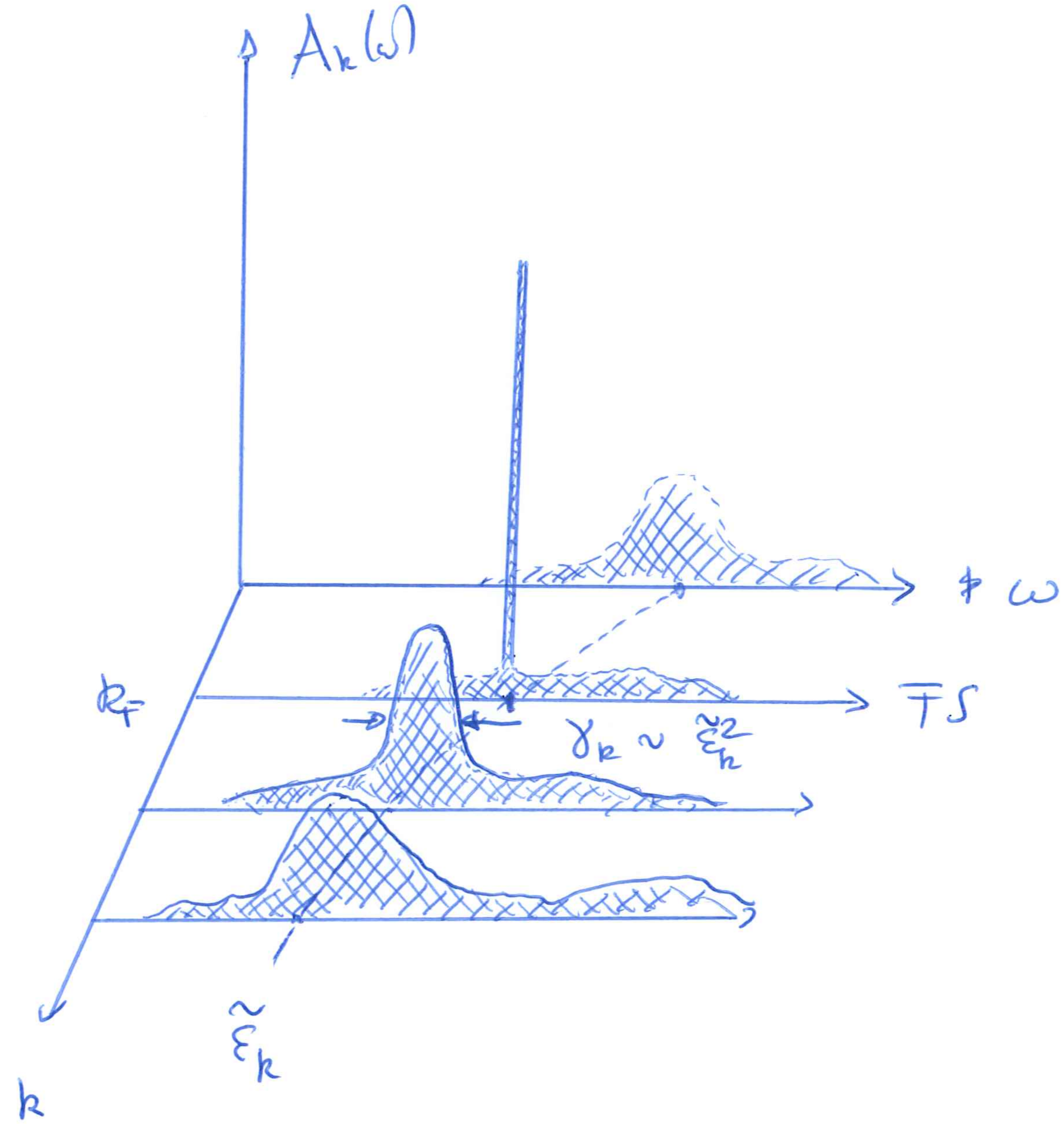
$$\Sigma''(\omega) \sim \omega^2 \quad \text{for } T=0, \quad \omega \to 0$$

Consequence:

- well-defined Fermi surface
  (defined by $\tilde{\varepsilon}_k = 0$)
- close to FS: $\gamma_k \sim \tilde{\varepsilon}_k^2$
  (lifetime $\ll 1/\tilde{\varepsilon}_k \Rightarrow$ well defined quasi-particle energy
- for $\vec{k} \to FS$:
  $$A_k(\omega) \to Z_k\, \delta(\omega - \tilde{\varepsilon}_k) + A_{incoh.}(k,\omega)$$
  (asymptotically independent particles.)

$\Rightarrow$ jump in momentum occupation at FS:
(check): $\langle n_k \rangle = \langle c_k^+ c_k \rangle = -i\, G_k^<(t,t)$





$A_k(\omega)$
ω
k_F
FS
$\gamma_k \sim \tilde{\varepsilon}_k^2$
$\tilde{\varepsilon}_k$
k