Field theory for nonequilibrium systems, WS 2016/17

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Sheet 3

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1 Nyquist-Johnson noise in a resistor

Consider an RC circuit (see Figure 1) with a voltage measurement across the capacitor C. The thermal charge fluctuations $Q(t)$ on R are also the charge fluctuations on C (why?). We measure voltage fluctuations $U(t) = Q(t)/C$ in a time interval $t \in [-\frac{t_0}{2}, \frac{t_0}{2}]$.

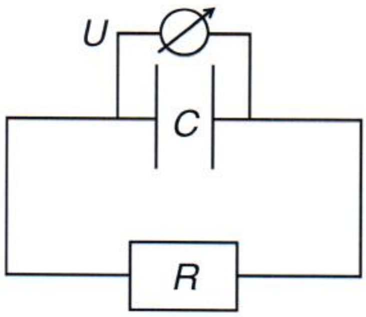


Figure 1: RC circuit with voltage fluctuations measured on C.

(a) Consider the voltage-voltage correlation function $S_U(t-t') \equiv \langle U(t)U(t') \rangle$ and the power spectrum $S_U(\omega) \equiv |U(\omega)|^2$. Assume that we perform $\nu = 1, \ldots, M$ measurements and the ν-th measurement gives the Fourier components

$$U^{(\nu)}(\omega_n) = \frac{1}{\sqrt{t_0}} \int_{-t_0/2}^{t_0/2} dt\ e^{i\omega_n t} U^{(\nu)}(t) \tag{1}$$

with $\omega_n \equiv \frac{2\pi n}{t_0}$. How would you obtain the correlation function $S_U(t-t')$ from these measurements in the limit $M \to \infty$? Find a relation between

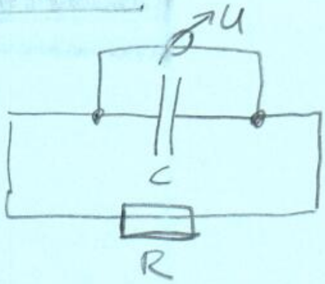
$$S_U(\omega_n) = \int_{-t_0/2}^{t_0/2} d\tau\ e^{i\omega_n \tau} S_U(\tau) \tag{2}$$

and $U(\omega_n)$. Then take the limit $t_0 \to \infty$ ($\omega_n \to \omega$).

(b) Now we go from the voltage response to the charge response. Consider the circuit in Figure 2 with an external voltage source V^{ext} and the resistor at temperature T.

Nyquist - Johnson noise

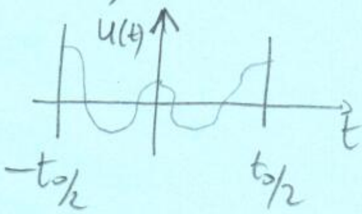
RC circuit



- voltage fluctuations $U(t) = \frac{Q(t)}{C}$

- charge fluctuations (thermal) on R are also charge fluctuations on C (charge conservation!)

- measure $U(t)$ in interval $\left[-t_0/2, t_0/2\right]$



- consider $S_u(t-t') \equiv \langle U(t)\, U(t') \rangle$

power spectrum $S_u(\omega) \equiv \langle |U(\omega)|^2 \rangle$

- measure Fourier components $U(\omega_n)$

ν-th measurement $U^{(\nu)}(\omega_n) = \frac{1}{\sqrt{t_0}} \int_{-t_0/2}^{t_0/2} dt\, e^{i\omega_n t}\, U^{(\nu)}(t)$, $\omega_n \equiv \frac{2\pi}{t_0} n$ ← integer

- correlation function via averaging

$$\langle U(t)\, U(t') \rangle_M = \frac{1}{M} \sum_{\nu=1}^{M} U^{(\nu)}(t)\, U^{(\nu)}(t')$$

$M \to \infty$: $S_u(t,t') \to S_u(t-t')$

- do time averaging

$$S_u(\tau) = \frac{1}{t_0} \int_{-t_0/2}^{t_0/2} d\bar{t} \left\langle u(\bar{t}+\tau)\, u(\bar{t}) \right\rangle$$ auto-correlation

- Fourier transform

$$S_u(\omega_n) = \int_{-t_0/2}^{t_0/2} d\tau \; e^{i\omega_n \tau} S_u(\tau)$$

- relation between $S_u(\omega_n)$ and $u(\omega_n)$?

$$S_u(\omega_n) = \frac{1}{t_0} \int_{-t_0/2}^{t_0/2} d\tau \int_{-t_0/2}^{t_0/2} d\bar{t} \; e^{i\omega_n \tau} \frac{1}{t_0} \sum_{ll'} u(\omega_l)\, u(\omega_{l'}) \times$$

$$\times e^{-i\omega_l \bar{t}} e^{-i\omega_{l'}(\bar{t}+\tau)}$$

$\bar{t}$ - integral: $\displaystyle \frac{1}{t_0} \int_{-t_0/2}^{t_0/2} d\bar{t} \; e^{-i(\omega_l+\omega_{l'})\bar{t}} = \delta(\omega_l, -\omega_{l'})$

τ - integral: $\delta(\omega_{l'}, \omega_n)$

$u(t)$ is real $\Rightarrow$ $u(-\omega_n) = u^*(\omega_n)$

$$\Rightarrow S_u(\omega_n) = \left\langle |u(\omega_n)|^2 \right\rangle$$

$$\xrightarrow{t_0 \to \infty} S_u(\omega) = \left\langle |u(\omega)|^2 \right\rangle$$

spectral density of voltage fluctuations

= intensity distribution in the frequency spectrum of $u(t)$

= power spectrum of the stochastic variable u

non-reproducible!

Summary: We measure the random ↓ $u(t)$ in a large time interval.

⇒ Fourier coefficients $u(\omega_n)$

⇒ $S_u(\omega_n) = |u(\omega_n)|^2$

⇒ $S_u(\omega)$ for $t_0 \to \infty$ is reproducible!

⇒ $S_u(\omega)$ is the Fourier transform of the correlation function $S_u(t-t') \equiv \langle u(t) u(t') \rangle$

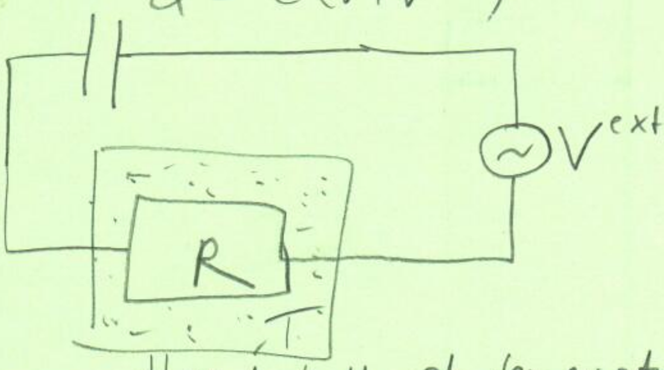
Wiener-Khintchine theorem (1930, 1934)

One can show that $\langle u(\omega) u^*(\omega') \rangle = 2\pi \, \delta(\omega - \omega') \, S_u(\omega)$

⇒ $u(\omega)$ and $u^*(\omega')$ are uncorrelated for $\omega \neq \omega'$ since $S_u(t, t')$ only depends on $t - t'$

Voltage response ⇒ charge response

$Q = C(V + V^{ext})$



Let V be the voltage drop along R.

" Q —"— charge on C.

$$V^{ext}(t) \Rightarrow \hat{H}_1(t) = -V^{ext}(t) \cdot \hat{Q}$$

⇒ measure average charge $\langle \hat{Q}(t) \rangle$

$$\langle \hat{Q}(t) \rangle = C \cdot \left(V(t) + V^{ext}(t) \right) \qquad (*)$$

$V(t)$ → time derivative of $\langle \hat{Q}(t) \rangle$

$$J = \frac{V}{R}$$

$$\Rightarrow \frac{d\langle \hat{Q} \rangle}{dt} = -J = -\frac{V}{R} \qquad (**)$$

$(**)$ in $(*)$: $$\frac{d\langle \hat{Q} \rangle}{dt} + \frac{\langle \hat{Q} \rangle}{RC} = \frac{V^{ext}}{R}$$

Fourier: $$\left[-i\omega + \frac{1}{RC} \right] \langle \hat{Q}(\omega) \rangle = \frac{V^{ext}(\omega)}{R}$$

- Generally: $\langle \hat{Q}(\omega) \rangle = \chi_{QQ}(\omega)\, V^{ext}(\omega)$

here: $$\chi_{QQ}(\omega) = \frac{i/R}{\omega + \frac{i}{RC}}$$

Dissipation: $$\chi''_{QQ}(\omega) = C^2 \omega \frac{R}{1+\omega^2(RC)^2}$$

$\Rightarrow$ charge fluctuations at equilibrium ($V^{ext}=0$):

$$S_{QQ}(\omega) = C^2\, 2k_BT \frac{R}{1+\omega^2(RC)^2} \quad (*)$$

follow from <u>FDT</u> for $\hbar\omega \ll k_BT$

[classically: $\chi'' \to S$ via $\omega \to 2k_BT$]

$\Rightarrow$ relation between charge fluctuations (= fluctuations of observable) and voltage fluctuations (= fluctuations of forces)

- for $V^{ext}=0$ we have $Q(t) = C\,u(t)$

$$\Rightarrow S_u(t) = \frac{1}{C^2} S_{QQ}(t)$$

$$\boxed{|u(\omega)|^2 = S_u(\omega) \overset{(*)}{=} \frac{2k_BTR}{1+\omega^2(RC)^2} \quad \text{for } \hbar\omega \ll k_BT}$$

(**)

Finally in a purely resistive case ($L \to 0$)

$$|U(\omega)|^2 = 2k_B T \cdot R \quad \text{for } L \to 0$$

"resistance noise"

(**) was measured by Johnson (1928) and derived by Nyquist (1928)

Remarks: (i) A resistor at temperature T has charge fluctuations, which generate an electrical current and therefore voltage fluctuations $U(t)$, which are dissipated as <u>heat</u>. The heat generated in the resistor has to be in balance with the energy that is taken from the fluctuations (remember that FDT follows from a detailed balance relation)

(ii) the power spectrum $S_U(\omega)$ for $L \to 0$ is frequency-independent! "white noise"

$\Rightarrow$ autocorrelation $\langle U(t+\tau)\, U(t)\rangle = 2k_B T R\, \delta(\tau)$

$\Rightarrow$ dissipated energy $\int_{-\infty}^{\infty} d\omega \frac{|U(\omega)|^2}{R} = 2k_B T \int_{-\infty}^{\infty} d\omega = \infty$ ↯

unphysical!

⇒ requires quantum corrections
(already proposed by Nyquist)

For the symmetrised form of FDT

$$\Phi_{uu}(\omega) = \frac{1}{C^2}\hbar\chi''_{QQ}(\omega)\coth\left(\frac{\hbar\omega}{2k_BT}\right)$$

$$= R\hbar\omega\coth\left(\frac{\hbar\omega}{2k_BT}\right)$$

$$= 2R\left[\frac{1}{2}\hbar\omega + \frac{\hbar\omega}{\exp\left[\frac{\hbar\omega}{k_BT}\right]-1}\right]$$

↑ zero-point fluctuations ← Bose distribution

↓ this term was actually missed by Nyquist, already appears in an earlier work by Planck (1911)

- 2nd term can be integrated; first term still divergent!

2 options: (1) $R(\omega) \equiv$ const is incorrect (just like ω-independent damping is incorrect for a damped oscillator)

or (2) symmetrised form cannot be used — depends on details of measurement