$$\ldots = (\pm)^{k-1}\,\mathrm{Tr}\Big[\hat{\psi}(x_k)\,\mathcal{T}\Big\{e^{-i\int_\gamma d\bar z\,\hat H(\bar z)}\,\psi(1)\ldots\psi(k-1)\,\psi(k+1)\ldots\hat\psi^\dagger(1')\Big\}\Big]$$

cyclic permutation under Tr

$$= (\pm)^{k-1}\,\mathrm{Tr}\Big[\mathcal{T}\Big\{e^{-i\int_\gamma d\bar z\,\hat H(\bar z)}\,\psi(1)\ldots\psi(k-1)\,\hat\psi(k+1)\ldots\hat\psi^\dagger(1')\,\psi(x_k,z_k)\Big\}\Big]$$

(anti-)commute back $(2n-k)$ times

"inserted as earliest"

$$= (\pm)^{k-1}(\pm)^{2n-k}\Big[\mathcal{T}\Big\{e^{-i\int_\gamma d\bar z\,\hat H(\bar z)}\,\psi(1)\ldots\psi(k-1)\,\psi(x_k,z_k)\,\psi(k+1)\ldots\psi^\dagger(1')\Big\}$$

//

IV.4 Truncation of the hierarchy

Goal: Motivation of self-energy Σ to truncate the infinite hierarchy

$\Rightarrow$ basis for many-body perturbation theory and Feynman diagrams

One-particle Green's function $G(1;2) \equiv G_1(1;2)$

$$\boxed{\begin{aligned}\Big[i\frac{d}{dz_1}-h(1)\Big]G(1;1') &= \delta(1;1') \pm i\int d2\; v(1;2)\,G_2(1,2;1',2^+)\\ G(1;1')\Big[-i\frac{\overleftarrow{d}}{dz_1'}-h(1')\Big] &= \delta(1;1') \pm i\int d2\; v(1';2)\,G_2(1,2^-;1',2)\end{aligned}}$$

Equations of motion for G (EOMs)

For G_2 we have the EOMs

$$\left[i\frac{d}{dz_1}-h(1)\right]G_2(1,2;1',2')=\delta(1;1')G(2;2')\pm\delta(1;2')G(2;1')$$
$$\pm i\int d3\, v(1;3)\, G_3(1,2,3;1',2',3^+)$$

$$\left[i\frac{d}{dz_2}-h(2)\right]G_2(1,2;1',2')=\pm\delta(2;1')G(1;2')+\delta(2;2')G(1;1')$$
$$\pm i\int d3\, v(2;3)\, G_3(1,2,3;1',2',3^+)$$

$$G_2(1,2;1',2')\left[-i\frac{\overleftarrow{d}}{dz_1'}-h(1')\right]=\delta(1;1')G(2;2')\pm\delta(2;1')G(1;2')$$
$$\pm i\int d3\, v(1';3)\, G_3(1,2,3^-;1',2',3)$$

$$G_2(1,2;1',2')\left[-i\frac{\overleftarrow{d}}{dz_2'}-h(2')\right]=\pm\delta(1;2')G(2;1')+\delta(2;2')G(1;1')$$
$$\pm i\int d3\, v(2';3)\, G_3(1,2,3^-;1',2',3)$$

Looking at δ-functions we write

$$\boxed{G_2(1,2;1',2')=G(1;1')G(2;2')\pm G(1;2')G(2;1')+\underbrace{\Upsilon(1,2;1',2')}_{\text{correlation function }\Upsilon}}$$

check: for $v=0$ this G_2 with $\Upsilon=0$ satisfies the EOMs.

For example:

$$\left[i\frac{d}{dz_1}-h(1)\right]\Big(G(1;1')G(2;2')\pm G(1;2')G(2;1')\Big)=\delta(1;1')\delta(2;2')$$
$$\pm\delta(1;2')\delta(2;1')$$

is fulfilled by G for $v=0$.

G_2 also fulfills the KMS boundary conditions when G does.

The approximation

$$G_2(1,2;1',2') = G(1,1')\,G(2;2') \pm G(1;2')\,G(2,1')$$

is called <u>Hartree-Fock approximation</u> for G_2.

Inserting the Hartree-Fock approximation into the EOMs for G:

$$(*)\quad \boxed{\begin{aligned} \left[i\frac{d}{dz_1} - h(1)\right] G(1,1') &= \delta(1;1') \pm i\int d2\, v(1;2)\left[G(1;1')G(2;2^+) \pm G(1;2^+)G(2,1')\right] \\ &\equiv \delta(1;1') + \int d2\, \Sigma(1;2)\, G(2;1') \\ G(1,1')\left[-i\frac{\overleftarrow{d}}{dz_1'} - h(1')\right] &= \delta(1,1') \pm i\int d2\, v(1';2)\left[G(1;1')G(2^-;2) \pm G(1;2)G(2^-;1')\right] \\ &\equiv \delta(1;1') + \int d2\, G(1;2)\,\Sigma(2;1') \end{aligned}}$$

Where we have implicitly defined the <u>Hartree-Fock self-energy</u>

$$(**)\quad \boxed{\begin{aligned} \Sigma(1;2) &= \delta(1,2)\,V_H(1) + i\,v(1;2)\,G(1;2^+) \\ \text{with}\quad V_H(1) &= \pm i\int d3\, v(1;3)\,G(3;3^+) = \int dx_3\, v(x_1,x_3,z_1)\,n(x_3,z_1). \end{aligned}}$$

We have used $v(1;3) = \delta(z_1,z_3)\,v(x_1,x_3,z_1)$

$$\pm i\,G(x_3,z_1;x_3,z_1^+) = n(x_3,z_1)$$

V_H: Hartree potential $\hat{=}$ classical electrostatics

e.g. $V_H(x_1,x_3,z_1) = \dfrac{1}{|\vec{r}_1 - \vec{r}_3|}$ for Coulomb interactions.

2nd term in Σ: "Fock" or "exchange potential" — local in time, but nonlocal in space!

No classical interpretation.

The equations (*) and (**) must be solved <u>self-consistently</u>, as $\Sigma = \Sigma[G]$

$\Rightarrow$ nonlinear equations

$\Rightarrow$ nonperturbative in v!

Using the KMS boundary condition the solution can be written in integral form.

<u>Definition:</u> Noninteracting Green's function G_0 via EOMs for $v=0$

$$\left[i\frac{d}{dz_1} - h(1)\right] G_0(1;1') = \delta(1;1')$$

$$G_0(1;1')\left[-i\frac{\overleftarrow{d}}{dz_1'} - h(1')\right] = \delta(1;1')$$

We use the KMS boundary conditions to write

$$\int d1\; G_0(2;1)\left[i\frac{d}{dz_1} - h(1)\right] G(1,1')$$

$$= \int d1\; G_0(2;1)\left[-i\frac{\overleftarrow{d}}{dz_1} - h(1)\right] G(1,1') + \underbrace{i\int dx_1\; G_0(2;x_1,z_1)G(x_1,z_1;1')\Big|_{z_1=z_i}^{z_1=z_f}}_{\text{vanishes because } G \text{ and } G_0 \text{ satisfy KMS}}$$

$$= \int d1\; \delta(2;1)\, G(1,1')$$

$$= G(2;1').$$

Similarly one can show that

$$\int d1'\; G(1,1')\left[-i\frac{\overleftarrow{d}}{dz_1'} - h(1')\right] G_0(1';2) = G(1;2).$$

Multiplying (*) from the right / left with G_0, we obtain

$$\boxed{\begin{aligned} G(1;2) &= G_0(1;2) + \int d3\, d4\; G_0(1;3)\,\Sigma(3;4)\,G(4;2) \\ G(1;2) &= G_0(1;2) + \int d3\, d4\; G(1;3)\,\Sigma(3;4)\,G_0(4;2) \end{aligned}}$$

which are equivalent. <u>Dyson equations</u>

KMS boundary conditions automatically incorporated via G_0

<u>Later:</u> The exact G fulfills the same equation but with a more complicated Σ.

Dyson equation = formal solution of Martin-Schwinger hierarchy for G.

[book (Stefanucci & van Leeuwen: T-matrix approximation]

IV.5 Exact solution of the hierarchy via Wick's theorem

For $v=0$ the hierarchy couples G_n only to G_{n-1}:

$$\left[i\frac{d}{dz_k} - h(k)\right] G_{0,n} = \sum_{j=1}^{n} (\pm)^{k+j}\, \delta(k,j')\, G_{0,n-1}(1,\ldots\hat{k},\ldots n; 1',\ldots\hat{j'}\ldots n')$$

$$G_{0,n}\left[-i\frac{\overleftarrow{d}}{dz_k'} - h(k')\right] = \sum_{j=1}^{n} (\pm)^{k+j}\, \delta(j,k)\, G_{0,n-1}(1,\ldots\hat{j}\ldots n; 1',\ldots\hat{k'}\ldots n')$$

which can be solved exactly. Below we prove that

$$\boxed{G_{0,n}(1,\ldots,n;1',\ldots,n') = \begin{vmatrix} G_0(1;1') & \cdots & G_0(1;n') \\ \vdots & & \\ G_0(n;1') & \cdots & G_0(n;n') \end{vmatrix}_{\pm}}$$

"Wick's theorem"

with $G_0 \equiv G_{0,1}$.

$$|A|_{\pm} \equiv \sum_{P} (\pm)^{P} A_{1P(1)} \cdots A_{nP(n)}$$

is the permanent / determinant of the $n\times n$ matrix A

$\downarrow$ bosons $\quad \downarrow$ fermions

Here $A_{ij} = G_0(i;j)$.

Note that we have already seen that this works for G_2 in the previous section.

General proof: Expand the permanent/determinant along row k

$$G_{0,n}(1,\dots,n;1',\dots,n) = \sum_{j=1}^{n} (\pm)^{k+j}\, G_0(k,j') \, G_{0,n-1}(1,\dots \overline{k} \dots n; 1',\dots \overline{j'} \dots n')$$

Apply $\left[i \frac{d}{dz_k} - h(k) \right]$ from left: This fulfills the EOM for $G_{0,n}$.

Expanding along column k gives the 2nd EOM.

$\Rightarrow$ the expansion in a permanent/determinant of G_0's is a solution of the noninteracting hierarchy.

Also: KMS boundary condition fulfilled by observing that $\pm$-multiplication of a row/column of G_0 in also multiplies the permanent/determinant by the same factor.

$\Rightarrow$ it is sufficient to solve the EOMs for G_0 to obtain all $G_{0,n}$'s!

Remark: This derivation of Wick's theorem is completely general. The expansion appears as a natural solution to a boundary value problem for the Martin-Schwinger hierarchy.

It is also the basis for many-body perturbation theory (MBPT). The interacting Green's function is obtained from a brute-force expansion

$$G_n = \frac{1}{i^n} \frac{\mathrm{Tr}\left[\mathcal{T}\left\{e^{-i\int_\gamma d\bar{z}\, \hat{H}_0(\bar{z})} e^{-i\int_\gamma d\bar{z}\, \hat{H}_{int}(\bar{z})} \hat{\psi}(1) \ldots \hat{\psi}^+(1')\right\}\right]}{\mathrm{Tr}\left[\mathcal{T}\left\{e^{-i\int_\gamma d\bar{z}\, \hat{H}_0(\bar{z})} e^{-i\int_\gamma d\bar{z}\, \hat{H}_{int}(\bar{z})}\right\}\right]}$$

with $\hat{H} = \hat{H}_0 + \hat{H}_{int}$, commuting under $\mathcal{T}$.

<u>Expansion</u> in powers of $\hat{H}_{int}$:

$$G_n = \frac{1}{i^n} \frac{\sum_{k=0}^{\infty} \frac{(-i)^k}{k!} \int_\gamma d\bar{z}_1 \ldots d\bar{z}_k \left\langle \mathcal{T}\left\{\hat{H}_{int}(\bar{z}_1) \ldots \hat{H}_{int}(\bar{z}_k)\, \hat{\psi}(1) \ldots \hat{\psi}^+(1')\right\}\right\rangle_0}{\sum_{k=0}^{\infty} \frac{(-i)^k}{k!} \int_\gamma d\bar{z}_1 \ldots d\bar{z}_k \left\langle \mathcal{T}\left\{\hat{H}_{int}(\bar{z}_1) \ldots \hat{H}_{int}(\bar{z}_k)\right\}\right\rangle_0}$$

with $\langle \mathcal{T}\{\ldots\}\rangle_0 \equiv \mathrm{Tr}\left[\mathcal{T}\left\{e^{-i\int_\gamma d\bar{z}\, \hat{H}_0(\bar{z})}\right\}\right]$

Consider a general two-body interaction:

$$\hat{H}_{int}(z) = \frac{1}{2}\int dz' \int dx\, dx'\, v(x,z;x',z')\, \hat{\psi}^+(x,z^+)\, \hat{\psi}^+(x',z'^+)\, \hat{\psi}(x',z')\, \hat{\psi}(x,z)$$

with $v(x,z;x',z') = \delta(z,z') \begin{cases} v(x,x',t) & \text{if } z = t_\pm \\ v^M(x,x') & \text{if } z = -i\tau \end{cases}$.

Using $a \equiv (x_a, z_a)$ and $b \equiv (x_b, z_b)$ and renaming $\bar{z}_k \to z_k$ we have for $n=1$

$$G(a,b) = \frac{1}{i} \frac{\sum_{k=0}^{\infty} \frac{(-i)^k}{k!} \int_\gamma dz_1 \ldots dz_k \left\langle \mathcal{T}\left\{\hat{H}_{int}(z_1) \ldots \hat{H}_{int}(z_k)\, \hat{\psi}(a)\, \hat{\psi}^+(b)\right\}\right\rangle_0}{\text{same without } \hat{\psi}(a)\, \hat{\psi}^+(b)}$$

Now we write the numerator using $\hat{H}_{int}$:

$$\sum_{k=0}^{\infty} \frac{1}{k!}\left(-\frac{i}{2}\right)^k \int d1 \dots dk\, d1' \dots dk'\, v(1;1') \dots v(k;k') \times$$

$$\times \langle T \{ \hat\varphi^+(1^+) \hat\varphi^+(1'^+) \hat\varphi(1') \hat\varphi(1) \dots \hat\varphi^+(k^+) \dots \hat\varphi^+(k'^+) \hat\varphi(k') \hat\varphi(k)\, \hat\varphi(a) \hat\varphi^+(b) \} \rangle_0$$

reordering:

$$\langle T \{ \hat\varphi(a) \hat\varphi(1) \hat\varphi(1') \dots \hat\varphi(k) \hat\varphi(k') \hat\varphi^+(k'^+) \hat\varphi^+(k) \dots \hat\varphi^+(1'^+) \hat\varphi^+(1^+) \hat\varphi^+(b) \} \rangle_0$$

Which is $\quad i^{2k+1} Z_0\, G_{0,2k+1}(a,1,1',\dots,k,k';b,1^+,1'^+,\dots,k^+,k'^+)$

and in the denominator the k-th order term is identified as

$$i^{2k} Z_0\, G_{0,2k}(1,1',\dots,k,k';1^+,1'^+,\dots,k^+,k'^+)$$

$$\Rightarrow G(a;b) = \frac{\sum_{k=0}^{\infty} \frac{1}{k!}\left(\frac{i}{2}\right)^k \int v(1;1') \dots v(k;k')\, G_{0,2k+1}(a,1,1',\dots;b,1^+,1'^+,\dots)}{\sum_{k=0}^{\infty} \frac{1}{k!}\left(\frac{i}{2}\right)^k \int v(1;1') \dots v(k;k')\, G_{0,2k}(1,1',\dots;1^+,1'^+,\dots)}$$

where $\quad \int \equiv \int d1 \dots dk\, d1' \dots dk'$

Using Wick's theorem we arrive at

$$\boxed{G(a;b) = \frac{\sum_{k=0}^{\infty} \frac{1}{k!}\left(\frac{i}{2}\right)^k \int v(1;1') \dots v(k;k') \begin{vmatrix} G_0(a;b) & G_0(a;1^+) & \dots & G_0(a;k'^+) \\ G_0(1;b) & G_0(1;1^+) & \dots & G_0(1;k'^+) \\ \vdots & \vdots & & \vdots \\ G_0(k';b) & G_0(k';1^+) & \dots & G_0(k';k'^+) \end{vmatrix}_\pm}{\sum_{k=0}^{\infty} \frac{1}{k!}\left(\frac{i}{2}\right)^k \int v(1;1') \dots v(k;k') \begin{vmatrix} G_0(1;1^+) & G_0(1;1'^+) & \dots & G_0(1;k'^+) \\ G_0(1';1^+) & G_0(1';1'^+) & \dots & G_0(1';k'^+) \\ \vdots & \vdots & & \vdots \\ G_0(k';1^+) & G_0(k';1'^+) & \dots & G_0(k';k'^+) \end{vmatrix}_\pm}}$$

exact expansion of G in terms of G_0.

IV.6 Many-body perturbation theory for the Green's function (MBPT)

Starting point: Expansion of interacting G in terms of noninteracting G_0

$$G(a;b) = \frac{\sum_{k=0}^{\infty} \frac{1}{k!}\left(\frac{i}{2}\right)^k \int v(1;1')\dots v(k;k') \begin{vmatrix} G_0(a;b) & G_0(a;1^+) & \dots & G_0(a;k'^+) \\ G_0(1;b) & G_0(1;1^+) & \dots & G_0(1;k'^+) \\ \vdots & \vdots & & \vdots \\ G_0(k';b) & G_0(k';1^+) & \dots & G_0(k';k'^+) \end{vmatrix}_\pm}{\sum_{k=0}^{\infty} \frac{1}{k!}\left(\frac{i}{2}\right)^k \int v(1;1')\dots v(k;k') \begin{vmatrix} G_0(1;1^+) & G_0(1;1'^+) & \dots & G_0(1;k'^+) \\ G_0(1';1^+) & G_0(1';1'^+) & \dots & G_0(1';k'^+) \\ \vdots & \vdots & & \vdots \\ G_0(k';1^+) & G_0(k';1'^+) & \dots & G_0(k';k'^+) \end{vmatrix}_\pm} \qquad (*)$$

Feynman 1948: invention of Feynman diagrams to represent each term in (*) graphically

IV.6.1 Getting started with Feynman diagrams

Denominator of (*) = ratio $\frac{Z}{Z_0}$

First order:

$$\left(\frac{Z}{Z_0}\right)^{(1)} = \frac{i}{2}\int d1\,d1'\, v(1;1') \begin{vmatrix} G_0(1;1^+) & G_0(1;1'^+) \\ G_0(1';1^+) & G_0(1';1'^+) \end{vmatrix}_\pm$$

$$= \frac{i}{2}\int d1\,d1'\, v(1;1')\left[G_0(1;1^+)G_0(1';1'^+) \pm G_0(1;1'^+)G_0(1';1^+)\right]$$

Basic Feynman diagrams:

$G_0(1;2^+) = 1 \longleftarrow 2$ direction to distinguish from $G_0(2;1^+)$

destroy at 1   create at 2

"+" only matters if $z_1 = z_2$

$v(1;2) = 1 \sim\sim\sim\sim 2 = v(2;1)$

Two terms in $\left(\frac{Z}{Z_0}\right)^{(1)}$:

[integration over internal vertices 1 and 1']

Diagrams correspond to Hartree-Fock approximation but with $G \to G_0$

> From now on: When we omit the "+" shift, if a diagram contains a G with the same contour-time arguments then the second argument is understood to be infinitesimally later than the first.

The prefactor is $\left(\frac{i}{2}\right)$ for the first diagram and $\left(\pm\frac{i}{2}\right)$ for the second diagram.

2nd order: perm./det of 4x4 matrix

$\Rightarrow 4! = 24$ terms

$$\left(\frac{Z}{Z_0}\right)^{(2)} = \frac{1}{2!}\frac{i^2}{2^2}\int d1\,d1'\,d2\,d2'\;v(1;1')\,v(2;2')\;\times$$

$$\times \sum_P (\pm)^P\, G_0(1;P(1))\,G_0(1';P(1'))\,G_0(2;P(2))\,G_0(2';P(2'))$$

<u>Vacuum diagrams:</u> only internal vertices, that are integrated over. "closed diagrams"

<u>Rules to convert vacuum diagrams into mathematical expressions:</u>

- number all vertices and assign an interaction line $v(i;j)$ to a wiggly line between i and j and a Green's function $G_0(i;j^+)$ to an oriented line from j to i
- integrate over all vertices and multiply by $\left[(\pm)^P \frac{1}{k!}\left(\frac{i}{2}\right)^k\right]$, where $(\pm)^P$ is the sign of the permutation and k is the number of interaction lines

<u>non-vacuum diagrams:</u>

Look at numerator of $(*)$, $N(a;b)$.

First order: 3×3 matrix

$$N^{(1)}(a;b) = \frac{i}{2} G_0(a;b) \int d1\, d1'\, v(1;1') \begin{vmatrix} G_0(1;1^+) & G_0(1;1'^+) \\ G_0(1';1^+) & G_0(1';1'^+) \end{vmatrix}_\pm$$

$$\pm \frac{i}{2} \int d1\, d1'\, v(1;1') G_0(1;b) \begin{vmatrix} G_0(a;1^+) & G_0(a;1'^+) \\ G_0(1';1^+) & G_0(1';1'^+) \end{vmatrix}_\pm$$

$$\pm \frac{i}{2} \int d1\, d1'\, v(1;1') G_0(1';b) \begin{vmatrix} G_0(a;1^+) & G_0(a;1'^+) \\ G_0(1;1^+) & G_0(1;1'^+) \end{vmatrix}_\pm$$

- first line: $\left(\frac{Z}{Z_0}\right)^{(1)}$ times $G_0(a;b)$

$\Rightarrow$ same diagrams with extra line $b \rightarrow a$

- remaining terms can be drawn similarly.

$(b, 1, 1')$

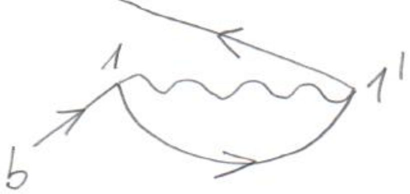
generating permutation
(for 2nd indices in G_0's)

$(b, 1', 1)$

interchanges

$(1, b, 1')$

$(1', 1, b)$



$(1', b, 1)$

$(1, 1', b)$

<u>Green's function diagrams</u> (no integration over external vertices)

How to simply determine the sign of a permutation?

IV.6.2 Loop rule

Sign $\Longleftrightarrow$ sign of permutation that changes the 2nd argument (starting point) of the corresponding Green's function

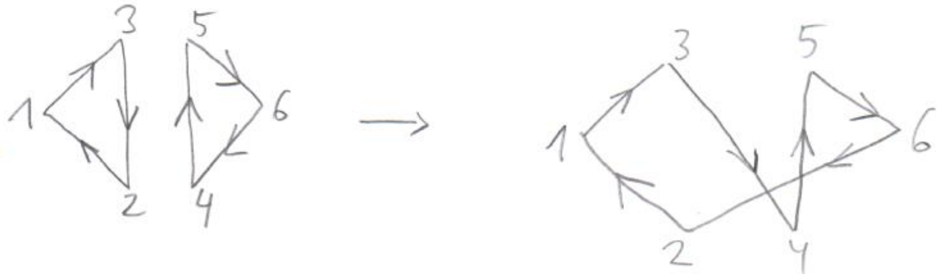
Vacuum diagram consists of loops:

interchange of labels (i,j) can occur either between two starting points of the same loop or between two starting points of different loops.

Case (i): same loop – interchange 2 and 4

$\Rightarrow$ # loops increases by 1

Case (ii): different loops – interchange 3 and 6

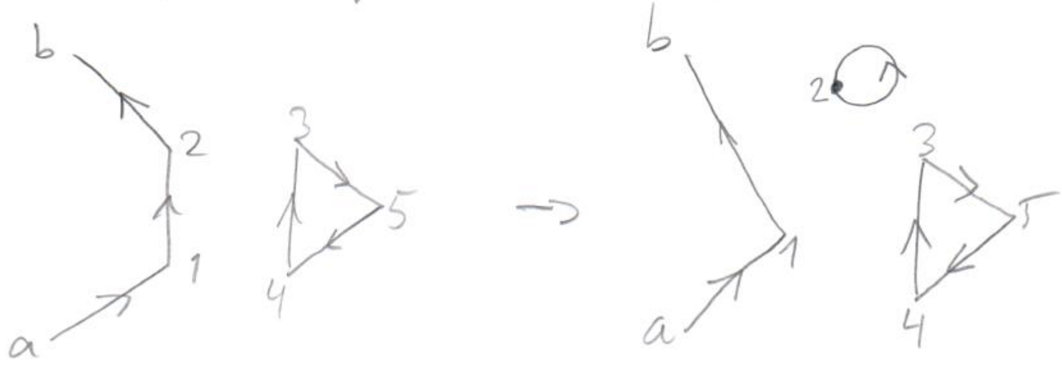


$\Rightarrow$ # loops <u>decreases</u> by 1

<u>General rule:</u> # loops changes by ± 1 for one interchange of starting points

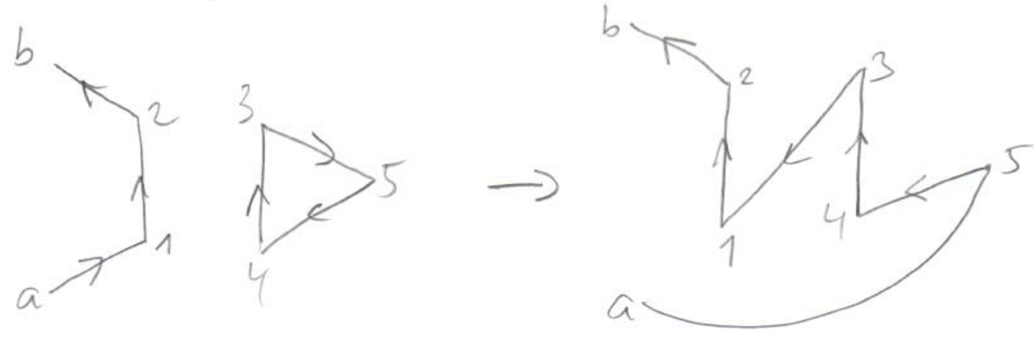
G-diagrams: two more possibilities

(iii) interchange between starting point on a–b path



interchange 1 and 2 $\Rightarrow$ # loops += 1

(iv) interchange between starting point on a-b path and starting point on a loop



interchange a and 3 $\Rightarrow$ # loops −= 1

identity permutation: sign +

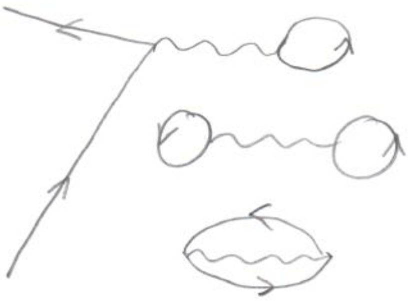
$\Rightarrow$ $\boxed{\text{loop rule: } (\pm)^P = (\pm)^l \text{ with } l = \#\text{loops}}$

IV.6.3 Cancellation of disconnected diagrams

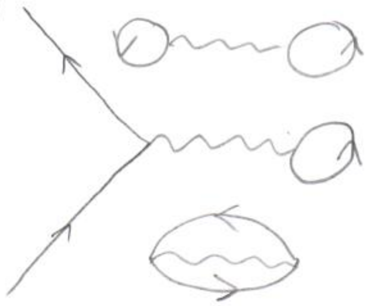
The disconnected vacuum diagrams in the numerator are exactly cancelled by the vacuum diagrams in the denominator.

Example: Some 3rd-order terms of the numerator

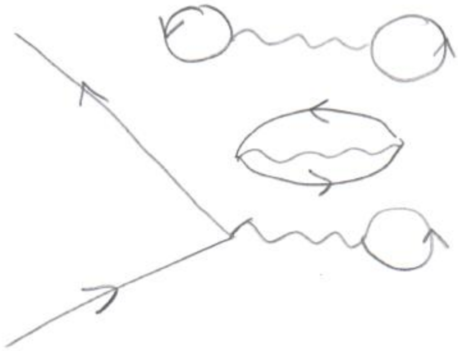
(1)



(2)



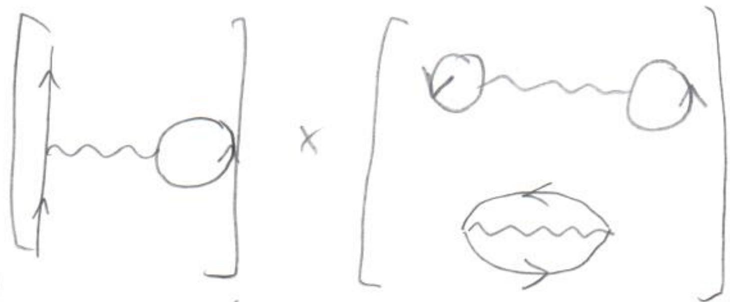
(3)



(1), (2), (3) related by permutation of interaction lines $v(i;i') \leftrightarrow v(j;j')$ that preserve the structure of disjoint pieces. They also all have the same prefactor, $(\pm)^4 \frac{1}{3!}\left(\frac{i}{2}\right)^3$. Total contribution:

$$3 \times \frac{1}{3!}\left(\frac{i}{2}\right)^3 \underbrace{\int[G_0 G_0 G_0 v]}_{G\text{-diagram}} \underbrace{\int[G_0 G_0 G_0 G_0 v v]}_{\text{vacuum diagram}}$$

Consider the product of diagrams



$$\underbrace{-\frac{i}{2}\int\left[G_0 G_0 G_0 V\right] \quad -\frac{1}{2!}\left(\frac{i}{2}\right)^2\int\left[G_0 G_0 G_0 G_0 VV\right]}$$

= same as the sum of the 3 diagrams above!

this argument can be generalized to all diagrams!
[see Stefanucci & van Leeuwen, Chapter 10.3]

Denote by $G_C^{(n)}(a;b)$ the sum of all n-th order connected diagrams of $N(a;b)$:

$$G_C^{(n)}(a;b) = \frac{1}{n!}\left(\frac{i}{2}\right)^n \sum_{\substack{i = \text{G-connected} \\ \text{diagrams}}} (\pm)^{\ell_i}\, G_{C,i}^{(n)}(a;b)$$

$$\Rightarrow N(a;b) = \sum_{k=0}^{\infty}\sum_{n=0}^{k} G_C^{(n)}(a;b)\left(\frac{Z}{Z_0}\right)^{(k-n)} = \sum_{n=0}^{\infty}\sum_{k=n}^{\infty} G_C^{(n)}(a;b)\left(\frac{Z}{Z_0}\right)^{(k-n)}$$

just shift of indices

$$= \left(\frac{Z}{Z_0}\right)\sum_{n=0}^{\infty} G_C^{(n)}(a;b)$$

Result:

$$\boxed{G(a;b) = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{i}{2}\right)^n \int v(1;1')\dots v(n;n') \begin{vmatrix} G_0(a;b) & G_0(a;1^+) & \dots & G_0(a;n'^+) \\ G_0(1;b) & G_0(1;1^+) & \dots & G_0(1;n'^+) \\ \vdots & \vdots & & \vdots \\ G_0(n';b) & G_0(n';1^+) & \dots & G_0(n';n'^+) \end{vmatrix}_{\pm, c}}$$

= Sum of connected diagrams

IV.6.4 Summing only topologically inequivalent diagrams

[details see Stefanucci & van Leeuwen, Chapter 10.4]

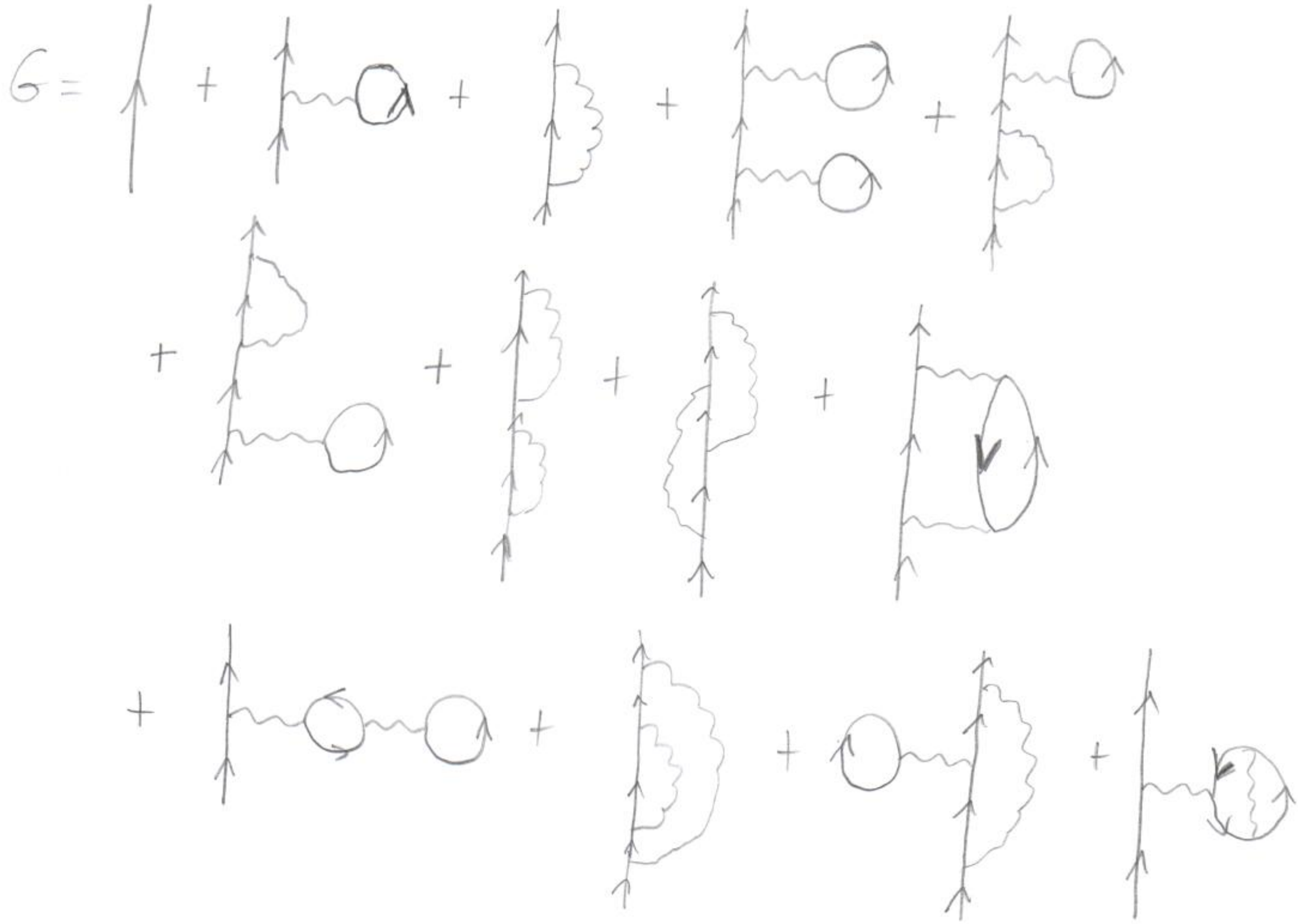
$$G(a;b) = \sum_{n=0}^{\infty} i^n \int v(1;1')\dots v(n;n') \begin{vmatrix} G_0(a,b) & G_0(a;1^+) & \dots & G_0(a;n'^+) \\ G_0(1,b) & G_0(1;1^+) & \dots & G_0(1;n'^+) \\ \vdots & \vdots & & \vdots \\ G_0(n',b) & G_0(n',1^+) & \dots & G_0(n';n'^+) \end{vmatrix}_{\pm, c, t.i.}$$

topologically inequivalent

$\hat{=}$ cannot be continously deformed into each other by mirroring interaction lines or interchanging interaction lines

New rules:
- number all vertices, assign G and v
- integrate over internal vertices and multiply by $(i)^n (\pm)^l$ n = # interaction lines, l = # loops

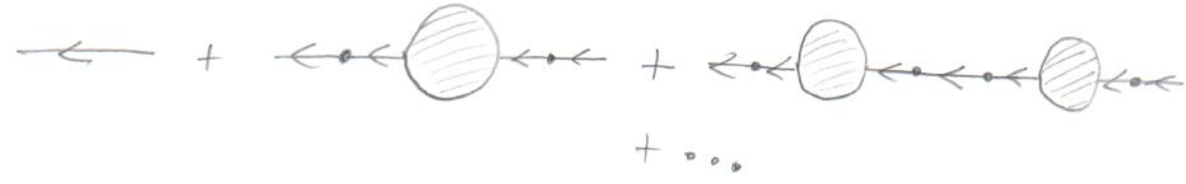
Second order expansion for G in terms of G_0:



$+ \mathcal{O}(V^3)$

IV. 6.5 Self-energy and Dyson equations

General form of G:

$G =$ 

where the self-energy

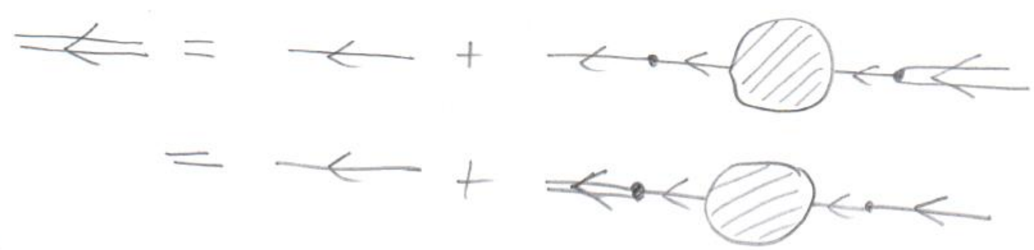
$\Sigma(1;2) \equiv$ 

= sum of all diagrams that do not break into disconnected pieces by cutting a single G_0-line.

= sum of all one-particle irreducible diagrams.

$$\Sigma = \Sigma[G_0; v]$$

interacting $G(1;2) =$ 

$\Rightarrow$ 

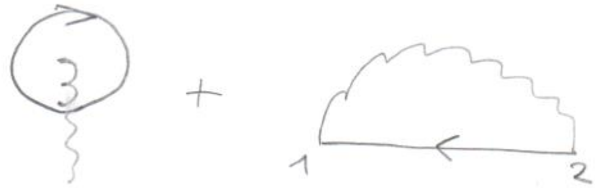
$$G(1;2) = G_0(1;2) + \int d3\, d4\; G_0(1;3)\,\Sigma(3;4)\,G(4;2)$$

$$= G_0(1;2) + \int d3\, d4\; G(1;3)\,\Sigma(3;4)\,G_0(4;2)$$

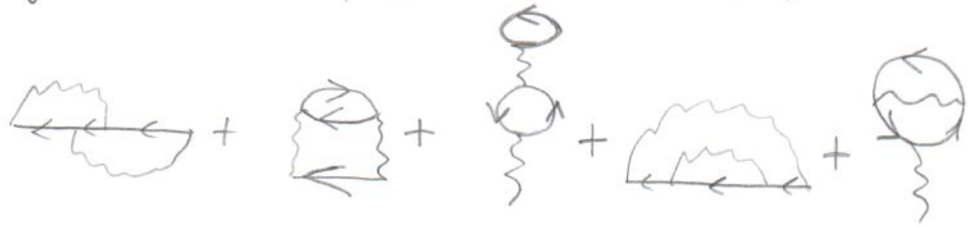
which we had derived before in Chapter IV.4

$\Rightarrow$ Dyson equation! [But now with MBPT for Σ]

Diagrams for Σ: same rules as for G

$\Sigma^{(1)}(1;2) =$  Hartree-Fock! (with $G \to G_0$)

$$= \pm i\delta(1;2)\int d3\, v(1;3)\, G_0(3;3^+) + i v(1;2)\, G_0(1;2^+)$$

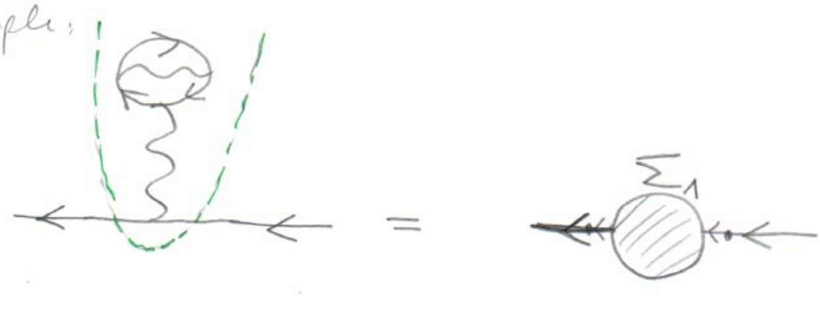
$\Sigma^{(2)}(1;2) =$ 

- All self-energies which have the same interaction vertex at start & end have a δ-function, like the Hartree term.
- 2nd-order for Σ: only 6 terms ($G^{(2)}$ has 10)

IV. 6.6 G-skeleton diagrams

G-skeleton diagram: remove all Σ-insertions from a diagram
Σ-insertion: piece that can be cut away by cutting <u>two</u> G-lines.

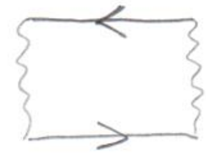
For example:

(1) 

(2) 

Corresponding G-skeleton diagrams:

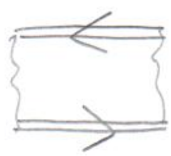
(1) $\Rightarrow$ 

(2) $\Rightarrow$ 

G-skeleton diagrams $\Rightarrow$ express Σ in terms of the <u>interacting</u> (dressed) Green's function G rather than the noninteracting (bare) Green's function G_0!

Summing over all possible insertions of self-energies:

(1) $\Rightarrow$ 

(2) $\Rightarrow$ 

yields

$$\boxed{\begin{array}{l} \Sigma = \Sigma_s [G, v] \\ = \text{functional of the interaction } v \\ \quad \text{and the dressed Green's function } G \end{array}}$$

"s" = skeleton

$$\Sigma[G_0, v] = \Sigma_s[G, v]$$

~~Next Section: how to construct the functional $\Phi[G]$,~~

~~"$\Sigma = \frac{\delta \Phi}{\delta G}$" exact self-energy~~

up to 2nd-order in v self-energy using G-skeleton diagrams:

$$\Sigma_s\,[G,v] =$$ 

only 2 in second order compared to 6 bare ones

("second-Born approximation")

Given approximation to Σ $\Rightarrow$ approximate G by solving the Dyson equation

$$G(1;2) = G_0(1;2) + \int d3\, d4\; G_0(1;3)\,\Sigma_s[G,v]\,G(4;2)$$

$\Rightarrow$ <u>nonlinear</u> integral equation!

$\Rightarrow$ solved <u>self-consistently</u> $\quad \Sigma \to G \to \Sigma \to G \to \ldots$

$\Rightarrow$ exercise: mean-field antiferromagnetism $\hat{=}$ self-consistent Hartree approximation

<u>Integro-differential form:</u> Kadanoff-Baym equations with KMS boundary conditions

$$\left[i\frac{d}{dz_1} - h(z_1)\right] G(z_1,z_2) = \delta(z_1,z_2) + \int_\gamma dz\; \Sigma(z_1,z)\,G(z,z_2)$$

$$G(z_1,z_2)\left[-i\frac{\overleftarrow{d}}{dz_2} - h(z_2)\right] = \delta(z_1,z_2) + \int_\gamma dz\; G(z_1,z)\,\Sigma(z,z_2)$$