Start lecture 3

Let $N \to \infty$ keeping $t = N\Delta t$ fixed:

Dense steps in time interval $[0, t]$,

continuous phase-space curve $x(t)$.

$$\Delta t \sum_{n=0}^{N-1} \mapsto \int_0^t dt' \,, \quad \frac{q_{n+1} - q_n}{\Delta t} \mapsto \partial_{t'} q\big|_{t'=t_n} \equiv \dot{q}\big|_{t'=t_n}$$

$$\left[V(q_n) + T(p_{n+1})\right] \mapsto \left[T(p|_{t'=t_n}) + V(q|_{t'=t_n})\right] \equiv H(x|_{t'=t_n})$$

$=$ classical Hamiltonian

$$\lim_{N\to\infty} \int \prod_{\substack{n=1 \\ q_0 = q_i \\ q_N = q_f}}^{N-1} dq_n \prod_{n=1}^{N} dp_n/2\pi\hbar \equiv \int_{\substack{q(t)=q_f \\ q(0)=q_i}} \mathcal{D}x \qquad (****)$$

Note: This integral measure is divergent. Tough problem! In practice this will not cause problems for our purposes though.

Using $(****)$ in $(***)$ we finally obtain

$$\boxed{\langle q_f | e^{-i\hat{H}t/\hbar} | q_i \rangle = \int_{\substack{q(t)=q_f \\ q(0)=q_i}} \mathcal{D}x \, \exp\left[\frac{i}{\hbar}\int_0^t dt' \left(p\dot{q} - H(p,q)\right)\right]}$$

This is the <u>Hamiltonian formulation</u> of the path integral.

For $T(p)$ with quadratic p-dependence, Gaussian integration leads to the Lagrangian form:

$$\boxed{\langle q_f | e^{-i\hat{H}t/\hbar} | q_i \rangle} = \int\limits_{\substack{q(t)=q_f \\ q(0)=q_i}} \mathcal{D}q \, e^{-\frac{i}{\hbar}\int_0^t dt' \, V(q)} \int \mathcal{D}p \, e^{-\frac{i}{\hbar}\int_0^t dt' \left(\frac{p^2}{2m} - p\dot{q}\right)}$$

Gaussian integral, see below

$$\boxed{= \int\limits_{\substack{q(t)=q_f \\ q(0)=q_i}} \mathcal{D}q \, \exp\left[\frac{i}{\hbar}\int_0^t dt' \, L(q,\dot{q})\right]}$$

with $\mathcal{D}q = \lim\limits_{N\to\infty} \left(\frac{Nm}{it\,2\pi\hbar}\right)^{N/2} \prod\limits_{n=1}^{N-1} dq_n$ the functional measure

and $L(q,\dot{q}) = \frac{m\dot{q}^2}{2} - V(q)$ the classical Lagrangian

skipped

II.3 Important Gaussian integrals

(i) 1D Gaussian integral

$$\boxed{\int_{-\infty}^{\infty} dx \, e^{-\frac{a}{2}x^2} = \sqrt{\frac{2\pi}{a}} \, , \quad \mathrm{Re}\, a > 0}$$

$\rightarrow$ normalization of Gaussian integrals

$$\boxed{\int_{-\infty}^{\infty} dx \, x^2 e^{-\frac{a}{2}x^2} = \sqrt{\frac{2\pi}{a^3}}}$$

$\rightarrow$ variance of Gaussian distributions

$$\boxed{\int_{-\infty}^{\infty} dx \; e^{-\frac{a}{2}x^2 + bx} = \sqrt{\frac{2\pi}{a}}\, e^{b^2/2a}}$$

shifted Gaussian integral
(change of variables: $x \to x + \frac{b}{a}$)

With complex arguments:

$$\boxed{\int d(z,\bar{z})\, e^{-\bar{z}wz} = \frac{\pi}{w}} \quad , \; \mathrm{Re}\, w > 0$$

where $\bar{z}$ is the complex conjugate of z.

Here, $\int d(z,\bar{z}) \equiv \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy$ with $z \doteq x + iy$

$z\bar{z} = x^2 + y^2$

$\Rightarrow$ two independent real integrals

Similarly, $\int d(z,\bar{z})\, e^{-\bar{z}wz + \bar{u}z + v\bar{z}} = \frac{\pi}{w} e^{\frac{\bar{u}v}{w}}$, $\mathrm{Re}\, w > 0$.

(ii) higher-dimensional Gaussian integrals

(a) real case:

$$\boxed{\int d\underline{v}\; e^{-\frac{1}{2}\underline{v}^T \underline{\underline{A}}\, \underline{v}} = (2\pi)^{N/2} (\det \underline{\underline{A}})^{-1/2}}$$

$\underline{\underline{A}}$ = positive definite real symmetric $N \times N$ matrix

$\underline{v}$ = N-component real vector

Proof: use $\underline{\underline{A}} = \underline{\underline{O}}^T \underline{\underline{D}}\, \underline{\underline{O}}$ with $\det \underline{\underline{O}} = 1$

Similarly

$$(*) \quad \boxed{\int d\underline{v}\; e^{-\frac{1}{2}\underline{v}^T \underline{\underline{A}}\,\underline{v} + \underline{j}^T \underline{v}} = (2\pi)^{N/2} (\det \underline{\underline{A}})^{-1/2} e^{\frac{1}{2} \underline{j}^T \underline{\underline{A}}^{-1} \underline{j}}}$$

Proof: shift $\underline{v} \to \underline{v} + \underline{\underline{A}}^{-1} \underline{j}$

This integral also generates other useful identities:

$$\partial^2_{j_m j_n} \Big|_{\underline{j}=0} (*)$$

$$\Rightarrow \int d\underline{v}\; e^{-\frac{1}{2}\underline{v}^T \underline{\underline{A}}\,\underline{v}}\, v_m v_n = (2\pi)^{N/2} (\det \underline{\underline{A}})^{-1/2} \left(\underline{\underline{A}}^{-1}\right)_{mn}$$

$$\Rightarrow \quad \langle v_m v_n \rangle = \left(\underline{\underline{A}}\right)^{-1}_{mn} \equiv A^{-1}_{mn}$$

$$\text{with} \quad \langle \ldots \rangle \equiv (2\pi)^{-N/2} (\det \underline{\underline{A}})^{1/2} \int d\underline{v}\; e^{-\frac{1}{2}\underline{v}^T \underline{\underline{A}}\,\underline{v}} (\ldots)$$

"probability distribution"

Four times:

$$\boxed{\langle v_m v_n v_q v_p \rangle = A^{-1}_{mn} A^{-1}_{qp} + A^{-1}_{mq} A^{-1}_{np} + A^{-1}_{mp} A^{-1}_{nq}}$$

$\Rightarrow$ <u>Wick's theorem</u> (for real bosonic fields)

"sum over all pairings"

More generally:

$$\langle v_{i_1} v_{i_2} \ldots v_{i_{2n}} \rangle = \sum_{\substack{\text{pairings of} \\ \{i_1, \ldots, i_{2N}\}}} A^{-1}_{i_{k_1} i_{k_2}} \ldots A^{-1}_{i_{k_{2n-1}} i_{k_{2n}}}$$

b) complex case

$$\boxed{\int d(\underline{v}^+,\underline{v})\, e^{-\underline{v}^+\underline{\underline{A}}\,\underline{v}} = \pi^N (\det \underline{\underline{A}})^{-1}}$$

with $d(\underline{v}^+,\underline{v}) = \prod_{i=1}^{N} d\,\mathrm{Re}\, v_i \; d\,\mathrm{Im}\, v_i$

$\underline{\underline{A}}$: complex $N \times N$ matrix with positive definite Hermitian part.

$$\underline{\underline{A}} = \underline{\underline{A}}^{herm} + \underline{\underline{A}}^{antiherm}$$

$$\underline{\underline{A}}^{herm} = \frac{1}{2}(\underline{\underline{A}} + \underline{\underline{A}}^+)$$

$$\underline{\underline{A}}^{antiherm} = \frac{1}{2}(\underline{\underline{A}} - \underline{\underline{A}}^+)$$

with linear contribution:

$$\int d(\underline{v}^+,\underline{v})\, e^{-\underline{v}^+\underline{\underline{A}}\,\underline{v} + \underline{w}^+\underline{v} + \underline{v}^+\cdot\underline{w}'} = \pi^N (\det \underline{\underline{A}})^{-1} e^{\underline{w}^+\underline{\underline{A}}^{-1}\underline{w}'}$$

Expectation values: Apply $\partial^2_{w'_m, \overline{w}_n}\big|_{\underline{w}=\underline{w}'=0}$

$$\langle \overline{v}_m v_n \rangle = A^{-1}_{nm}$$

$$\langle \ldots \rangle \equiv \pi^{-N} \det \underline{\underline{A}} \int d(\underline{v}^+,\underline{v})\, e^{-\underline{v}^+\underline{\underline{A}}\,\underline{v}} (\ldots)$$

$$\boxed{\langle \overline{v}_{i_1} \overline{v}_{i_2} \ldots \overline{v}_{i_n} \, v_{j_1} v_{j_2} \ldots v_{j_n} \rangle = \sum_P A^{-1}_{j_1 i_{P_1}} \ldots A^{-1}_{j_n i_{P_n}}}$$

Complex case: only pairings of complex conjugates

e.g. $\langle \overline{v}_n \overline{v}_m v_p v_q \rangle = A^{-1}_{pm} A^{-1}_{qn} + A^{-1}_{pn} A^{-1}_{qm}$

Gaussian functional integration

Starting from real version

$$(*) \quad \int d\underline{v}\; e^{-\frac{1}{2}\underline{v}^T \underline{\underline{A}}\,\underline{v} + \underline{j}^T \underline{v}} = (2\pi)^{N/2} (\det \underline{\underline{A}})^{-1} e^{\frac{1}{2}\underline{j}^T \underline{\underline{A}}^{-1} \underline{j}}$$

we go to the continuum limit

$$\{v_i\} \longrightarrow v(x)$$

$$A_{ij} \longrightarrow \text{operator kernel or propagator } A(x,x').$$

$$(*) \Rightarrow \int \mathcal{D}v(x)\; \exp\left[-\frac{1}{2}\int dx \int dx'\; v(x) A(x,x') v(x') + \int dx\; j(x) v(x)\right]$$

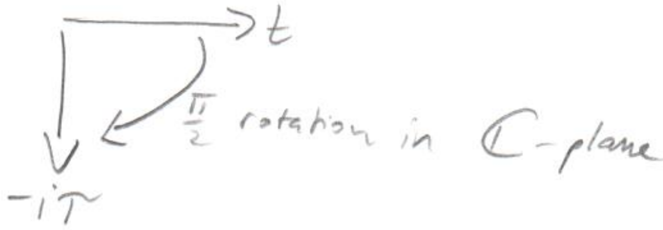
$$\propto (\det A)^{-1/2} \exp\left[\frac{1}{2}\int dx \int dx'\; j(x) A^{-1}(x,x') j(x')\right]$$

with $\boxed{\int dx'\; A(x,x') A^{-1}(x',x'') = \delta(x-x'')}$ ("identity matrix")

Remarks:
- $A^{-1}(x,x')$ is the Green's function of $A(x,x')$
- $\mathcal{D}v(x)$ is the measure of integration. $(2\pi)^N$ is formally divergent for $N\to\infty$ but can be left out since we only need derivatives of the functional integral in the end.

skipped:

Nomenclature: • $t \to -i\tau$ "Wick rotation"



• imaginary time Lagrangian actions = "Euclidean actions"

• real time Lagrangian actions = "Minkowski actions"

Semiclassical limit

"$\hbar \to 0$" = "semiclassical limit"

Classical solutions: extremal path configurations, i.e.

Euler-Lagrange equation:

Lagrangian $\delta S[q]=0 \Rightarrow (d_t \partial_{\dot q} - \partial_q) L(q,\dot q) = 0$

$\Rightarrow$ "stationary phase" for the functional integral

next step: Saddle point approximation

= Gaussian integration over fluctuations around the stationary phase configuration(s)

Consider a functional integral

$$\int \mathcal{D}x \; e^{-F[x]} \quad \text{where} \quad \mathcal{D}x = \lim_{N\to\infty} \prod_{n=1}^{N} dx_n$$

with action $F[x]$ = arbitrary complex functional of x such that the integral converges.

and continuum limit $\{x_n\} \xrightarrow{N\to\infty} x(t)$

Steps for stationary phase approximation:

1. Find stationary configurations $\bar{x}$

$$\mathcal{D}F[x] = 0 \iff \forall t: \left.\frac{\delta F[x]}{\delta x(t)}\right|_{x=\bar{x}} = 0$$

2. Taylor expansion

$$F[x] = F[\bar{x}+y] = F[\bar{x}] + \frac{1}{2}\int dt \int dt' \, y(t') A(t,t') y(t) + \dots$$

$$A(t,t') \equiv \left.\frac{\delta^2 F[x]}{\delta x(t)\,\delta x(t')}\right|_{x=\bar{x}}$$

3. Positive definiteness of $A(t,t')$?

If yes : $\int \mathcal{D}x \; e^{-F[x]} \simeq e^{-F[\bar{x}]} \det\left(\frac{\hat{A}}{2\pi}\right)^{-1/2}$

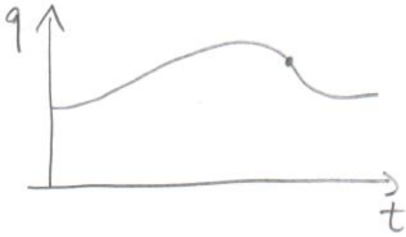
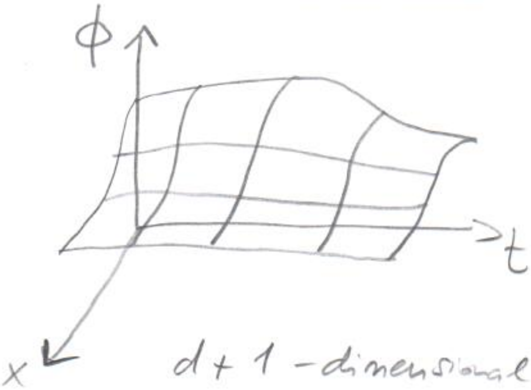
4. For multiple stationary configurations (solutions of 1.) add the solutions:

$$\boxed{\int \mathcal{D}x \; e^{-F[x]} \simeq \sum_i e^{-F[\bar{x}_i]} \det\left(\frac{\hat{A}_i}{2\pi}\right)^{-1/2}}$$

skipped

Lecture 4:

II.4 Functional field integral

Quantum mechanics	Quantum field theory
particle trajectory	field configurations
$\cdot q$	
	
1-dimensional	$d+1$-dimensional

formalism: almost the same!

II.4.1 Construction of many-body path integral

Reminder: Construction of path integral via insertion of position and momentum eigenstates

Now: eigenstates of annihilation operators

$\longrightarrow$ <u>coherent states</u>

Note: will be done only for bosons, since fermionic coherent states require the introduction of Grassmann numbers for anticommutation. Not important for field integral formalism.

Boson coherent states

Consider a general state in a bosonic Fock space

$$|\phi\rangle = \sum_{n_1,n_2\dots} C_{n_1,n_2\dots} |n_1,n_2,\dots\rangle$$

$$|n_1,n_2,\dots\rangle \equiv \frac{(a_1^\dagger)^{n_1}(a_2^\dagger)^{n_2}\dots}{\sqrt{n_1!}\sqrt{n_2!}\dots} \underbrace{|0\rangle}_{\text{vacuum}}$$

$a_i^\dagger$ creates a boson in state i

Observation: if the minimum number of particles in state $|\phi\rangle$ is n_0, the minimum of $a_i^\dagger|\phi\rangle$ must be n_0+1. $\Rightarrow$ $a_i^\dagger$ cannot possess eigenstates!

But: a_i possesses eigenstates

$$\boxed{|\phi\rangle \equiv \exp\left(\sum_i \phi_i a_i^\dagger\right)|0\rangle}$$

boson coherent states

$\{\phi_i\}$ = set of complex numbers

We have $\boxed{a_i|\phi\rangle = \phi_i|\phi\rangle}$, which can be shown via $a_i \exp(\phi_i a_i^\dagger)|0\rangle = \phi_i \exp(\phi_i a_i^\dagger)|0\rangle$.

Properties of coherent states:

(i) $\langle\phi| a_i^{\dagger} = \langle\phi| \bar{\phi}_i$ where $\bar{\phi}_i$ is complex conjugate of ϕ_i

(ii) $a_i^{\dagger}|\phi\rangle = \partial_{\phi_i}|\phi\rangle$ (shown by Taylor expansion)

which is consistent with canonical commutators:

$$[a_i^{\dagger}, a_j]|\phi\rangle = (\partial_{\phi_j}\phi_i - \phi_i\partial_{\phi_j})|\phi\rangle = \delta_{ij}|\phi\rangle$$

(iii) $\langle\theta|\phi\rangle = \exp\left(\sum_i \bar{\theta}_i \phi_i\right)$

(iv) Norm of a coherent state: $\langle\phi|\phi\rangle = \exp\left(\sum_i |\phi_i|^2\right)$

(v) Coherent state form an overcomplete set of states in Fock space $\mathcal{F}$
(not pairwise orthogonal, see (iii))

$$\int \underbrace{\prod_i \frac{d\bar{\phi}_i\, d\phi_i}{\pi}}_{\equiv d(\bar{\phi},\phi)} e^{-\sum_i \bar{\phi}_i \phi_i} |\phi\rangle\langle\phi| = \underbrace{\mathbb{1}_{\mathcal{F}}}_{\text{Unit operator in Fock space}}$$

where $d\bar{\phi}_i\, d\phi_i = d\,\mathrm{Re}\,\phi_i\; d\,\mathrm{Im}\,\phi_i$

Proof: $\mathbb{1}_{\mathcal{F}}$ is the operator that commutes with $\{a_i\}$ and $\{a_i^{\dagger}\}$, it is therefore sufficient to show that this commutation holds for the l.h.s. of (v) due to Schur's lemma using that these operators create all states in Fock space.

$$a_i \int d(\bar{\phi},\phi)\, e^{-\sum_i \bar{\phi}_i \phi_i} |\phi\rangle\langle\phi| = \int d(\bar{\phi},\phi)\, e^{-\sum_i \bar{\phi}_i \phi_i}\, \phi_i |\phi\rangle\langle\phi|$$

$$= -\int d(\bar{\phi},\phi) \left(\partial_{\bar{\phi}_i} e^{-\sum_i \bar{\phi}_i \phi_i}\right)|\phi\rangle\langle\phi|$$

$$\overset{\text{by parts}}{=} \int d(\bar{\phi},\phi)\, e^{-\sum_i \bar{\phi}_i \phi_i} |\phi\rangle \left(\partial_{\phi_i}\langle\phi|\right)$$

$$= \int d(\bar{\phi},\phi)\, e^{-\sum_i \bar{\phi}_i \phi_i} |\phi\rangle\langle\phi| a_i.$$

The constant of proportionality is fixed by the overlap with the vacuum:

$$\int d(\bar{\phi},\phi) e^{-\sum_i \bar{\phi}_i \phi_i} \langle 0|\phi\rangle\langle\phi|0\rangle = \int d(\bar{\phi},\phi) e^{-\sum_i \bar{\phi}_i \phi_i} = 1.$$

II.4.2 Field integral for the quantum partition function

Goal: Compute correlation functions of the form

$$\langle a^+ a \dots \rangle$$

= expectation values of products of operators

Strategy: compute the partition function, from which correlation functions can be obtained.

Partition function for the density matrix of a Gibbs ensemble, $\hat{\rho} \equiv \frac{e^{-\beta(\hat{H}-\mu\hat{N})}}{Z}$, is

$$Z = \operatorname{tr} e^{-\beta(\hat{H}-\mu\hat{N})} = \sum_n \langle n| e^{-\beta(\hat{H}-\mu\hat{N})}|n\rangle$$

with a complete set of Fock states $\{|n\rangle\}$, $\beta = \frac{1}{k_B T}$ inverse temperature, and μ the chemical potential. We use the identity (v) of coherent states

$$Z = \int d(\bar{\psi},\psi) e^{-\sum_i \bar{\psi}_i \psi_i} \sum_n \langle n|\psi\rangle \langle\psi| e^{-\beta(\hat{H}-\mu\hat{N})}|n\rangle.$$

Now we want to remove the $\sum_n |n\rangle\langle n| = \mathbb{1}_{\mathcal{F}}$. Care has to be taken for fermions (see Altland & Simons, Chap. 4.2), therefore we do this for bosons, for which

$\langle n|\psi\rangle\langle\psi|n\rangle = \langle\psi|n\rangle\langle n|\psi\rangle$ because complex numbers commute:

$$\mathcal{Z} = \int d(\bar{\psi},\psi)\, e^{-\sum_i \bar{\psi}_i \psi_i} \sum_n \langle\psi|e^{-\beta(\hat{H}-\mu\hat{N})}|n\rangle\langle n|\psi\rangle$$

$$= \int d(\bar{\psi},\psi)\, e^{-\sum_i \bar{\psi}_i \psi_i} \langle\psi|\, e^{-\beta(\hat{H}-\mu\hat{N})}|\psi\rangle .$$

Now we can apply the general construction scheme of the path integral.

Assume a Hamiltonian (*)

$$\hat{H}(a^+,a) = \sum_{ij} h_{ij}\, a_i^+ a_j + \sum_{ijkl} V_{ijkl}\, a_i^+ a_j^+ a_k a_l$$

that is normal-ordered, i.e. can be diagonalized using coherent states.

We divide the imaginary-time interval $[0,\beta]$ into N segments and insert identities:

$$\mathcal{Z} = \int_{\substack{\bar{\psi}^0=\bar{\psi}^N \\ \psi^0=\psi^N}} \prod_{n=1}^{N} d(\bar{\psi}^n,\psi^n)\, e^{-\delta \sum_{n=0}^{N-1}\left[\underbrace{\frac{\bar{\psi}^n-\bar{\psi}^{n+1}}{\delta}\psi^n}_{\substack{\to\ \bar{\psi}^{n+1}(\psi^{n+1}-\psi^n)+\dots \\ \to\ \bar{\psi}\partial_\tau\psi}} + H(\bar{\psi}^{n+1},\psi^n) - \mu N(\bar{\psi}^{n+1},\psi^n)\right]}$$

where $\delta = \frac{\beta}{N}$ and $H(\bar{\psi},\psi') \equiv \frac{\langle\psi|\hat{H}(a^+,a)|\psi'\rangle}{\langle\psi|\psi'\rangle}$

$$= \sum_{ij} h_{ij}\,\bar{\psi}_i\psi_j' + \sum_{ijkl} V_{ijkl}\,\bar{\psi}_i\bar{\psi}_j\psi_k'\psi_l'$$

$\psi^n \equiv \{\psi_i^n\}$ the fields at n-th time step

Taking the limit $N \to \infty$ we obtain the continuum path integral

$$\boxed{\begin{aligned} &\mathcal{Z} = \int \mathcal{D}(\bar{\psi},\psi)\, e^{-S[\bar{\psi},\psi]} \qquad \text{Functional integral in imaginary time representation} \\ &S[\bar{\psi},\psi] = \int_0^\beta d\tau \left[\bar{\psi}(\tau)\partial_\tau \psi(\tau) + H(\bar{\psi}(\tau),\psi(\tau)) - \mu N(\bar{\psi}(\tau),\psi(\tau)) \right] \\ &\mathcal{D}(\bar{\psi},\psi) \equiv \lim_{N\to\infty} \prod_{n=1}^{N} d(\bar{\psi}^n,\psi^n) \\ &\bar{\psi}(0) = \bar{\psi}(\beta), \quad \psi(0) = \psi(\beta) \qquad \text{periodic boundary conditions (for bosons)} \end{aligned}}$$

Explicitly for the Hamiltonian (*):

$$S[\bar{\psi},\psi] = \int_0^\beta d\tau \left[\sum_{ij} \bar{\psi}_i(\tau) \left[(\partial_\tau - \mu)\delta_{ij} + h_{ij} \right] \psi_j(\tau) + \right.$$

$$\left. + \sum_{ijkl} V_{ijkl}\, \bar{\psi}_i(\tau)\, \bar{\psi}_j(\tau)\, \psi_k(\tau)\, \psi_l(\tau) \right].$$

end lecture 4
(1.5 hrs)