Partitioning of 2- and 3-dimensional point spaces based on Fibo- and Tribonacci

Terminology

A Fibonacci sequence is normally said to start with 0,1 or 1,1 and from there extended upwards and downwards. In the following, the Fibonacci sequence (or other similar sequences with any other integers as offset) is regarded as a set of pairs, (a,b), where a and b are integers (both not equal to 0). Each pair defines by extension one (and only one) sequence, and a sequence consists of infinitively many integer pairs, of which each pair can be extended to include any other pair in the sequence. Similarly, integer triples define the Tribonacci or similar sequences.

Example: (2,3) and (13,21) can both be extended as the following sets of integer pairs: …, (0,1), (1,1), (1,2), (2,3), (3,5), (5,8), (8,13), (13,21), … and …, (2,3), (3,5), (5,8), (8,13), (13,21), (21,34),…, and therefore they define the same sequence. (1,3) defines another sequence, often referred to as the Lucas sequence. In the 2-dimensional plane every point with integer coordinates thus defines a single sequence consisting of an infinite number of points.

In the following, all integer sequences where each term is the sum of the two preceding terms will be called a fibosequence, and a tribosequence is an integer sequence where each term is the sum of the three preceding terms.

Points in the plane

It is well known that the ratio between corresponding values of the trad. Fibonacci and Lucas sequences for large values tends to the value of $\sqrt{5}$. This ratio, however, also exists between any fibosequence and a sequence called the superior sequence to the fibosequence.

In a fibosequence with the adjacent terms a and b, b will approx. be equal to $a*\varphi = a(1+\sqrt{5})/2$. The corresponding term to b in the superior sequence, b_1, is then given by:

$$b_1 = b*\sqrt{5} = a*\sqrt{5}(1+\sqrt{5})/2 = a(5+\sqrt{5})/2 = a(4+1+\sqrt{5})/2 = 2a+ \varphi a = 2a+b$$

Using this formula more terms in the superior sequence may be calculated:

Initial seq.:	a	b	a+b	a+2b	2a+3b
Sup. Seq.		2a+b	a+3b	3a+4b	4a+7b

As a and b are integers, the terms in the superior sequence will also be integers, and furthermore they show the fibosequence property with each term being the sum of the two preceding terms. The corresponding value to a may therefore be calculated backward as:

$a_1 = (a + 3b) – (2a+b) = -a +2b$

The superior value (a_1,b_1) of an integer pair (a,b) can therefore be found using the matrix calculation:

$$\begin{pmatrix} a1 \\ b1 \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ 2 & 1 \end{pmatrix} \text{x} \begin{pmatrix} a \\ b \end{pmatrix}$$

Repeating this calculation gives:

$$\begin{pmatrix} a2 \\ b2 \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ 2 & 1 \end{pmatrix} \text{x} \begin{pmatrix} a1 \\ b1 \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ 2 & 1 \end{pmatrix} \text{x} \begin{pmatrix} -a + 2b \\ 2a + b \end{pmatrix} = \begin{pmatrix} -(-a + 2b) + 2(2a + b) \\ 2(-a + 2b) + (2a + b) \end{pmatrix} = \begin{pmatrix} 5a \\ 5b \end{pmatrix}$$

The fibosequences will always have the form (e,u), (u,u) or (u,e) where ’e’ stands for an even integer and ’u’ for an uneven integer. The last digit of the terms in the fibosequence (a_2,b_2) above will have the form (…5,5,0,5,5,0,5,5,…)

The plane consisting of integer points (a,b) may now be partitioned into three sets:

<u>Set 1</u>: Points (a_1,b_1) where a and b are non-primic.

<u>Set 2</u>: Primic points (a_2,b_2) defining a sequence having a primic superior sequence.

<u>Set 3</u>: Primic points (a_3,b_3) defining a superior sequence. The superior sequence of a Set 3-sequence will be non-primic, and 5 will be a common divisor for all of its terms.

Examples:

(2,4), (3,27) are points that belong to non-primic sequences in Set 1

(4,5) is a primic Set 2-sequence, and it can be extended as: (4,5), (5,9), (9,14), (14,23), (23,37), … The corresponding Set 3-sequence is (6,13), (13,19), (19,32), (32,51), (51,83), … where e.g. 83/37 gives an estimated value of 2,24 for $\sqrt{5}$.

Repeating this calculation with the Set 3-sequence found above, we find the non-primic Set1-sequence (20,25), (25,45), (45,70), (70,115), (115,185),… which is exactly 5 times (or twice multiplied by $\sqrt{5}$) the Set 2-sequence in this example. Therefore, integers such as 37, 83 and 185 can be said to have an implicit relationship around $\sqrt{5}$ because 83/37 = 2,24 and 185/83 = 2,23 are both reasonable approximations of $\sqrt{5}$.

Points in the space

The points here are denoted (a,b,c), and a, b, c are integers (not all equal to 0), and each point defines a tribosequence.

We can distinguish between four sets in space:

<u>Set 1</u>: Points (a_1,b_1,c_1) where a_1,b_1 and c_1 are non-primic

<u>Set 2</u>: Primic points (a_2,b_2,c_2) of one of the forms:(e,e,u), (u,e,e), (u,u,e) or (e,u,u). A tribosequence of Set 1 points will have a superior tribosequence.

<u>Set 3</u>: Primic points (a_3,b_3,c_3) of one of the forms: (u,e,u) or (e,u,e). A tribosequence of Set 3 points is superior to a tribosequence of Set 2 points.

<u>Set 4</u>: Primic points (a_4,b_4,c_4) of the form (u,u,u). A tribosequence of Set 3 point is superior to a tribosequence of Set 2 points.

Whereas the ratio between two consecutive integers for a fibosequence approximatively is φ, the corresponding value for a tribosequence is τ = 1,83928675… where the exact value contains both third roots and square roots.

We now set τ' = 1/ τ +1 = 1,54368901... An exact formula for τ' is similar to the formula of τ'.

A way to calculate a Set 3- or Set 4-sequence, superior to a sequence of the preceding set using τ' is not as trivial as for points in the plane. Instead, large values of a, b and c may be multiplied with τ' for large values, rounded to the nearest integers and extended backwards. Having done this several times and solved nine linear equations with nine variables, the following matrix can be found (here shown for a Set 3-sequence superior to a Set 2-sequence, but with a similar calculation for the Set 3- and Set 4-reltionship):

$$\begin{pmatrix} a3 \\ b3 \\ c3 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \times \begin{pmatrix} a2 \\ b2 \\ c2 \end{pmatrix}$$

Examples:

A Set 2-sequence is based on the point (1,1,0) and looks like: 1, 1, 0, 2, 3, 5, 10, 18, 33, 61, …

The superior Set 3-sequence will be based on (-1,2,1) and may be extended to -1, 2, 1, 2, 5, 8, 15, 28, 51, 94,

and the superior Set 4-sequence based on (-1,1,3) extends to -1, 1, 3, 3, 7, 13, 23, 43, 79, 145, …

From the example τ’ may be correctly estimated to 94/61 = 1,54 and 145/94 = 1,54 after the first ten calculated values in each sequence.

A further calculation based on the last sequence leads to a non-primic Set 1-sequence with only even integers, in this case: 2, 0, 4, 6, 10, 20, 36, 66, 122, … After dividing with 2 as common divisor we get: 1, 0, 2, 3, 5, 10, 18, 33, 61, … which is the original Set 2-sequence, just one term displaced. This means that 2τ = τ’* τ’* τ’ or τ’ = $\sqrt[3]{2\tau}$, because τ corresponds to a displacement in a tribosequence, and τ’ multiplied three times corresponds to going from a Set 2-sequence over Set 3 and Set 4 to the Set 1-sequence.

Conclusion

Above, a way to partition a 2- and a 3-dimensional space of points with integer coordinates into a number of sets is shown.

In the plane, the result is a partition of the primic points into two sets showing infinitively more occurrences of the known relationship between the classical Fibonacci- and Tribonacci sequences based on $\sqrt{5}$. Moreover, a formula is presented that connects the participants of any such occurrence.

In the space, a similar partitioning of the primic points results in three sets. This partitioning into these sets is not surprising, as it reflects the ways to distribute even and uneven integers in a triple. It is, however, surprising that there is a connection between the members of the three sets via the value of τ’, where τ’ is a simple function of the Tribonacci constant, τ.

March 2024, Jens Rasmussen